An Algorithm for Testing the Half-plane Property of Matroids

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Journées Nationales de Calcul Formel

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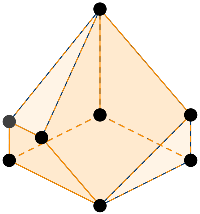
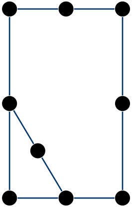
1. Hyperbolic Polynomials and Spectrahedral Cones

2. Connection to Matroids

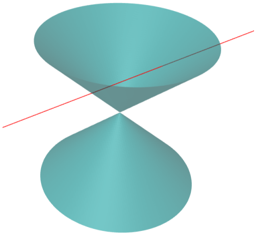
3. An Algorithm for the Half-Plane Property of Matroids





Hyperbolic Polynomials

Definition: A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is called hyperbolic with respect to $e \in \mathbb{R}^n$ if $h(e) \neq 0$ and for all $v \in \mathbb{R}^n$, $h(et - v)$ in $\mathbb{R}[t]$ has only real roots.



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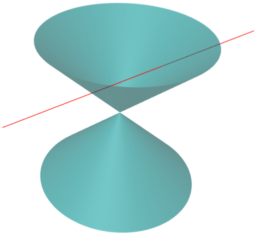
The hyperbolicity cone of h at e is

$$C_h(e) = \{v \in \mathbb{R}^n : h(et - v) = 0 \implies t \in \mathbb{R}_{\geq 0}\}.$$



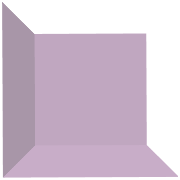
Hyperbolic Polynomials

$$h(et - v) = h(e_1 t - v_1, \dots, e_n t - v_n)$$

$$C_h(e) = \left\{ v \in \mathbb{R}^n : h(et - v) = 0 \implies t \in \mathbb{R}_{\geq 0} \right\}$$

$$h = x_1 x_2 x_3, \ e = (1, 1, 1)$$

$$h = (t - v_1)(t - v_2)(t - v_3)$$



$$C_h(e) = \mathbb{R}^3_{\geq 0}$$

$$h = \det(X), \ X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, \ e = \mathrm{I}$$

$$\det \begin{pmatrix} v_1 - t & v_2 \\ v_2 & v_3 - t \end{pmatrix}$$

$$C_h(e) = \left\{ v \in \mathbb{R}^3 : \begin{pmatrix} v_1 & v_2 \\ v_2 & v_3 \end{pmatrix} \succeq 0 \right\}$$

Cone of PSD 2×2 matrices

Determinantal Representability

Definition: A homogeneous polynomial $f \in \mathbb{R}[x_1, \cdots, x_n]$ is said to have a determinantal representation if there are PSD matrices $A_1, \ldots, A_n$ such that

$$f = \lambda \det(x_1 A_1 + \cdots + x_n A_n)$$

for some $\lambda \in \mathbb{R}$.

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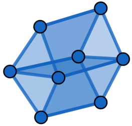
f is weakly determinantal $\overset{\Longrightarrow}{\not\Longleftarrow}$ f is hyperbolic with respect to all $e \in \mathbb{R}^n_{>0}$



Counter example: The basis generating polynomial of the Vamós matroid

Spectrahedral Cones

Definition: A convex cone C is called spectrahedral if

$$C = \{v \in \mathbb{R}^n : A(v) = v_1 A_1 + \ldots + v_n A_n \succeq 0\}$$

where $A_1, \ldots, A_n$ are real symmetric $d \times d$ matrices.

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- Spectrahedral cones are hyperbolicity cones. (consider $h = \det(A_1 x_1 + \cdots + A_n x_n)$).

Note: For the rest of the talk "hyperbolic" refers to hyperbolic with respect to every point in the positive orthant.

Spectrahedral Representability

Question: Given a homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ that is hyperbolic. When is $C_h(e)$ spectrahedral?

h has a determinantal representation $\Longrightarrow$ C_h is spectrahedral.

Theorem(Helton-Vinnikov, 2007) Let $h \in \mathbb{R}[x_1 \cdots, x_n]$ be hyperbolic. The hyperbolicity cone C_h is spectrahedral if and only if there exists a hyperbolic polynomial g with $C_h \subset C_g$ such that $h \cdot g$ has a determinantal representation.

$$C_{h \cdot g} = C_h \cap C_g$$

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Question: Are all hyperbolicity cones spectrahedral?

Generalized Lax Conjecture

Conjecture: Every hyperbolicity cone is spectrahedral.
Every hyperbolic program can be written as a semi-definite program.

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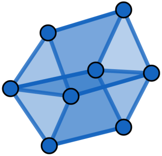
- The conjecture is true for quadratic polynomials (Netzer-Thom, 2012).
- The conjecture is true for matching polynomials of simple graphs. (Amini, 2019).

C spectrahedral $\implies$ C is a hyperbolicity cone for some h.

h has a determinantal representation $\implies$ C_h is spectrahedral.

C_h is spectrahedral $\iff$ $\exists\, g$ hyperbolic with $C_h \subset C_g$ such that $h \cdot g$ has a determinantal representation.

The basis generating polynomial of the Vamos matroid is hyperbolic, but not weakly determinantal!



Connection to Matroids

Definition: A matroid M is $E = [n]$ with a collection $\mathcal{B}$ of its subsets (bases) satisfying

$$\text{If } B_1, B_2 \in \mathcal{B} \text{ and } x \in B_1 \setminus B_2,$$

$$\text{then } \exists y \in B_2 \setminus B_1 \text{ such that } (B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}.$$

The basis generating polynomial of M is $h_M := \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i$.

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$E = \{1, 2, 3, 4\}$$
$$\mathcal{B} = \{\{2, 3, 4\}, \{2, 1, 4\}, \{1, 3, 4\}\}$$
$$h_M = x_2 x_3 x_4 + x_2 x_1 x_4 + x_1 x_3 x_4$$

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h_M is homogeneous and multiaffine

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Definition: A homogeneous polynomial $f \in \mathbb{R}[x_1, \cdots, x_n]$ is said to have the half-plane property if there exists an open half-plane $\mathcal{H} \subset \mathbb{C}$ with $0 \in \partial\mathcal{H}$ such that $f(x_1, \cdots, x_n) \neq 0$ for $x_1, \cdots, x_n \in \mathcal{H}$.

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Let M be a matroid with the basis generating polynomial $h_M \in \mathbb{R}[x_1, \ldots, x_n]$.

h_M is weakly determinantal $\Longrightarrow$ h_M has the half-plane property $\Longleftrightarrow$ h_M is hyperbolic

Connection to Matroids

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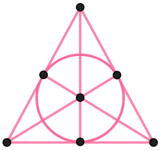
Theorem(Choe et. al., 2004): The half-plane property is closed under taking minors and direct sums of matroids.

Theorem(Kummer and S., 2021): Being weakly determinantal and having a spectrahedral hyperbolicity cone are closed under taking minors.

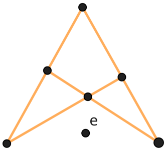
Classification of Matroids

Theorem(Choe et. al., 2004):

- All matroids on at most 6 elements have the half-plane property.
- Matroids that have rank or corank 2 have the half-plane property.
- Fano matroid F_7, F_7^-, F_7^{--}, F_7^{-3}, $M(K_4)+e$, P_8, P_8^-, P_8^{--} don't have the half-plane property.
- 6th root of unity matroids are weakly determinantal.



F_7



M(K$_4$)+e

A Criteria for the Half-plane Property

Theorem(Brändén, 2007 - Wagner and Wei, 2009):
Let h_M be the basis generating polynomial of a matroid M. The following are equivalent:

- h_M has the half-plane property.
- For all $1 \leq i, j \leq n$, the Rayleigh difference

$$\Delta_{ij}(h_M) := \frac{\partial h_M}{\partial x_i}(x)\frac{\partial h_M}{\partial x_j}(x) - \frac{\partial^2 h_M}{\partial x_i \partial x_j}(x)h_M(x) \geq 0$$

  for all $x \in \mathbb{R}^n$
  (We call them SOS-Rayleigh if $\Delta_{ij}(h_M)$ is SOS for all i,j).

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  for all $x \in \mathbb{R}^n$
  (We call them SOS-Rayleigh if $\Delta_{ij}(h_M)$ is SOS for all i,j).
- All of its proper minors have the half-plane property and for some $1 \leq i, j \leq n$, $\Delta_{ij}(h_M) \geq 0$ for all $x \in \mathbb{R}^n$.

A Criteria for Being Weakly Determinantal

Theorem(Kummer-Plaumann-Vinzant, 2015):
Let h_M be a basis generating polynomial of some matroid M. If h_M is weakly determinantal, then it is SOS-Rayleigh.

$$\begin{array}{ccccc}
\begin{array}{l} h_M \text{ is} \\ \text{weakly determinantal} \end{array} & \Longrightarrow & \begin{array}{l} h_M \text{ has} \\ \text{the half-plane property} \end{array} & \Longleftrightarrow & h_M \text{ is hyperbolic} \\
\Downarrow & & \Updownarrow & & \\
h_M \text{ is SOS-Rayleigh} & \Longrightarrow & \Delta_{ij} h_M \geq 0 \text{ for all } i, j \in [n] & &
\end{array}$$

An Algorithm for the Half-plane Property

Input: A matroid M on ground set $E = [n]$ with the collection of bases $\mathcal{B}$

all of whose proper minors have the HPP

$h_M := \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i$

$J := \{(i,j) : 0 \leq i,j \leq n, i \neq j\}$

For (i,j) in J Do

$$\Delta_{ij} := \frac{\partial h_M}{\partial x_i}(x)\frac{\partial h_M}{\partial x_j}(x) - \frac{\partial^2 h_M}{\partial x_i \partial x_j}(x)h_M(x)$$

solveSOS Δ_{ij}

SDP that attempts to find a PSD Gram matrix G with rational entries s.t. $m^T G m = \Delta_{ij}$

Use M2 package "SumsOfSquares"

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solveSOS Δ_{ij}

If a PSD Gram matrix G with entries in $\mathbb{Q}$ is found $\longrightarrow$ Δ_{ij} is SOS

M has the HPP – Stop

else $\longrightarrow$ Continue

End

Use M2 package "SumsOfSquares"

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If a PSD Gram matrix G with entries in $\mathbb{Q}$ is found $\longrightarrow$ Δ_{ij} is SOS
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else $\longrightarrow$ Continue

End

$\longrightarrow$ Go to Julia for finding negative points

For each (i,j) finds critical points of Δ_{ij} using "HomotopyContinuation.jl" and evaluates Δ_{ij} at the critical points

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solveSOS Δ_{ij}

If a PSD Gram matrix G with entries in $\mathbb{Q}$ is found → Δ_{ij} is SOS, **M has the HPP** – Stop

else → Continue

End

→ Go to Julia for finding negative points

If a negative value is found → **M doesn't have HPP**

Else → **Undetected**

An Algorithm for SOS-Rayleigh

Input: A matroid M on ground set $E = [n]$ with the collection of bases $\mathcal{B}$

with the HPP

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solveSOS Δ_{ij}

If a PSD Gram matrix G with entries in $\mathbb{Q}$ is found → Δ_{ij} is SOS, Continue

else → Stop → Try to produce a non-SOS certificate

Use M2 package "SumsOfSquares"

End

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If a PSD Gram matrix G with entries in $\mathbb{Q}$ is found $\longrightarrow$ Δ_{ij} is SOS. Continue

else $\longrightarrow$ Stop $\longrightarrow$ Try to produce a non-SOS certificate $\longrightarrow$

SDP that attempts to find a PD matrix M with rational entries s.t. $trace(MG_i) = 0$ for all matrices G_i that define the Gram Spectrahedra

End

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solveSOS Δ_{ij}

If a PSD Gram matrix G with entries in $\mathbb{Q}$ is found $\longrightarrow$ Δ_{ij} is SOS. Continue

else $\longrightarrow$ Stop $\longrightarrow$ Try to produce a non-SOS certificate

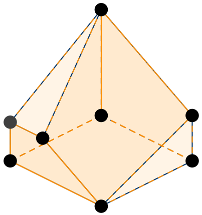
- If certified $\longrightarrow$ **M isn't SOS-Rayleigh**
- Else $\longrightarrow$ **Undetected**

End

$\longrightarrow$ **M is SOS-Rayleigh**

Matroids on 8 Elements of Rank 3 or 4

Properties	Count
Simple	685
Simple, connected and without the 10 forbidden minors	309
Having the HPP	287
SOS-Rayleigh	256
With the HPP and not SOS-Rayleigh	14
With the HPP and SOS-Rayleigh undetected	17
Without the HPP	22



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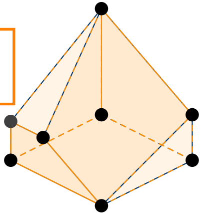
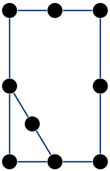
In particular, they are not weakly determinantal.
Good candidates for seaching for a counter example.



Matroids on 9 Elements of rank 3

Properties	Count
Simple	383
Simple, connected and without the 10 forbidden minors	119
Having the HPP	116
With the HPP and SOS-Rayleigh	106
With the HPP and SOS-Rayleigh undetected	10
Without the HPP	3



Matroids on 9 Elements of rank 4

Properties	Count
Simple	185982
Simple, connected and without the 35 excluded minors	6718
Having the HPP	4125
Candidates for having the HPP	819
Without the HPP	1218
HPP undetected	556

Merci pour votre attention!