Polynomial description for the[1] $\mathbb{T}$–Orbit Spaces of Multiplicative Actions

Evelyne Hubert*, Tobias Metzlaff*, Cordian Riener†

*INRIA Méditerranée, Université Côte d'Azur, France

†UiT The Arctic University, Norway

Les Journées Nationales de Calcul Formel
February 28, 2022

[1] This work has been supported by European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Actions, grant agreement 813211 (POEMA)

Contribution.

$\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Z})$ Weyl group of type $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.
$\mathbb{T}^n := \{\mathrm{x} \in \mathbb{C}^n \,|\, \forall\, 1 \leq i \leq n : \, |\mathrm{x}_i| = 1\}$.

Main Result [Hubert, M, Riener; 2022]

The set of orbits $\mathbb{T}^n/\mathcal{G}$ has the structure of a *compact basic semi–algebraic set*, consisting of all $\mathrm{z} \in \mathbb{R}^n$ with

$$H(\mathrm{z}) \succeq 0$$

for an *explicit* symmetric matrix polynomial $H \in \mathbb{R}[\mathrm{z}]^{n \times n}$.

Content:

1. Multiplicative Actions.
2. Motivation.
3. Main Result.
4. Conjecture.
5. Chromatic Number.

Contribution.

$\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Z})$ Weyl group of type $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.
$\mathbb{T}^n := \{\mathrm{x} \in \mathbb{C}^n \,|\, \forall\, 1 \leq i \leq n : \ |\mathrm{x}_i| = 1\}$.

Main Result [Hubert, M, Riener; 2022]

The set of orbits $\mathbb{T}^n/\mathcal{G}$ has the structure of a *compact basic semi–algebraic set*, consisting of all $\mathrm{z} \in \mathbb{R}^n$ with

$$H(\mathrm{z}) \succeq 0$$

for an *explicit* symmetric matrix polynomial $H \in \mathbb{R}[\mathrm{z}]^{n \times n}$.

Content:

1. Multiplicative Actions.
2. Motivation.
3. Main Result.
4. Conjecture.
5. Chromatic Number.

Multiplicative Actions.

A finite group $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{Z})$ has a *nonlinear* action

$$\begin{array}{rccc} \star: & \mathcal{G} \times (\mathbb{C}^*)^n & \to & (\mathbb{C}^*)^n, \\ & (B, \mathrm{x}) & \mapsto & B \star \mathrm{x} := (\mathrm{x}_1, \ldots, \mathrm{x}_n)^{B^{-1}} = (\mathrm{x}^{B^{-1}_{\cdot 1}}, \ldots, \mathrm{x}^{B^{-1}_{\cdot n}}), \end{array}$$

which induces an action on the coordinate ring

$$\begin{array}{rccc} \cdot: & \mathcal{G} \times \mathbb{K}[\mathrm{x}^{\pm}] & \to & \mathbb{K}[\mathrm{x}^{\pm}], \\ & (B, f = \sum_\alpha f_\alpha \, \mathrm{x}^\alpha) & \mapsto & B \cdot f := \sum_\alpha f_\alpha \, \mathrm{x}^{B\,\alpha}. \end{array}$$

with $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$ and $\mathbb{K}[\mathrm{x}^{\pm}] := \mathbb{K}[\mathrm{x}_1, \mathrm{x}_1^{-1}, \ldots, \mathrm{x}_n, \mathrm{x}_n^{-1}]$.

Definition

$f \in \mathbb{K}[\mathrm{x}^{\pm}]$ is **$\mathcal{G}$–invariant** $\Leftrightarrow$ for all $B \in \mathcal{G}$: $B \cdot f = f$.
$\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}}$ is the ring of $\mathcal{G}$–invariants, the **multiplicative invariants**.

Multiplicative Actions.

A finite group $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{Z})$ has a *nonlinear* action

$$\begin{array}{rccl} \star: & \mathcal{G} \times (\mathbb{C}^*)^n & \to & (\mathbb{C}^*)^n, \\ & (B, \mathrm{x}) & \mapsto & B \star \mathrm{x} := (\mathrm{x}_1, \ldots, \mathrm{x}_n)^{B^{-1}} = (\mathrm{x}^{B^{-1}_{\cdot 1}}, \ldots, \mathrm{x}^{B^{-1}_{\cdot n}}), \end{array}$$

which induces an action on the coordinate ring

$$\begin{array}{rccl} \cdot: & \mathcal{G} \times \mathbb{K}[\mathrm{x}^{\pm}] & \to & \mathbb{K}[\mathrm{x}^{\pm}], \\ & (B,\, f = \sum_\alpha f_\alpha \, \mathrm{x}^\alpha) & \mapsto & B \cdot f := \sum_\alpha f_\alpha \, \mathrm{x}^{B\,\alpha}. \end{array}$$

with $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$ and $\mathbb{K}[\mathrm{x}^{\pm}] := \mathbb{K}[\mathrm{x}_1, \mathrm{x}_1^{-1}, \ldots, \mathrm{x}_n, \mathrm{x}_n^{-1}]$.

Definition

$f \in \mathbb{K}[\mathrm{x}^{\pm}]$ is **$\mathcal{G}$–invariant** $\Leftrightarrow$ for all $B \in \mathcal{G}$: $B \cdot f = f$.
$\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}}$ is the ring of $\mathcal{G}$–invariants, the **multiplicative invariants**.

Multiplicative Actions.

A finite group $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{Z})$ has a *nonlinear* action

$$\begin{array}{rrcl} \star : & \mathcal{G} \times (\mathbb{C}^*)^n & \to & (\mathbb{C}^*)^n, \\ & (B, \mathrm{x}) & \mapsto & B \star \mathrm{x} := (\mathrm{x}_1, \ldots, \mathrm{x}_n)^{B^{-1}} = (\mathrm{x}^{B^{-1}_{\cdot 1}}, \ldots, \mathrm{x}^{B^{-1}_{\cdot n}}), \end{array}$$

which induces an action on the coordinate ring

$$\begin{array}{rrcl} \cdot : & \mathcal{G} \times \mathbb{K}[\mathrm{x}^\pm] & \to & \mathbb{K}[\mathrm{x}^\pm], \\ & (B,\, f = \sum_\alpha f_\alpha\, \mathrm{x}^\alpha) & \mapsto & B \cdot f := \sum_\alpha f_\alpha\, \mathrm{x}^{B\,\alpha}. \end{array}$$

with $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$ and $\mathbb{K}[\mathrm{x}^\pm] := \mathbb{K}[\mathrm{x}_1, \mathrm{x}_1^{-1}, \ldots, \mathrm{x}_n, \mathrm{x}_n^{-1}]$.

Definition

$f \in \mathbb{K}[\mathrm{x}^\pm]$ is **$\mathcal{G}$–invariant** $\Leftrightarrow$ for all $B \in \mathcal{G}$: $B \cdot f = f$.
$\mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}}$ is the ring of $\mathcal{G}$–invariants, the **multiplicative invariants**.

Multiplicative Actions.

Assume $\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}} = \mathbb{K}[\theta_1, \dots, \theta_m]$ and define
$\mathbb{T}^n := \{\mathrm{x} \in \mathbb{C}^n \,|\, \forall\, 1 \leq i \leq n : \, |\mathrm{x}_i| = 1\}$.

Theorem [Hubert, M, Riener; 2022]

The map

$$\begin{array}{rccl} \vartheta : & \mathbb{T}^n & \to & \mathbb{C}^m, \\ & \mathrm{x} & \mapsto & (\theta_1(\mathrm{x}), \dots, \theta_m(\mathrm{x})), \end{array}$$

separates the orbits $\mathbb{T}^n/\mathcal{G}$.

Definition

$\mathcal{T} := \mathrm{im}(\vartheta)$ is called the $\mathbb{T}$–**orbit space** of $\mathcal{G}$.

Claim: $\mathcal{T}$ is a compact basic semi–algebraic set.

Multiplicative Actions.

Assume $\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_m]$ and define $\mathbb{T}^n := \{\mathrm{x} \in \mathbb{C}^n \,|\, \forall\, 1 \leq i \leq n : \, |\mathrm{x}_i| = 1\}$.

Theorem [Hubert, M, Riener; 2022]

The map

$$\begin{array}{rccl} \vartheta : & \mathbb{T}^n & \to & \mathbb{C}^m, \\ & \mathrm{x} & \mapsto & (\theta_1(\mathrm{x}), \ldots, \theta_m(\mathrm{x})), \end{array}$$

separates the orbits $\mathbb{T}^n/\mathcal{G}$.

Definition

$\mathcal{T} := \mathrm{im}(\vartheta)$ is called the $\mathbb{T}$–**orbit space** of $\mathcal{G}$.

Claim: $\mathcal{T}$ is a compact basic semi–algebraic set.

Multiplicative Actions.

Assume $\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_m]$ and define
$\mathbb{T}^n := \{\mathrm{x} \in \mathbb{C}^n \,|\, \forall\, 1 \leq i \leq n : \; |\mathrm{x}_i| = 1\}$.

Theorem [Hubert, M, Riener; 2022]

The map

$$\begin{array}{rccl} \vartheta : & \mathbb{T}^n & \to & \mathbb{C}^m, \\ & \mathrm{x} & \mapsto & (\theta_1(\mathrm{x}), \ldots, \theta_m(\mathrm{x})), \end{array}$$

separates the orbits $\mathbb{T}^n/\mathcal{G}$.

Definition

$\mathcal{T} := \mathrm{im}(\vartheta)$ is called the $\mathbb{T}$–**orbit space** of $\mathcal{G}$.

Claim: $\mathcal{T}$ is a compact basic semi–algebraic set.

Motivation.

"linear" [1]

G compact Lie group on $\mathbb{C}^n$
coordinate ring $\mathbb{K}[\mathrm{X}]$
$\mathbb{K}[\mathrm{X}]^G = \mathbb{K}[\pi_1, \ldots \pi_m]$
restriction to $\mathbb{R}^n$: $\overline{\mathrm{X}} = \mathrm{X}$
$\mathbb{R}^n / G$: $M(\mathrm{X}) \succeq 0$

"nonlinear"

$\mathcal{G}$ finite int. group on $(\mathbb{C}^*)^n$
coordinate ring $\mathbb{K}[\mathrm{x}^{\pm}]$
$\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_m]$
restriction to $\mathbb{T}^n$: $\overline{\mathrm{x}} = 1/\mathrm{x}$
$\mathbb{T}^n / \mathcal{G}$: ?

Applications

- differential geometry [Dubrovin; 1993]
- equivariant dynamical systems [Gatermann; 2000]
- **poly. opt.** [Riener, Theobald, Anrdén, Lasserre; 2013]
- quantum systems [Gerdt, Khvedelidze, Palii; 2013]

[1] [Procesi, Schwarz: *Inequalities defining orbit spaces*; 1985]

Motivation.

"linear"[1]

G compact Lie group on $\mathbb{C}^n$
coordinate ring $\mathbb{K}[\mathrm{X}]$
$\mathbb{K}[\mathrm{X}]^G = \mathbb{K}[\pi_1, \ldots \pi_m]$
restriction to $\mathbb{R}^n$: $\overline{\mathrm{X}} = \mathrm{X}$
$\mathbb{R}^n/G$: $M(\mathrm{X}) \succeq 0$

"nonlinear"

$\mathcal{G}$ finite int. group on $(\mathbb{C}^*)^n$
coordinate ring $\mathbb{K}[\mathrm{x}^\pm]$
$\mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_m]$
restriction to $\mathbb{T}^n$: $\overline{\mathrm{x}} = 1/\mathrm{x}$
$\mathbb{T}^n/\mathcal{G}$: ?

Applications

- differential geometry [Dubrovin; 1993]
- equivariant dynamical systems [Gatermann; 2000]
- **poly. opt.** [Riener, Theobald, Anrdén, Lasserre; 2013]
- quantum systems [Gerdt, Khvedelidze, Palii; 2013]

[1] [Procesi, Schwarz: *Inequalities defining orbit spaces*; 1985]

Motivation.

"linear" [1]

G compact Lie group on $\mathbb{C}^n$
coordinate ring $\mathbb{K}[\mathrm{X}]$
$\mathbb{K}[\mathrm{X}]^G = \mathbb{K}[\pi_1, \ldots \pi_m]$
restriction to $\mathbb{R}^n$: $\overline{\mathrm{X}} = \mathrm{X}$
$\mathbb{R}^n/G$: $M(\mathrm{X}) \succeq 0$

"nonlinear"

$\mathcal{G}$ finite int. group on $(\mathbb{C}^*)^n$
coordinate ring $\mathbb{K}[\mathrm{x}^{\pm}]$
$\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_m]$
restriction to $\mathbb{T}^n$: $\overline{\mathrm{x}} = 1/\mathrm{x}$
$\mathbb{T}^n/\mathcal{G}$: ?

Applications

- differential geometry [Dubrovin; 1993]
- equivariant dynamical systems [Gatermann; 2000]
- **poly. opt.** [Riener, Theobald, Anrdén, Lasserre; 2013]
- quantum systems [Gerdt, Khvedelidze, Palii; 2013]

[1] [Procesi, Schwarz: *Inequalities defining orbit spaces*; 1985]

Main Result.

For $\alpha \in \mathbb{Z}^n$, define

$$\Theta_\alpha(\mathrm{x}) := \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} \mathrm{x}^{B\,\alpha} \in \mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}} \quad \text{and} \quad \theta_i := \Theta_{e_i}.$$

Assume $\mathcal{G} \cong$ Weyl group of a *crystallographic* root system, such that the weight lattice $\Omega := \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$ is $\mathcal{G}$–invariant.

Theorem[1]

$\theta_1, \ldots, \theta_n$ are algebraically independent and

$$\mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_n].$$

Definition

Define the map

$$\begin{array}{rccl} \mathfrak{c}: & \mathbb{R}^n & \to & \mathbb{C}^n, \\ & u & \mapsto & \vartheta(\exp(-2\pi\mathrm{i}\,\langle \omega_1, u\rangle), \ldots, \exp(-2\pi\mathrm{i}\,\langle \omega_n, u\rangle)). \end{array}$$

[1] [N. Bourbaki: *Systèmes de racines* §3 Théorème 1; 1968]

Main Result.

For $\alpha \in \mathbb{Z}^n$, define

$$\Theta_\alpha(\mathrm{x}) := \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} \mathrm{x}^{B\,\alpha} \in \mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}} \quad \text{and} \quad \theta_i := \Theta_{e_i}.$$

Assume $\mathcal{G} \cong$ Weyl group of a *crystallographic* root system, such that the weight lattice $\Omega := \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$ is $\mathcal{G}$–invariant.

Theorem[1]

$\theta_1, \ldots, \theta_n$ are algebraically independent and

$$\mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_n].$$

Definition

Define the map

$$\begin{array}{rccl} \mathfrak{c}: & \mathbb{R}^n & \to & \mathbb{C}^n, \\ & u & \mapsto & \vartheta(\exp(-2\pi\mathrm{i}\,\langle \omega_1, u\rangle), \ldots, \exp(-2\pi\mathrm{i}\,\langle \omega_n, u\rangle)). \end{array}$$

[1] [N. Bourbaki: *Systèmes de racines* §3 Théorème 1; 1968]

Main Result.

For $\alpha \in \mathbb{Z}^n$, define

$$\Theta_\alpha(\mathrm{x}) := \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} \mathrm{x}^{B\,\alpha} \in \mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}} \quad \text{and} \quad \theta_i := \Theta_{e_i}.$$

Assume $\mathcal{G} \cong$ Weyl group of a *crystallographic* root system, such that the weight lattice $\Omega := \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$ is $\mathcal{G}$–invariant.

Theorem[1]

$\theta_1, \ldots, \theta_n$ are algebraically independent and

$$\mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_n].$$

Definition

Define the map

$$\begin{array}{rccl} \mathfrak{c}: & \mathbb{R}^n & \to & \mathbb{C}^n, \\ & u & \mapsto & \vartheta(\exp(-2\pi\mathrm{i}\,\langle \omega_1, u\rangle), \ldots, \exp(-2\pi\mathrm{i}\,\langle \omega_n, u\rangle)). \end{array}$$

[1] [N. Bourbaki: *Systèmes de racines* §3 Théorème 1; 1968]

Main Result.

For $\alpha \in \mathbb{Z}^n$, define

$$\Theta_\alpha(\mathrm{x}) := \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} \mathrm{x}^{B\,\alpha} \in \mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}} \quad \text{and} \quad \theta_i := \Theta_{e_i}.$$

Assume $\mathcal{G} \cong$ Weyl group of a *crystallographic* root system, such that the weight lattice $\Omega := \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$ is $\mathcal{G}$–invariant.

Theorem[1]

$\theta_1, \ldots, \theta_n$ are algebraically independent and

$$\mathbb{K}[\mathrm{x}^\pm]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_n].$$

Definition

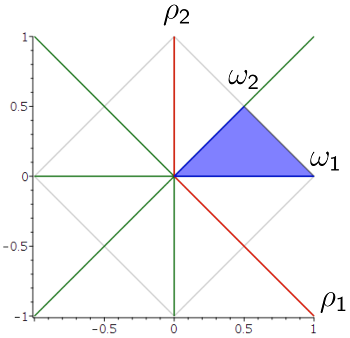
Define the map

$$\begin{array}{rccl} \mathfrak{c}: & \mathbb{R}^n & \to & \mathbb{C}^n, \\ & u & \mapsto & \vartheta(\exp(-2\pi\mathrm{i}\,\langle \omega_1, u\rangle), \ldots, \exp(-2\pi\mathrm{i}\,\langle \omega_n, u\rangle)). \end{array}$$

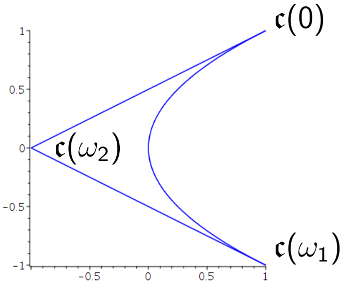
[1] [N. Bourbaki: *Systèmes de racines* §3 Théorème 1; 1968]

Example

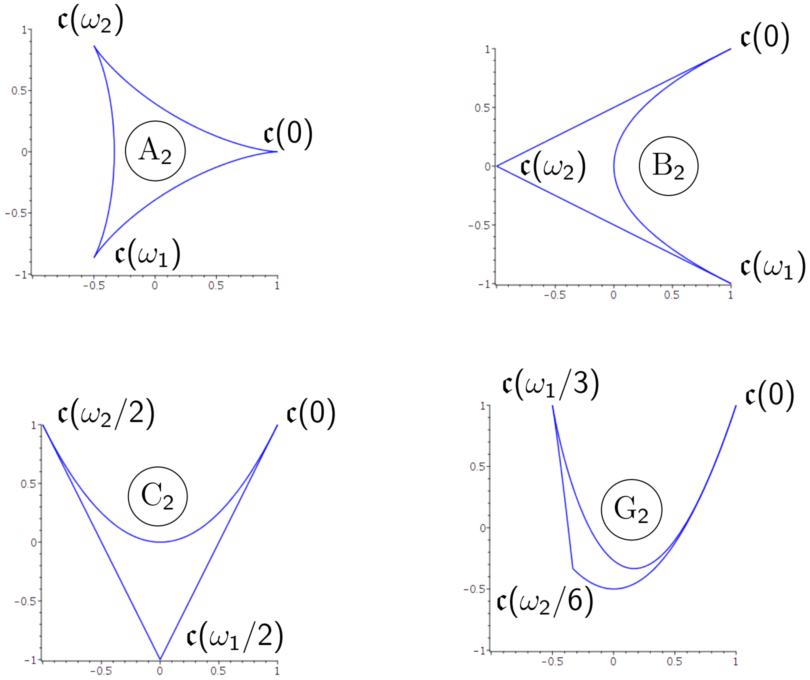
$\mathfrak{c} : \mathbb{R}^2 \to \mathbb{R}^2, u \mapsto ((\cos(2\pi u_1) + \cos(2\pi u_2))/2, \cos(\pi u_1)\cos(\pi u_2))$

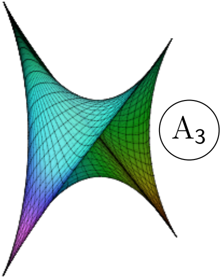


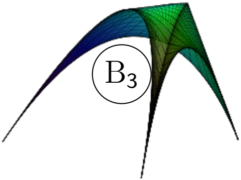
fundamental domain
of the affine Weyl group

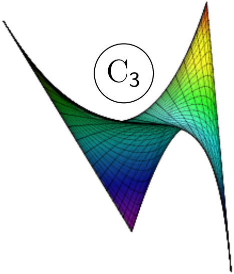


$\mathcal{T} = \mathrm{im}(\mathfrak{c})$


$\mathfrak{c}(\omega_2)$
$\mathfrak{c}(0)$
A_2
$\mathfrak{c}(\omega_1)$
$\mathfrak{c}(0)$
$\mathfrak{c}(\omega_2)$
B_2
$\mathfrak{c}(\omega_1)$
$\mathfrak{c}(\omega_2/2)$
$\mathfrak{c}(0)$
C_2
$\mathfrak{c}(\omega_1/2)$
$\mathfrak{c}(\omega_1/3)$
$\mathfrak{c}(0)$
G_2
$\mathfrak{c}(\omega_2/6)$







Applications

- cubature nodes [Li, Xu; 2010]
- sampling points
  [Munthe–Kaas et al; 2012]
- sparse interpolation
  [Hubert, Singer; 2021]

Main Result.

Assume $\mathcal{G} \cdot \mathrm{x}_1 = \{\mathrm{y}_1, \mathrm{y}_2, \ldots\}$ is the orbit of $\mathrm{x}_1 \rightsquigarrow \mathrm{y}(\mathrm{x}) \in [\mathrm{x}^{\pm}]^n$

$\mathcal{A}_{n-1}$

$\mathcal{G} \cong \mathfrak{S}_n$

$\sigma_i(\mathrm{y}(\mathrm{x})) = \binom{n}{i} \Theta_{\omega_i}(\mathrm{x})$

$\sigma_n(\mathrm{y}(\mathrm{x})) = 1$

$\mathcal{C}_n$

$\mathcal{G} \cong \mathfrak{S}_n \ltimes \{\pm 1\}^n$

$\sigma_i(\mathrm{y}(\mathrm{x})) = 2^i \binom{n}{i} \Theta_{\omega_i}(\mathrm{x})$

$\mathcal{B}_n$

$\mathcal{G} \cong \mathfrak{S}_n \ltimes \{\pm 1\}^n$

$\sigma_i(\mathrm{y}(\mathrm{x})) = 2^i \binom{n}{i} \Theta_{\omega_i}(\mathrm{x})$

$\sigma_n(\mathrm{y}(\mathrm{x})) = 2^n \Theta_{2\omega_n}(\mathrm{x})$

$\mathcal{D}_n$

$\mathcal{G} \cong \mathfrak{S}_n \ltimes \{\pm 1\}^n_+$

$\sigma_i(\mathrm{y}(\mathrm{x})) = 2^i \binom{n}{i} \Theta_{\omega_i}(\mathrm{x})$

$\sigma_{n-1}(\mathrm{y}(\mathrm{x})) = 2^{n-1} \Theta_{\omega_{n-1}+\omega_n}(\mathrm{x})$

$\sigma_n(\mathrm{y}(\mathrm{x})) = 2^{n-1} \left(\Theta_{2\omega_{n-1}}(\mathrm{x}) + \Theta_{2\omega_n}(\mathrm{x})\right)$

σ_i elementary symmetric polynomials in n indeterminates

Main Result.

Assume $\mathcal{G} \cdot \mathrm{x}_1 = \{\mathrm{y}_1, \mathrm{y}_2, \ldots\}$ is the orbit of $\mathrm{x}_1 \rightsquigarrow \mathrm{y}(\mathrm{x}) \in [\mathrm{x}^{\pm}]^n$

$\mathcal{A}_{n-1}$

$\mathcal{G} \cong \mathfrak{S}_n$

$\sigma_i(\mathrm{y}(\mathrm{x})) = \binom{n}{i} \Theta_{\omega_i}(\mathrm{x})$

$\sigma_n(\mathrm{y}(\mathrm{x})) = 1$

$\mathcal{B}_n$

$\mathcal{G} \cong \mathfrak{S}_n \ltimes \{\pm 1\}^n$

$\sigma_i(\mathrm{y}(\mathrm{x})) = 2^i \binom{n}{i} \Theta_{\omega_i}(\mathrm{x})$

$\sigma_n(\mathrm{y}(\mathrm{x})) = 2^n \Theta_{2\omega_n}(\mathrm{x})$

$\mathcal{C}_n$

$\mathcal{G} \cong \mathfrak{S}_n \ltimes \{\pm 1\}^n$

$\sigma_i(\mathrm{y}(\mathrm{x})) = 2^i \binom{n}{i} \Theta_{\omega_i}(\mathrm{x})$

$\mathcal{D}_n$

$\mathcal{G} \cong \mathfrak{S}_n \ltimes \{\pm 1\}^n_+$

$\sigma_i(\mathrm{y}(\mathrm{x})) = 2^i \binom{n}{i} \Theta_{\omega_i}(\mathrm{x})$

$\sigma_{n-1}(\mathrm{y}(\mathrm{x})) = 2^{n-1} \Theta_{\omega_{n-1}+\omega_n}(\mathrm{x})$

$\sigma_n(\mathrm{y}(\mathrm{x})) = 2^{n-1} \left(\Theta_{2\omega_{n-1}}(\mathrm{x}) + \Theta_{2\omega_n}(\mathrm{x})\right)$

σ_i elementary symmetric polynomials in n indeterminates

Main Result.

Proposition

There exists a unique $f \in \mathbb{K}[\mathrm{z}][t]$, such that for all $\mathrm{x} \in \mathbb{T}^n$,

$$f(\vartheta(\mathrm{x}))(t) = \sum_{i=0}^{n} (-1)^i \sigma_i(\mathrm{y}(\mathrm{x}))\, t^{n-i} \in \mathbb{R}[t].$$

With Sturm's version of Sylvester's Theorem[1] we obtain the main result.

Theorem [Hubert, M, Riener; 2022]

Define the matrix $H \in \mathbb{K}[\mathrm{z}]^{n \times n}$ with

$$\begin{aligned} H(\mathrm{z})_{ij} &= \mathrm{Trace}(4\,(C(\mathrm{z}))^{i+j-2} - (C(\mathrm{z}))^{i+j}), \quad \text{where} \\ C(\mathrm{z}) &= \mathrm{CompanionMatrix}(f(\mathrm{z})) \end{aligned}$$

Then $\mathcal{T} = \{\mathrm{z} \in \mathbb{R}^n \mid H(\mathrm{z}) \succeq 0\}$.

[1] Basu, Pollack, Roy: *Algorithms in Real Algebraic Geometry*, Theorem 4.57; 2006

Main Result.

Proposition

There exists a unique $f \in \mathbb{K}[\mathrm{z}][t]$, such that for all $\mathrm{x} \in \mathbb{T}^n$,

$$f(\vartheta(\mathrm{x}))(t) = \sum_{i=0}^{n} (-1)^i \sigma_i(\mathrm{y}(\mathrm{x}))\, t^{n-i} \in \mathbb{R}[t].$$

With Sturm's version of Sylvester's Theorem[1] we obtain the main result.

Theorem [Hubert, M, Riener; 2022]

Define the matrix $H \in \mathbb{K}[\mathrm{z}]^{n \times n}$ with

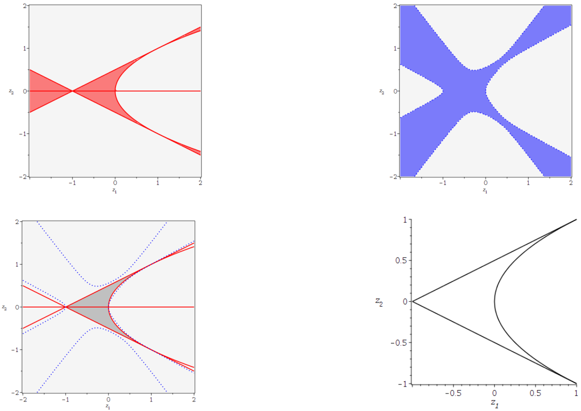
$$\begin{aligned} H(\mathrm{z})_{ij} &= \mathrm{Trace}(4\,(C(\mathrm{z}))^{i+j-2} - (C(\mathrm{z}))^{i+j}), \quad \text{where} \\ C(\mathrm{z}) &= \mathrm{CompanionMatrix}(f(\mathrm{z})) \end{aligned}$$

Then $\mathcal{T} = \{\mathrm{z} \in \mathbb{R}^n \,|\, H(\mathrm{z}) \succeq 0\}$.

[1] Basu, Pollack, Roy: *Algorithms in Real Algebraic Geometry*, Theorem 4.57; 2006

Example: $\mathcal{G} \cong \mathcal{B}_2$.

$$H(z) = 16 \begin{pmatrix} -z_1^2 + 2z_2^2 - z_1 & -4z_1^3 + 12z_1\, z_2^2 - 6z_1^2 - 2z_1 \\ -4z_1^3 + 12z_1\, z_2^2 - 6z_1^2 - 2z_1 & -16z_1^4 + 64z_1^2\, z_2^2 - 32z_2^4 - 32z_1^3 + 32z_1\, z_2^2 - 20z_1^2 + 8z_2^2 - 4z_1 \end{pmatrix}$$



$\mathrm{Det}(H(z)) = 0$ (solid) $\mathrm{Trace}(H(z)) = 0$ (dots)

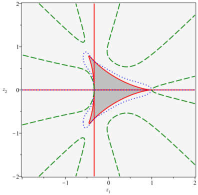
Conjecture.

$\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Z})$ finite, $\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_m]$.
Define $\tilde{\nabla} := [\mathrm{x}_1\, \partial/\partial \mathrm{x}_1, \ldots, \mathrm{x}_n\, \partial/\partial \mathrm{x}_n]^t$ and,
for $f \in \mathbb{K}[\mathrm{x}^{\pm}]$, $\widehat{f}(x) := f(1/\mathrm{x}) \in \mathbb{K}[\mathrm{x}^{\pm}]$.

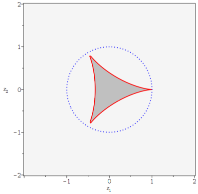
Conjecture [Hubert, M, Riener; 2022]

Let $\tilde{M} = (\langle \tilde{\nabla}\, \theta_i, \tilde{\nabla}\, \widehat{\theta_j} \rangle)_{ij} \in (\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}})^{m\times m}$ and $M \in \mathbb{K}[\mathrm{z}]^{m\times m}$, such that $\tilde{M} = M(\theta_1, \ldots, \theta_m)$. Then

$$\mathcal{T} = \{\mathrm{z} \in \mathbb{R}^n \mid M(\mathrm{z}) \preceq 0\}.$$



$H(\mathrm{z}) \succeq 0$



$M(\mathrm{z}) \preceq 0$

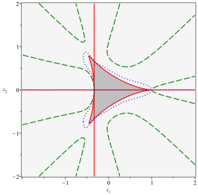
Conjecture.

$\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Z})$ finite, $\mathbb{K}[\mathrm{x}^\pm]^\mathcal{G} = \mathbb{K}[\theta_1, \ldots, \theta_m]$.
Define $\tilde{\nabla} := [\mathrm{x}_1\, \partial/\partial \mathrm{x}_1, \ldots, \mathrm{x}_n\, \partial/\partial \mathrm{x}_n]^t$ and,
for $f \in \mathbb{K}[\mathrm{x}^\pm]$, $\widehat{f}(x) := f(1/\mathrm{x}) \in \mathbb{K}[\mathrm{x}^\pm]$.

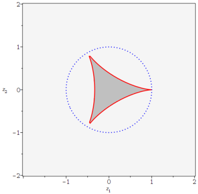
Conjecture [Hubert, M, Riener; 2022]

Let $\tilde{M} = (\langle \tilde{\nabla}\, \theta_i, \tilde{\nabla}\, \widehat{\theta_j} \rangle)_{ij} \in (\mathbb{K}[\mathrm{x}^\pm]^\mathcal{G})^{m \times m}$ and $M \in \mathbb{K}[\mathrm{z}]^{m \times m}$, such that $\tilde{M} = M(\theta_1, \ldots, \theta_m)$. Then

$$\mathcal{T} = \{\mathrm{z} \in \mathbb{R}^n \mid M(\mathrm{z}) \preceq 0\}.$$



$H(\mathrm{z}) \succeq 0$



$M(\mathrm{z}) \preceq 0$

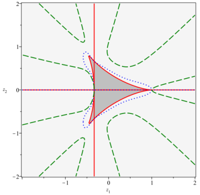
Conjecture.

$\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Z})$ finite, $\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}} = \mathbb{K}[\theta_1, \ldots, \theta_m]$.
Define $\tilde{\nabla} := [\mathrm{x}_1 \, \partial/\partial \mathrm{x}_1, \ldots, \mathrm{x}_n \, \partial/\partial \mathrm{x}_n]^t$ and,
for $f \in \mathbb{K}[\mathrm{x}^{\pm}]$, $\widehat{f}(x) := f(1/\mathrm{x}) \in \mathbb{K}[\mathrm{x}^{\pm}]$.

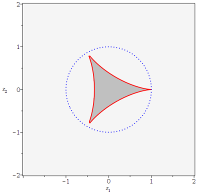
Conjecture [Hubert, M, Riener; 2022]

Let $\tilde{M} = (\langle \tilde{\nabla} \, \theta_i, \tilde{\nabla} \, \widehat{\theta_j} \rangle)_{ij} \in (\mathbb{K}[\mathrm{x}^{\pm}]^{\mathcal{G}})^{m \times m}$ and $M \in \mathbb{K}[\mathrm{z}]^{m \times m}$, such that $\tilde{M} = M(\theta_1, \ldots, \theta_m)$. Then

$$\mathcal{T} = \{\mathrm{z} \in \mathbb{R}^n \mid M(\mathrm{z}) \preceq 0\}.$$



$H(\mathrm{z}) \succeq 0$



$M(\mathrm{z}) \preceq 0$

Chromatic Number.

Definition

Let $||\cdot||$ be a norm on $\mathbb{R}^n$. Consider the graph (V, E) with

$$V = \mathbb{R}^n \quad \{u_1, u_2\} \in E \Leftrightarrow ||u_1 - u_2|| = 1.$$

The minimal number of colors needed to color this graph is called the measurable **chromatic number** $\chi_m(\mathbb{R}^n, ||\cdot||)$ of $\mathbb{R}^n$.

- [Hardwiger, Nelson; 1950]: $n = 2$ and $||\cdot||$ Euclidean norm
- [Bachoc, Decorte, Oliviera, Vallentin; 2014]: spectral (lower) bound for infinite graphs
- [Bachoc, Bellitto, Moustrou, Pêcher; 2019]: $\chi_m(\mathbb{R}^n, ||\cdot||) \leq 2^n$ for tiling polytope norms ("$=$" conjectured by Bachoc, Robins)

Chromatic Number.

Definition

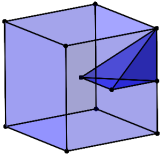
Let $||\cdot||$ be a norm on $\mathbb{R}^n$. Consider the graph (V, E) with

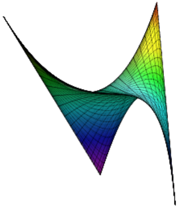
$$V = \mathbb{R}^n \quad \{u_1, u_2\} \in E \Leftrightarrow ||u_1 - u_2|| = 1.$$

The minimal number of colors needed to color this graph is called the measurable **chromatic number** $\chi_m(\mathbb{R}^n, ||\cdot||)$ of $\mathbb{R}^n$.

- [Hardwiger, Nelson; 1950]: $n = 2$ and $||\cdot||$ Euclidean norm
- [Bachoc, Decorte, Oliviera, Vallentin; 2014]:
  spectral (lower) bound for infinite graphs
- [Bachoc, Bellitto, Moustrou, Pêcher; 2019]:
  $\chi_m(\mathbb{R}^n, ||\cdot||) \leq 2^n$ for tiling polytope norms
  ("$=$" conjectured by Bachoc, Robins)

Example: $\mathcal{G} \cong \mathcal{C}_n$.





Weyl group: $\mathcal{G} \cong \mathfrak{S}_n \ltimes \{\pm 1\}^n$

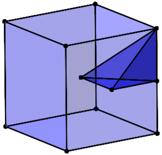
fundamental weights: $\omega_i = e_1 + \ldots + e_i$
(vertices and centers of edges/ faces/ facets/... of the cube)

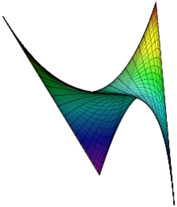
$$\mathfrak{c}(u) = \binom{n}{i}^{-1} \sigma_i(\cos(2\pi u_k)|_{k=1\ldots n})$$

Theorem [Hubert, M, Moustrou, Riener]

$$\chi_m(\mathbb{R}^n, ||\cdot||) \geq 1 - \frac{1}{\min\limits_{z \in \mathcal{T}} F(z)} = 2^n$$

Example: $\mathcal{G} \cong \mathcal{C}_n$.





Weyl group: $\mathcal{G} \cong \mathfrak{S}_n \ltimes \{\pm 1\}^n$

fundamental weights: $\omega_i = e_1 + \ldots + e_i$
(vertices and centers of edges/ faces/ facets/... of the cube)

$$\mathfrak{c}(u) = \binom{n}{i}^{-1} \sigma_i(\cos(2\pi u_k)|_{k=1\ldots n})$$

Theorem [Hubert, M, Moustrou, Riener]

$$\chi_m(\mathbb{R}^n, ||\cdot||) \geq 1 - \frac{1}{\min_{z \in \mathcal{T}} F(z)} = 2^n$$

Merci.

- N. Bourbaki: *Groupes et algèbres de Lie* Chapitre VI, Éléments de Mathématique, Springer 1981, springer.com
- S. Basu, R. Pollack, M. F. Roy: *Algorithms in Real Algebraic Geometry*, Springer 2006, springer.com
- H. Z. Munthe-Kaas, M. Nome, B. N. Ryland: *Through the Kaleidoscope*, FoCM 2012, cambridge.org
- E. Hubert, M. Singer: *Sparse Interpolation in Terms of Multivariate Chebyshev Polynomials*, FoCM 2021, springer.com
- E. Hubert, T. Metzlaff, C. Riener: *Polynomial description for the* $\mathbb{T}$*–Orbit Spaces of Multiplicative Actions*, preprint 2022, hal.inria.fr