Computational real algebraic geometry and applications to robotics

JNCF lecture, part 2

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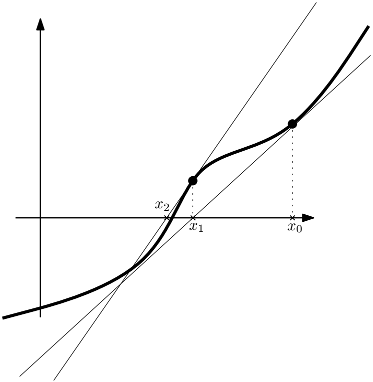
Solving systems

1. With initial point
   - Newton
2. Without initial point
   - Symbolic approaches
   - Numerical approaches

Newton

$$f(x) = 0$$

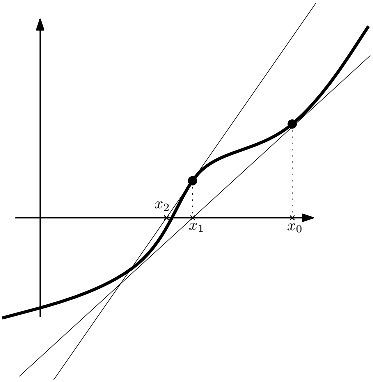


$$x_0 = \text{initial point}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Good for path tracking

Newton

$$f(x) = 0$$



$$x_0 = \text{initial point}$$
$$x_{n+1} = x_n - Df(x_n)^{-1} f(x_n)$$

Good for path tracking

Reliable Newton with Kantorovich

$$K(f, x_0) = \sup_x \|Df(x_0)^{-1} D^2 f(x)\|$$
$$\beta(f, x_0) = \|Df(x_0)^{-1} f(x_0)\|$$

Theorem (Kantorovich)

If $\beta(f, x_0) K(f, x_0) \leqslant 1/2$,
then f has a unique solution in $B(x_0, 2\beta(f, x_0))$.

Reliable Newton with Smale

$$\gamma(f, x_0) = \sup_{k \geqslant 2} \| Df(x_0)^{-1} \frac{D^k f(x_0)}{k!} \|^{\frac{1}{k-1}}$$

$$\beta(f, x_0) = \| Df(x_0)^{-1} f(x_0) \|$$

Theorem (Smale)

If $\beta(f, x_0)\gamma(f, x_0) \leqslant 3 - 2\sqrt{2}$,

then f has a unique solution in $B(x_0, \frac{1-\sqrt{2}/2}{\gamma(f,x_0)})$.

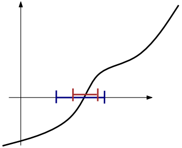
Reliable interval Newton

B a box interval containing p

1 variable

$$N(x) = p - \left(\frac{f(x)-f(p)}{x-p}\right)^{-1} f(p)$$

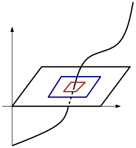
$$N(B) \subset B \Rightarrow N \text{ has a fix point in } B$$
$$\Rightarrow f \text{ has a solution in } B$$



2 variables or more

$$N(B) = p - \square J(B)^{-1} \begin{pmatrix} f(p) \\ g(p) \end{pmatrix}$$

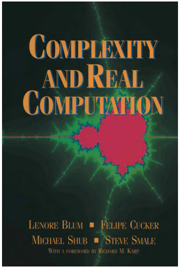
$$N(B) \subset B \Rightarrow f = g = 0 \text{ has a solution in } B$$

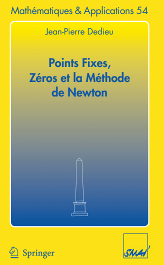


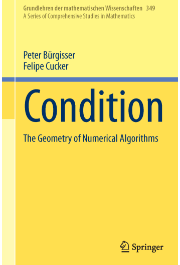
Interval operations

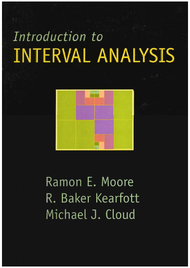
$$[a,b] \oplus [c,d] = [a+c, b+d]$$
$$[a,b] \otimes [c,d] = [min(ac,ad,bc,bd), max(ac,ad,bc,bd)]$$

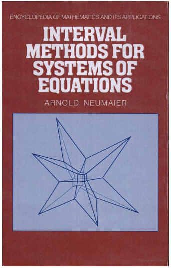
Further reading

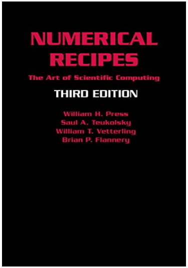












Further computing

Computer Algebra Systems

- Maple
- Mathematica
- Mathemagix (mmx)
- Matlab
- SageMath
- Xcas
- ...

Numerical Newton

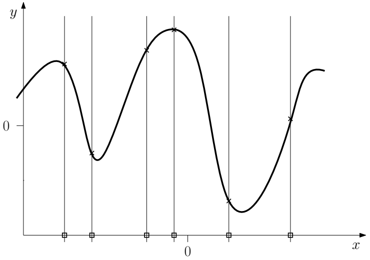
- Boost (C++)
- MPSolve (C) [Bini, Fiorentino, Robol]
- GNU Scientific Library (C)
- Roots, Optim, NLsolve (Julia)
- Scipy (Python)
- ...

Interval and ball arithmetic Libraries

- Arb (C, Python) [Johansson]
- GAOL (C++) [Goualard]
- Intlab (Octave/Matlab) [Rump]
- mpfi (C) [Revol]
- numerix (mmx, C++) [van der Hoeven]
- IntervalArithmetic (Julia) [Benet,Sanders]
- ...

Univariate Representation

Reduce problem to univariate polynomial



$$p(x) = 0$$
$$y = q(x)$$

Resultant definition

Given two polynomials in $\mathbb{C}[y]$:

- $\boxed{P = p_0 y^d + \cdots + p_d}$ with roots $\sigma_1, \ldots, \sigma_d$
- $\boxed{Q = q_0 y^d + \cdots + q_d}$ with roots $\tau_1, \ldots, \tau_d$

Definition: Resultant

$$\begin{aligned} Res(P, Q) &= p_0^d q_0^d \prod_{i,j} (\sigma_i - \tau_j) \\ &= p_0^d \boxed{\quad Q \quad}(\sigma_1) \cdots \boxed{\quad Q \quad}(\sigma_d) \\ &= (-1)^{d^2} q_0^d \boxed{\quad P \quad}(\tau_1) \cdots \boxed{\quad P \quad}(\tau_d) \end{aligned}$$

Resultant definition

Bezout

$$\begin{array}{rcccl} \varphi : & \mathbb{C}[y]_{d-1} & \times & \mathbb{C}[y]_{d-1} & \to \mathbb{C}[y]_{2d-1} \\ & U & , & V & \mapsto UP + VQ \end{array}$$

$$\begin{aligned} \varphi(U,V) = 1 \quad & \Leftrightarrow \gcd(P,Q) = 1 \\ & \Leftrightarrow \varphi \text{ invertible} \end{aligned}$$

- Resultant r is the determinant of the Sylvester Matrix
- $r \in \langle P, Q \rangle = I$

Definition: Sylvester matrix

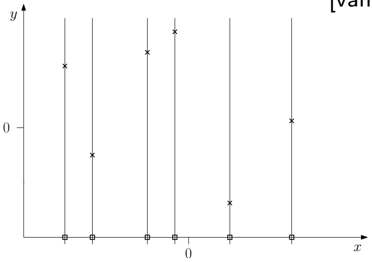
$$\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & \cdots & & & d+1 & d+2 & d+3 & \cdots & \end{array} \\ \begin{pmatrix} p_0 & & & & & q_0 & & & & \\ p_1 & p_0 & & & & q_1 & q_0 & & & \\ p_2 & p_1 & p_0 & & & q_2 & q_1 & q_0 & & \\ p_3 & p_2 & p_1 & p_0 & & q_3 & q_2 & q_1 & q_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

Bivariate case

Given two polynomials in $\mathbb{C}[x, y]$ of degree d in x and y:

- $P = p_0(x)y^d + \cdots + p_d(x)$
- $Q = q_0(x)y^d + \cdots + q_d(x)$

- $Res(P, Q)$ is a polynomial in x
- $O(n^{3-1/\omega+\varepsilon})$ arithmetic operations [Villard 2018]
- $O(n^{2+\varepsilon})$ randomized in a finite field [van der Hoeven, Lecerf 2019]



$$Res_d(P(x, y), Q(x, y))(\alpha) = Res_d(P(\alpha, y), Q(\alpha, y))$$

First subresultant definition

Given two polynomials in $\mathbb{C}[y]$:

- $\boxed{P = p_0 y^d + \cdots + p_d}$ with roots $\sigma_1, \ldots, \sigma_d$
- $\boxed{Q = q_0 y^d + \cdots + q_d}$ with roots $\tau_1, \ldots, \tau_d$

Properties

- $Sres_1 = s_1 y + s_0$ has degree at most 1
- $Sres_1 \in \left\langle \boxed{\text{P}}, \boxed{\text{Q}} \right\rangle = I$

Definition: first subresultant

$$Sres_1(P, Q) = p_0^{d-1} \left((y - \sigma_1) \prod_{j \neq 1} \frac{\boxed{Q}(\sigma_j)}{\sigma_j - \sigma_1} + \cdots + (y - \sigma_d) \prod_{j \neq d} \frac{\boxed{Q}(\sigma_j)}{\sigma_j - \sigma_d} \right)$$

First subresultant definition

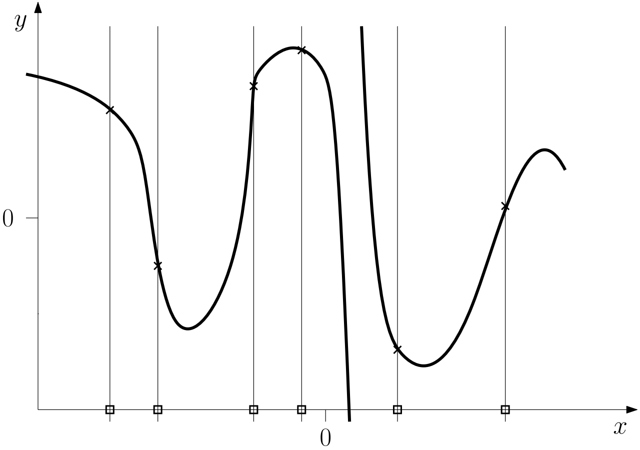
$$\begin{array}{rcccl} \varphi: & \mathbb{C}[y]_{d-2} & \times & \mathbb{C}[y]_{d-2} & \to \mathbb{C}[y]_{2d-2} \\ & U & , & V & \mapsto UP + VQ \end{array}$$

The s_0 and s_1 are the determinants of minors of the Sylvester Matrix

Sylvester matrix

$$\begin{array}{ccccccccccc} 1 & 2 & \cdots & & & d+1 & d+2 & d+3 & \cdots & \\ \end{array}$$
$$\begin{pmatrix} p_0 & & & & & q_0 & & & & \\ p_1 & p_0 & & & & q_1 & q_0 & & & \\ p_2 & p_1 & p_0 & & & q_2 & q_1 & q_0 & & \\ p_3 & p_2 & p_1 & p_0 & & q_3 & q_2 & q_1 & q_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Parametrization of y



$$s_1(x)y + s_0(x) = 0$$
$$r(x) = 0$$

Three variables and more

$$f_1(x_1, \cdots, x_n) = \cdots = f_n(x_1, \cdots, x_n) = 0$$

Problem

$$\text{Find } p(x_1) = q_1 f_1 + \cdots + q_n f_n$$

Matrix of $1, x_1, \ldots, x_1^D$ modulo $\langle f_1, \ldots, f_n \rangle$

$$M = \begin{array}{c} \\ m_1 \\ \vdots \\ m_k \end{array} \begin{array}{c} \begin{array}{ccc} 1 & \cdots & x_1^D \end{array} \\ \left(\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right) \end{array}$$

$$\Rightarrow \quad v \in Ker(M) \text{ iff } v_0 + \cdots + v_D x^D \in \langle f_1, \ldots, f_n \rangle$$

Three variables and more

$$f_1(x_1, \cdots, x_n) = \cdots = f_n(x_1, \cdots, x_n) = 0$$

Problem

Find $p(x_1) = q_1 f_1 + \cdots + q_n f_n$

Matrix of multiplication by x_1 modulo $\langle f_1, \ldots, f_n \rangle$

$$M = \begin{array}{c} \\ m_1 \\ \vdots \\ m_k \end{array} \begin{array}{c} \begin{array}{ccc} x_1 m_1 & \cdots & x_1 m_k \end{array} \\ \left(\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right) \end{array}$$

$$\Rightarrow \quad \chi_M(x_1) \in \langle f_1, \ldots, f_n \rangle$$

Three variables and more

$$f_1(x_1, \cdots, x_n) = \cdots = f_n(x_1, \cdots, x_n) = 0$$

Problem

Find $p(x_1) = q_1 f_1 + \cdots + q_n f_n$

Matrix of multiplication by x_1 modulo $f_1(x_1)$

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{d-1} \end{pmatrix}$$

$$\Rightarrow \quad \chi_M(x_1) = f_1(x_1) \in \langle f_1 \rangle$$

Normal form and univariate representation

Normal form

$$x_1 m_i \mod \langle f_1, \dots, f_n \rangle$$

- Euclidean division by Gröbner basis

Groebner Bases computation

- $n > 2$ in $\widetilde{O}((nd)^{(\omega+1)n})$ operations [Bardet, Faugère, Salvy 2015]
- $n = 2$ in $\widetilde{O}(d^2)$ with terse representation [van der Hoeven, Larrieu 2018]

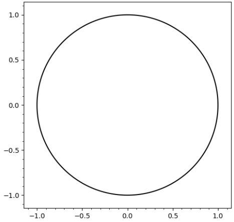
Univariate Representation

- Multivariate subresultant
- Rational Univariate Representation [Kronecker 1882, Rouillier 1999]
  - u-resultant and its derivatives at point $(t, a_1, \dots, a_n)$

$$\text{u-resultant} = C \prod_{\zeta | F(\zeta)=0} (u_0 + u_1\zeta_n + \cdots + u_n\zeta_n)$$

- Lexicographical Gröbner bases
  - Polynomial parametrization generically, bigger size [Dahan, Schost 2004]

Univariate Representation with Geometric Resolution



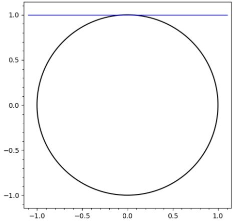
$$f(x, y) = 0$$

Theorem (Hensel lifting)

$$\begin{cases} y_0 & = \text{root of } f(0, y) = 0 \\ y_{n+1}(x) & = y_n(x) - \frac{f(x, y_n(x))}{\frac{\partial f}{\partial y}(x, y_n(x))} \mod x^{2^{n+1}} \end{cases}$$

Then $y_n(x)$ is a root of $f(x, y) = 0 \mod x^{2^n}$.

Univariate Representation with Geometric Resolution



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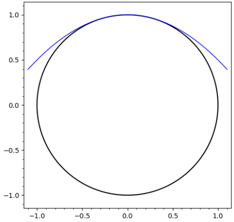

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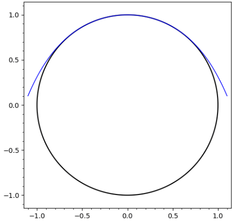

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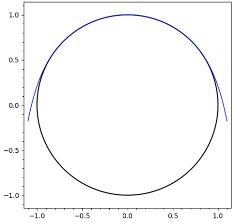

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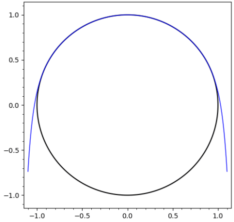

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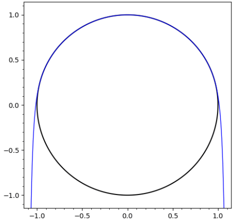

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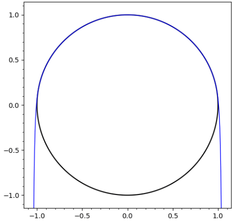

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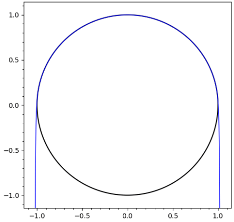

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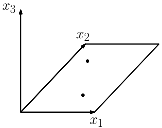
Then $y_n(x)$ is a root of $f(x,y) = 0 \mod x^{2^n}$.

Univariate Representation with Geometric Resolution

$$f_1(x_1, x_2, x_3) = \cdots = f_3(x_1, x_2, x_3) = 0$$

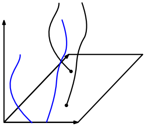
- RUR of $f_1(x_1, x_2, 0)$ and $f_2(x_1, x_2, 0)$

$$p_1(x_1) = 0 \quad x_2 = q_{12}(x_1)$$



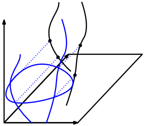
- Lift x_3 in $f_1(x_1, x_2, x_3) = f_2(x_1, x_2, x_3) = 0$

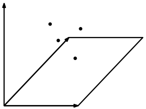
$$\widetilde{p}_1(x_1, x_3) = 0 \quad x_2 = \widetilde{q}_{12}(x_1, x_3)$$



- RUR of $\widetilde{p}_{11}(x_1, x_3)$ and $f_3(x_1, \widetilde{q}_{12}(x_1, x_3), x_3)$

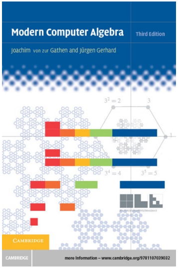
$$p_2(x_1) = 0 \quad x_2 = q_{22}(x_1) \quad x_3 = q_{23}(x_1)$$

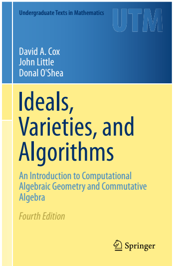


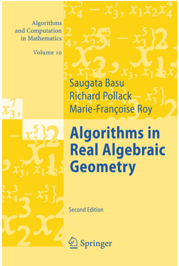


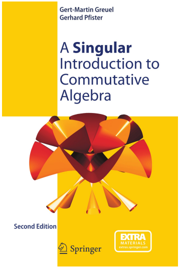
$\widetilde{O}(d^{3n})$ operations [Giusti, Lecerf, Salvy 2001]

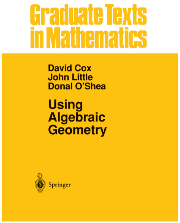
Further reading

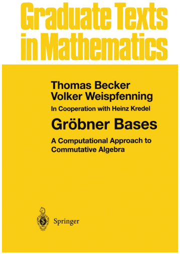


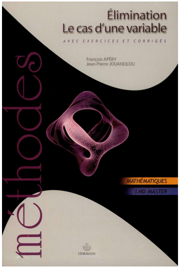


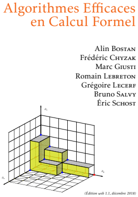












Further computing

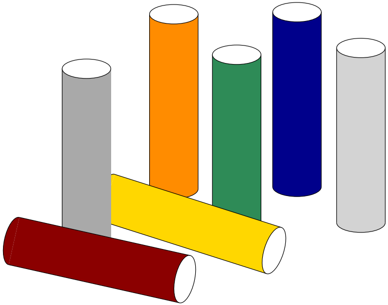
Computer Algebra systems

- Giac/xcas
- Maple
- Magma
- Mathemagix
- Mathematica
- Mcaulay2
- MuPAD
- SageMath
- Singular
- ...

Special purpose libraries/software

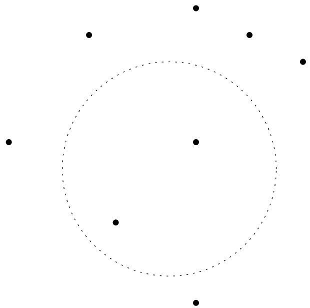
- FGb (C) [Faugère]
- Flint (C) [Hart]
- msolve [Berthomieux, Eder, Safey El Din]
- RS/RS3 (C) [Rouillier]
- borderbasix (mmx, C++) [Trebuchet, Mourrain]
- algebramix (mmx, C++) [van der Hoeven, Lecerf]
- geomsolvex (mmx, C++) [Lecerf]
- larrix (mmx, C++) [Larrieu]
- ...

Path continuation



Is it possible to arrange 7 infinite cylinders of unit radius such that they are mutually touching?

Path continuation



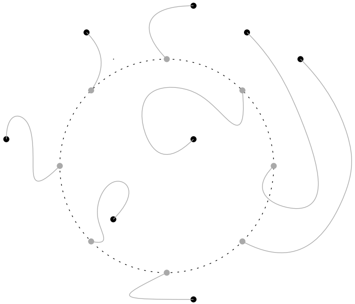
- $f(x) = 0$
- deg d

- We know the solutions of $x^d - 1 = 0$

$$(1 - t)(x^d - 1) + tf(x) = 0$$

- Multivariate case: compute the mixed volume first
- Find 1 solution by following only one path
  Polynomial time in $\binom{n+d}{d}$ average [Beltran, Pardo 2009], [Lairez 2017]
- Roots distribution of random polynomials [Edelman, Kostlan 1995]

Path continuation



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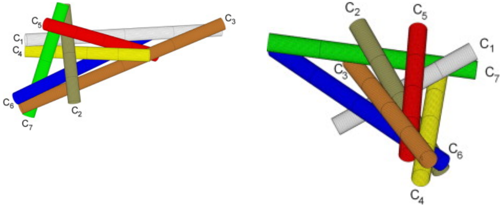
- Find 1 solution by following only one path
  Polynomial time in $\binom{n+d}{d}$ average [Beltran, Pardo 2009], [Lairez 2017]

- Roots distribution of random polynomials [Edelman, Kostlan 1995]

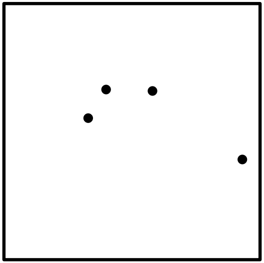
Path continuation

2 arrangements of 7 cylinders mutually touching [Bozóki et al. 2015]



- Modeled with 20 equations in 20 variables
- Mixed volume 121 098 993 664
- First real solution after 80 000 000 paths
- Solution certified with Smale theorem

Subdivision

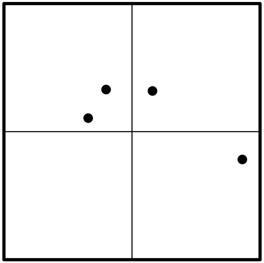


Test each cell

- 1 solution, guaranteed $\Rightarrow$ keep
- 0 solution, guaranteed $\Rightarrow$ remove
- don't know $\Rightarrow$ subdivide

Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

Subdivision



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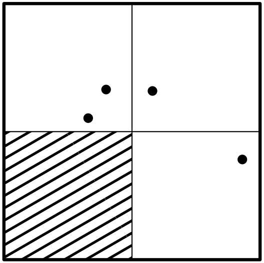
Subdivision



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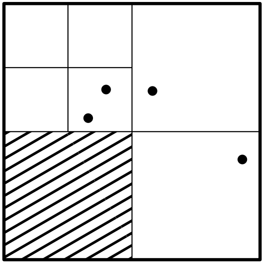
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

Subdivision



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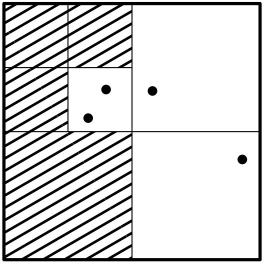
Subdivision



Test each cell

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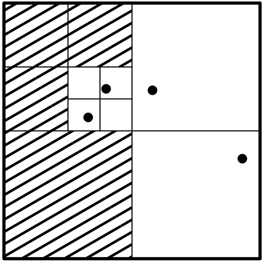
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

Subdivision



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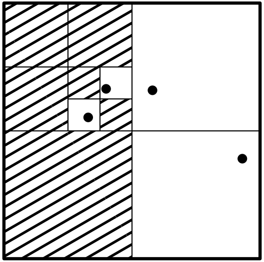
Subdivision



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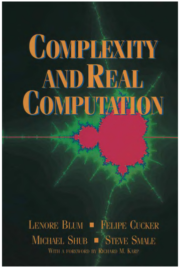
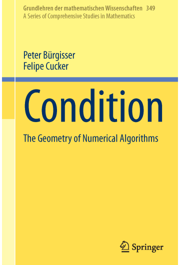
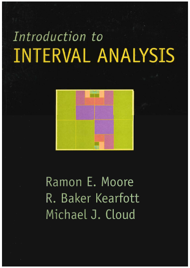
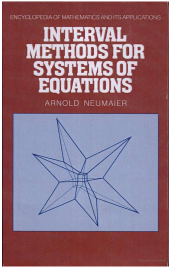
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

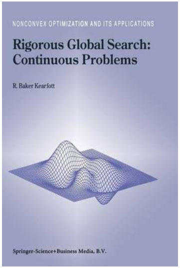
Further reading











Further computing

Path continuation

- Bertini (C++, Python) [Bates, Amethyst, Hauenstein, Sommese, Wampler]
- Hom4PS (C++) [Tien-Yien Li, Tianran Chen, Tsung-Lin Lee]
- PHCpack (Ada) [Verschelde]
- analyziz (mmx, C++) [van der Hoeven]
- ...

Subdivision multivariate

- ibex (C++) [Chabert]
- Realpaver (C) [Granvilliers]
- realroot (mmx, C++) [Mourrain]
- subdivision_solver (C++,Python) [Imbach]
- voxelize (C++) [M.]
- ...

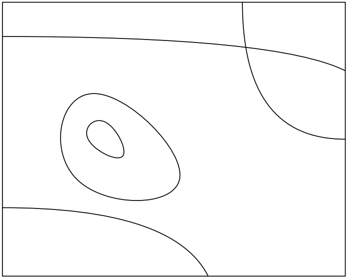
Subdivision univariate

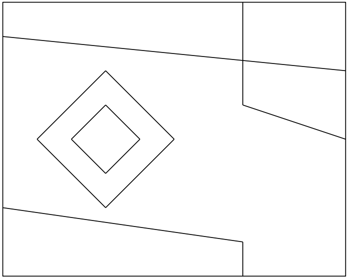
- ANewDsc (C) [Kobel,Rouillier,Sagraloff]
- Clenshaw (C, Python) [M.]
- Ccluster (C, Julia) [Imbach,Pan,Yap]
- real_roots (sage) [Witty]
- RS (C) [Rouillier]
- SLV (C) [Tsigaridas]
- ...

Drawing reliably the projection of the solution

1. Isotopy
2. Projection
3. Drawing reliably

Drawing hypersurfaces





Definition (isotopy)

A triangulation T is isotope to $V \subset \mathbb{R}^n$ if there exists $\varphi : [0,1] \times T \to \mathbb{R}^n$ such that:

- φ is continuous
- $\varphi(0, T) = T$ and $\varphi(1, T) = V$
- $\varphi_{t_0} : T \to \varphi(t_0, T)$ is an homeomorphism

Size of the triangulation [Kerber and Sagraloff 2011]

$$f(x_1, \ldots, x_n) \text{ of degree } d$$

2 variables

Theorem

In the worst case $\Omega(d^2)$ vertices and $O(d^2)$ segments

3 variables

Theorem

In the worst case $\Omega(d^3)$ vertices and $O(d^5)$ triangles

n variables

Theorem

In the worst case $\Omega(d^n)$ vertices and $O(d^{3/4 \cdot 2^n - 1})$ simplices

Size of the triangulation [Kerber and Sagraloff 2011]

f of degree d and vertices on the surface

2 variables

Theorem

In the worst case $\Omega(d^{\cancel{2}\,3})$ vertices and $O(d^{\cancel{2}\,3})$ segments

3 variables

Theorem

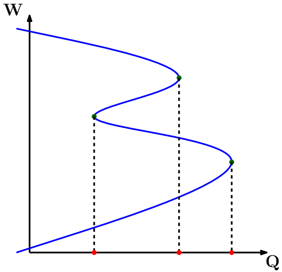
In the worst case $\Omega(d^{\cancel{3}\,4})$ vertices and $O(d^{\cancel{5}\,7})$ triangles

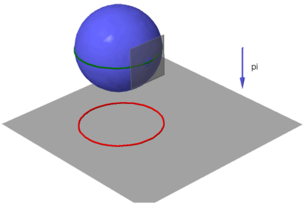
n variables

Theorem

In the worst case $\Omega(d^{n+1})$ vertices and $O(d^{\cancel{3/4}\,2^n-1})$ simplices

In Robotic

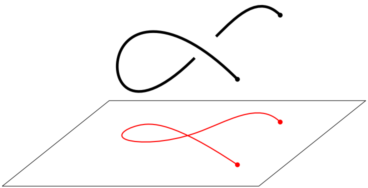




$$V \text{ solutions of } f_1 = \cdots = f_k = \det\left(\frac{\partial f_i}{\partial x_j}\right) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}^k$$

Goal: draw the projection of V in Q

Projection



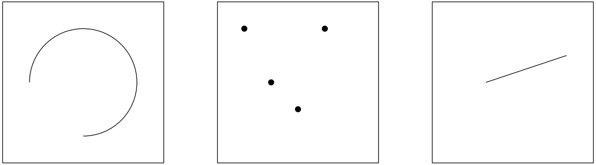
Definition (Semi-algebraic)

A semi-algebraic set is a set solution of a system of equalities and inequalities.

Theorem (Tarski-Seidenberg)

The projection of a semi-algebraic set is semi-algebraic.

Zariski closure



Definition (Zariski closure)

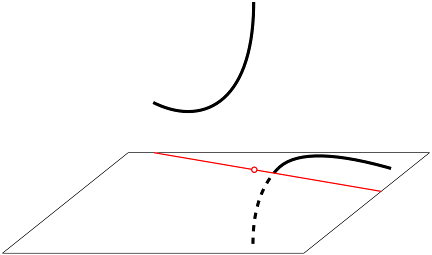
The Zariski closure of E is the minimal set $\overline{E}$ containing E that is solution of a system of equalities.

Theorem (elimination)

- V solution of $p_1(q, x) = \cdots = p_{k+1}(q, x) = 0$
- G Gröbner basis of F with respect to the lexdeg ordering with $x > q$
- G_q the polynomials in $G \cap \mathbb{Q}[q]$

$$\Rightarrow \overline{\pi_Q(V)} = \text{ solutions of } G_q$$

Projective elimination



$$p_1(q,x) = \cdots = p_{k+1}(q,x) = 0$$

Theorem

If V_p is the projective closure of V in $\mathbb{C}^n \times \mathbf{P}_k$, then

$$\overline{\pi_Q(V)} = \pi_Q(V_p)$$

Geometry of the elimination

$$x, y \in \mathbb{R}, z = a + ib \in \mathbb{C}$$

$$\begin{cases} 0 = f_1(x,y,a+ib) = & f_1(x,y,a) - \frac{b^2}{2} f_1''(x,y,a) + \cdots \\ & +ib\left(f_1'(x,y,a) - \frac{b^2}{6} f_1'''(x,y,a) + \cdots\right) \\ 0 = f_2(x,y,a+ib) = & f_2(x,y,a) - \frac{b^2}{2} f_2''(x,y,a) + \cdots \\ & +ib\left(f_2'(x,y,a) - \frac{b^2}{6} f_2'''(x,y,a) + \cdots\right) \end{cases}$$

Case $b = 0$: 2 equation in x, y, a

Case $b \neq 0$: 4 equation in x, y, a, b

$\Rightarrow$ possibly isolated points with $x, y \in \mathbb{R}$ and $z \in \mathbb{C} \backslash \mathbb{R}$

In general: $\overline{\pi_Q(V)}$ can have stable real components of dim $n - 2$

Geometry of the projection

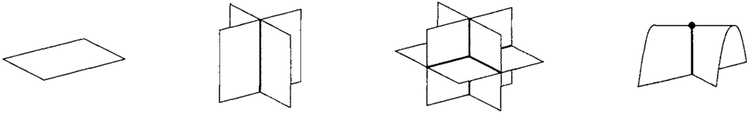


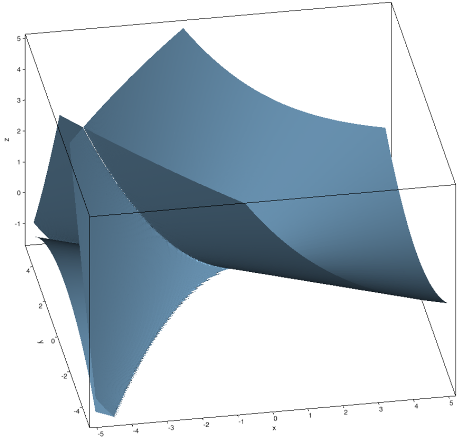
Figure 1: *Local singularities of images of generic maps of surfaces into 3-space*

[Guryonov 1997]

There exists a hypersurface $\Delta \subset C^\infty(M, \mathbb{R}^3)$ s.t. if $f \in C^\infty(M, \mathbb{R}^3) \backslash \Delta$, then the neighborhood of any point of $f(M)$ is one of the 4 above.

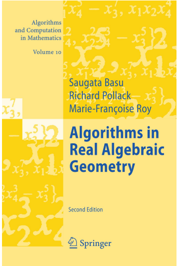
- Classification of generic singularities started with Whitney
- Thom introduced the transversality theorem
- Arnold and others provided numerous classifications

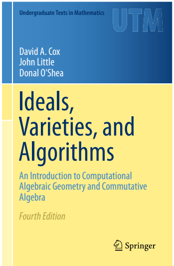
Geometry of the projection

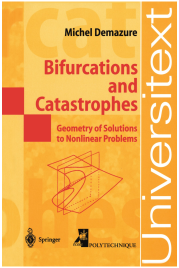


For the projection of critical points, we also have swallowtail singularities

Further reading







Further computing

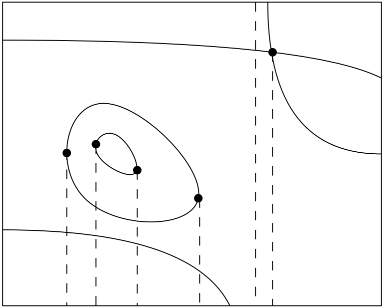
Quantifier elimination

- Redlog (Reduce) [Dolzmann, Sturm]
- RegularChains (Maple) [Moreno Maza et al]
- Resolve (Mathematica) [Strzebonski]
- QEPCAD (C) [Brown]
- Tarski (C++) [Brown]
- ...

Satisfiability of formula

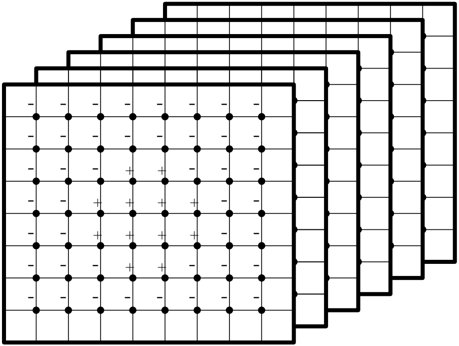
- RAGLib (Maple) [Safey el Din]
- RealCertify (Maple) [Magron, Safey el Din]
- RSolver (OCaml) [Ratschan]
- TSSOS (Julia) [Lassere, Magron, Wang]
- ...

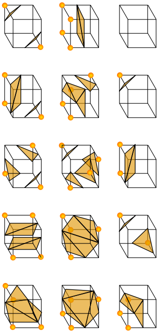
Drawing with symbolic approach



- Topology of curves in $\widetilde{O}(d^6 + \tau d^5)$ operations
  [Kobel, Sagraloff, 2015], [Niang Diatta, Diatta, Rouillier, Roy, Sagraloff, submitted 2018]
- Topology of surfaces with $O(d^5)$ simplices
  [Berberich, Kerber, Sagraloff 2009]

Marching cubes [Lorensen and Cline 1987]



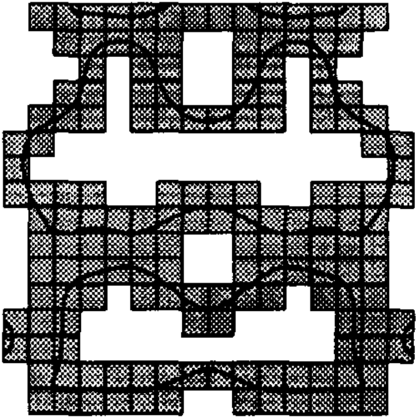


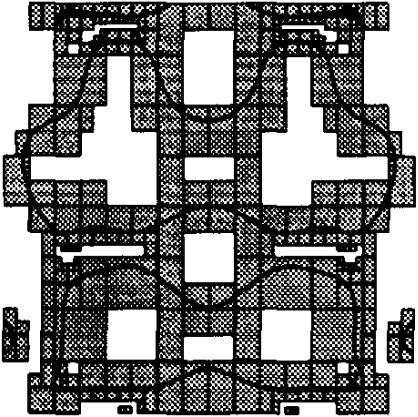
1. Evaluate $f(x, y, z)$ of deg d on a grid of size $N = n \times n \times n$
2. For each cube, compute a triangle of the surface to plot

Complexity: Cost of evaluating f on 1 point with Hörner: $O(d^3)$
Total naive: $O(d^3 n^3)$
Reuse computation by coordinates: $O(dn^3)$

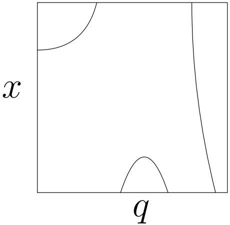
Subdivision pruning

Interval arithmetic to remove boxes [Snyder 1992, Plantinga-Vegter 2006]





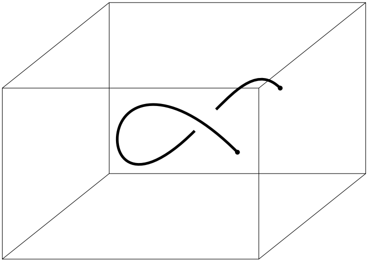
Jacobian



$$0 \notin \square\det(J_x(B)) = \square\det\left(\quad \frac{\partial f_i}{\partial x_j}(B) \quad \right)$$

$\Rightarrow$ V is globally parametrized by q in B

Surface tracking



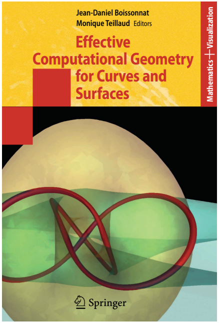
Regular case

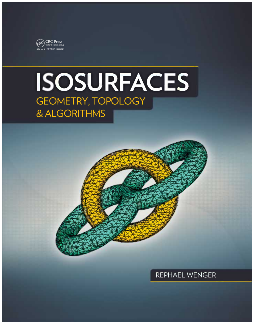
- First marching segments [Dobkin et al. 1990]
- First marching triangles [Hilton et al. 1996]
- Recent result on marching simplices, topology guaranteed [Boissonnat, Kachanovich, Wintraecken, 2020]

Singular surfaces

- Use isosingular deflation near singularities [Bates, Brake, Hauenstein, Sommese, Wampler, 2014]

Further reading





Further computing

Drawing with symbolic approach

- algcurve (Maple) [Deconink, Patterson, van Hoeij]
- isotop (C, Maple) [Peñeranda, Pouget, Lazard, Rouillier]
- EXACUS (C++) [Melhorn et al]
- ...

Drawing with subdivision

- axl (C++, mmx) [Christoflorou, Mantzaflaris, Mourrain, Wintz]
- ibex (C++) [Chabert]
- Realpaver (C) [Granvilliers]
- voxelize (C++) [M.]
- ...

Drawing with marching cube

- JuliaGeometry (Julia) [Kelly]
- scikit-image (C, python) [Lewiner]
- MathMod (C++) [Taha]
- ...

Drawing with continuation

- bertini_real [Brake et al]
- CGAL (C++) [Rineau, Yvinec]
- GUDHI (C++) [Kachanovich]
- ...

Merci !

Open problems

- Complexity of gröbner bases in dimension 3 with terse representation?
- What are the actual bounds on the number of simplices in drawings in 3D, in nD?
- What if we use polynomial pieces of degree k instead of linear pieces?
- Reliable drawing of singular surfaces with prescribed singularities?