Computational real algebraic geometry and applications to robotics

JNCF lecture, part 1

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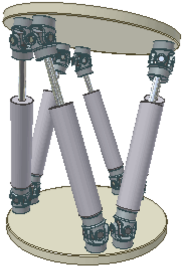
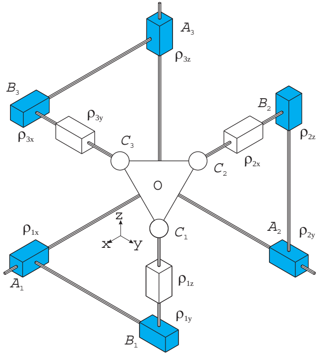
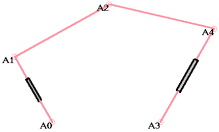
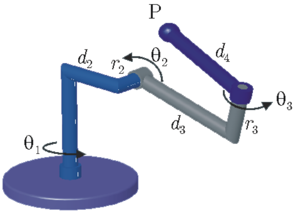
Mechanisms




A_3
ρ_{3z}
B_3
ρ_{3y}
ρ_{3x}
C_3
B_2
ρ_{2z}
ρ_{2x}
C_2
O
z
x
y
ρ_{1x}
C_1
A_2
ρ_{2y}
A_1
ρ_{1z}
ρ_{1y}
B_1

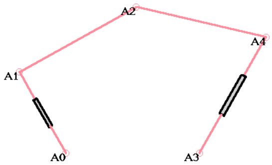

A2
A4
A1
A0
A3


P
d_4
d_2
r_2
θ_2
θ_3
d_3
r_3
θ_1

Modeling

Parallel P̲R-P̲RR

- Joint variables
  - r_1, r_2
- Pose variables
  - x, y
- Passive variables
  - θ_1, θ_2



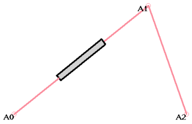
- Equations

$$(F)\begin{cases} x = cos(\frac{2\pi}{3})r_1 + cos(\theta_1) \\ x = 1 + cos(\frac{\pi}{3})r_2 + cos(\theta_2) \\ y = sin(\frac{2\pi}{3})r_1 + sin(\theta_1) \\ y = 1 + sin(\frac{\pi}{3})r_2 + sin(\theta_2) \end{cases}$$

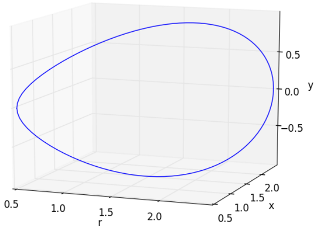
Workspace, Joint space

- Q: joint space
- W: workspace

- Total space: $Q \times W$
  - solutions of F: $V(F) \subset W \times Q$



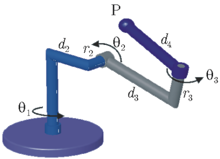
Parallel R<u>P</u>R-R



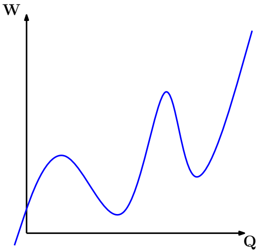
- Canonical projections:
  - $\pi_W : V(F) \rightarrow W$
  - $\pi_Q : V(F) \rightarrow W$

Serial robot

- Glossary:
  - P: prismatic joint
  - R: rotation joint
  - U: Cardan joint
  - S: spherical joint



Serial 3R

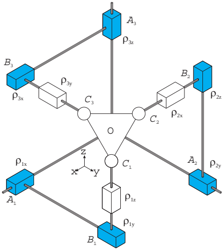


Properties

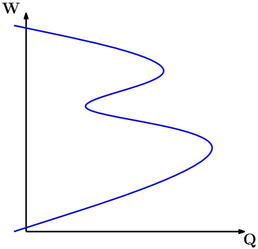
- Inverse Kinematics (IK) hard
- Forward Kinematics (FK) easy: 1 solutions

Parallel robot

- Glossary:
  - P: prismatic joint
  - R: rotation joint
  - U: Cardan joint
  - S: spherical joint



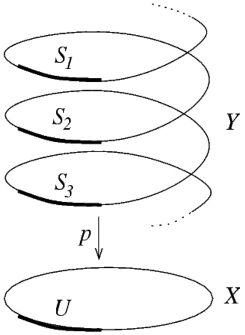
Parallel 3-PPPS



Properties

- Inverse Kinemaics (IK) easy
- Forward Kinematics (FK) hard: several solutions
- 2 solutions can cross
  - loose of control
  - break

Covering map



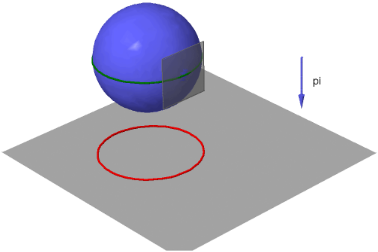
Definition

The continuous map $f : S \to U$ is a covering map if:

$$f^{-1}(U) = S_1 \cup \cdots \cup S_k \text{ where } \begin{cases} S_i \overset{f}{\simeq} U \\ S_i \text{ pairwise disjoint.} \end{cases}$$

Critical points

- $V \subset \mathbb{R}^n$ smooth variety of dimension p
- $\pi : V \rightarrow \mathbb{R}^p$ canonical projection



Critical points

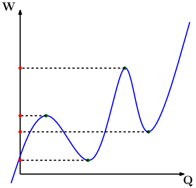
Let $T(a)$ be the linear space tangent to V at point a.
The critical points a of V for the projection π satisfy:

$$dim(\pi(T(a))) < p$$

Case of the serial robot

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

- FK: always 1 solution $\Rightarrow \pi_Q : V(F) \to Q$ invertible
- IK: partition W in $W_0, W_1, \ldots, W_k$ s.t.:
  - W_0 are the critical values of π_W
  - $W_1, \ldots, W_k$ are the connected components of $W \backslash W_0$
- Critical points of π_W: serial singularities



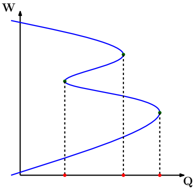
Theorem (covering map and critical values)

For all $1 \leqslant i \leqslant k$, the restriction of π_W to $\pi_W^{-1}(W_i)$ is a covering map above W_i.

Case of parallel robot

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

- IK: always 1 solution $\Rightarrow \pi_W : V(F) \to W$ invertible
- FK: partition Q in $Q_0, Q_1, \ldots, Q_k$ s.t.:
  - Q_0 are the critical values of π_Q
  - $Q_1, \ldots, Q_k$ are the connected components of $Q \backslash Q_0$
- Critical points of π_Q: parallel singularities



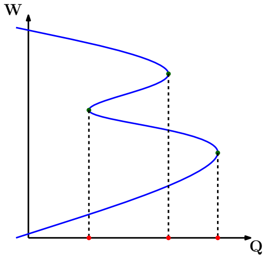
Theorem (covering map and critical values)

For all $1 \leqslant i \leqslant k$, the restriction of π_Q to $\pi_Q^{-1}(Q_i)$ is a covering map above Q_i.

Properties

Case of parallel robots

- FK: $F_q(x) = 0$, system parametrized by q
- For fixed q, finitely many solutions (0-dimensional)
- π_Q is not a covering map near q
  $\Rightarrow$ two sheets of solutions cross
  $\Rightarrow$ $F_q(x) = 0$ has singular solutions



- Remark: $V(F)$
  - not bounded: take asymptotes into account
  - not smooth: take singularities into account

Computation

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

$$\underbrace{\frac{\partial F}{\partial q}}_{A} dq + \underbrace{\frac{\partial F}{\partial x}}_{B} dx = 0$$

- Serial singularities

$$(S_s) : F = 0, det(A) = 0$$

  - W_0, critical values of π_W: projection on x_i of solutions of (S_s)
  - $W_1, \ldots, W_k$, complement of critical values

Computation

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

$$\underbrace{\frac{\partial F}{\partial q}}_{A} dq + \underbrace{\frac{\partial F}{\partial x}}_{B} dx = 0$$

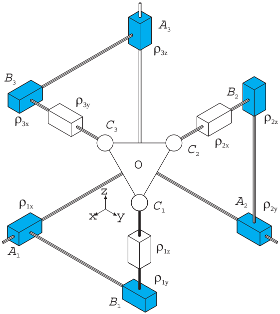
- Parallel singularities

$$(S_p) : F = 0, det(B) = 0$$

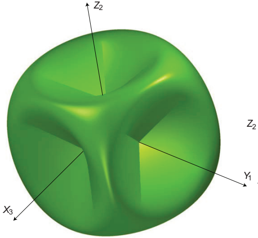
  - Q_0, critical values of π_Q: projection on q_i of solutions of (S_p)
  - $Q_1, \ldots, Q_k$, complement of critical values

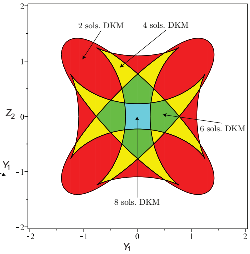
Example

- 3-PPPS:
  - Parallel
  - Joint variables: $x_1, y_1, y_2, z_2, x_3, z_3$
  - Pose variables: $p_x, p_y, p_z, \varphi, \theta, \sigma$



- Critical values of π_Q, and partition of Q



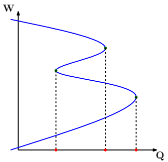


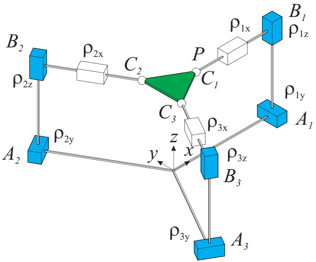
Design challenges

- $E \subset W$ given shape
  - Design a parallel robot without singularities in E

$$\pi_W(\text{critical points of } \pi_Q) \cap E = \emptyset$$

  - Maximise the volume of E (lecture P. Lairez)
- $\rightarrow$ design variables



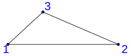


Demo

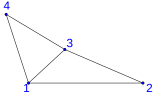
Modeling equations

1. Linkages
2. Rotations 3D
3. Singularities

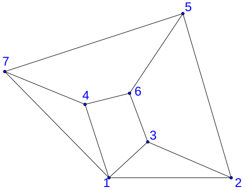
Planar Rigid Linkage: Laman Graph



3-bar



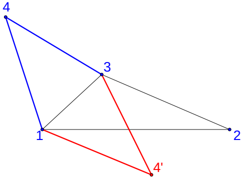
5-bar



11-bar

Constraints

- Fixed length bars: c_{ij}
- Free revolute joints
- Zero degree of freedom



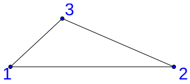
- Several assembly modes
- Number depends on c_{ij}
- Max number of assembly modes?

Properties of Rigid Linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 3-bar rigid linkage

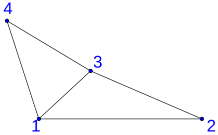


Properties of Rigid Linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 5-bar rigid linkage

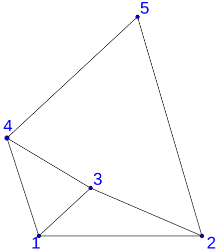


Properties of Rigid Linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 7-bar rigid linkage

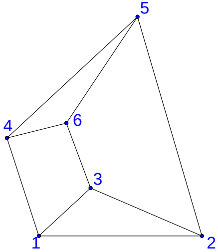


Properties of Rigid Linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 9-bar rigid linkage

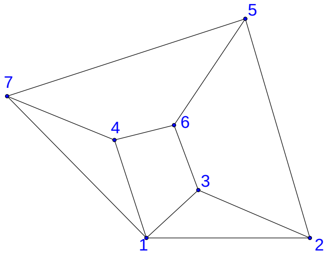


Properties of Rigid Linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 11-bar rigid linkage



Known properties

Theorem

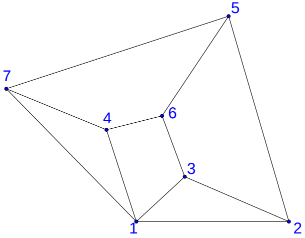
A linkage is rigid $\Leftrightarrow$ It can be constructed with H_1 and H_2

Corollary

$$\#Links = 2\#Joints - 3$$

Algebraic Modeling I

- c_{ij}: 10 parameters
- x_i, y_i: 14 variables



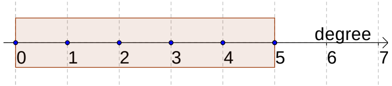
$$\begin{cases} x_1 = 0, y_1 = 0 \\ x_2 = 1, y_2 = 0 \end{cases}$$

$$\begin{cases} x_3{}^2 + y_3{}^2 &= c_{13} \\ (x_3 - 1)^2 + y_3{}^2 &= c_{23} \\ (x_5 - 1)^2 + y_5{}^2 &= c_{25} \\ (x_6 - x_3)^2 + (y_6 - y_3)^2 &= c_{36} \\ x_4{}^2 + y_4{}^2 &= c_{14} \end{cases} \qquad \begin{cases} x_7{}^2 + y_7{}^2 &= c_{17} \\ (x_6 - x_4)^2 + (y_6 - y_4)^2 &= c_{46} \\ (x_5 - x_6)^2 + (y_5 - y_6)^2 &= c_{56} \\ (x_7 - x_5)^2 + (y_7 - y_5)^2 &= c_{57} \\ (x_4 - x_7)^2 + (y_4 - y_7)^2 &= c_{47} \end{cases}$$

Number of solutions

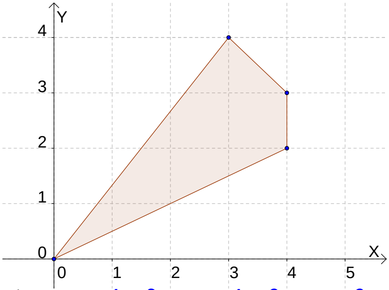
- Mixed Volume: $n!\,Volume(Support)$ (same support)

1 variable



$$1 - X + 3X^2 - X^3 + 6X^4 - 5X^5 = 0$$

2 variables



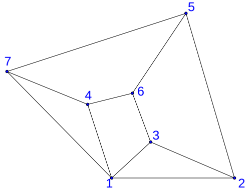
$$\begin{cases} 1 - X^4Y^2 + 7X^4Y^3 - 4X^3Y^4 = 0 \\ 8 + 6X^4Y^2 - 5X^4Y^3 - X^3Y^4 = 0 \end{cases}$$

- Our system: 2^{10}

Algebraic Modeling II

- Cayley-Menger matrix or distance matrix

$$
\begin{array}{c} \\ \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array}
\begin{array}{c} \quad\ \ v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \\
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & c_{12} & c_{13} & c_{14} & x_{15} & x_{16} & c_{17} \\
1 & c_{12} & 0 & c_{23} & x_{24} & c_{25} & x_{26} & x_{27} \\
1 & c_{13} & c_{23} & 0 & x_{34} & x_{35} & c_{36} & x_{37} \\
1 & c_{14} & x_{24} & x_{34} & 0 & x_{45} & c_{46} & c_{47} \\
1 & x_{15} & c_{25} & x_{35} & x_{45} & 0 & c_{56} & c_{57} \\
1 & x_{16} & x_{26} & c_{36} & c_{46} & c_{56} & 0 & x_{67} \\
1 & c_{17} & x_{27} & x_{37} & c_{47} & c_{57} & x_{67} & 0
\end{bmatrix}
\end{array}
$$



Theorem

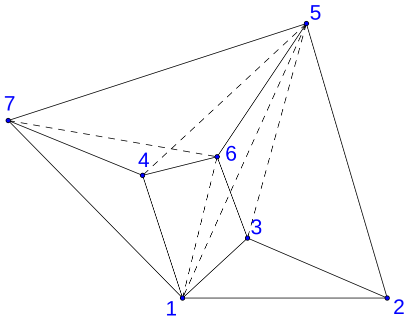
The distance matrix has rank 4.

Corollary

All the 5x5 minors vanish.

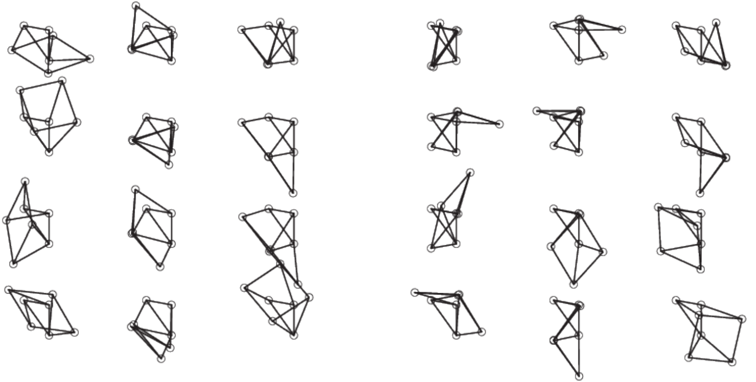
Algebraic Modeling II

$$\begin{cases} D(0,4,5,6,7)(c_{46}, c_{47}, c_{56}, c_{57}, x_{45}, x_{67}) & = & 0 \\ D(0,1,4,6,7)(c_{14}, c_{17}, c_{46}, c_{47}, x_{16}, x_{67}) & = & 0 \\ D(0,1,4,5,7)(c_{14}, c_{17}, c_{47}, c_{57}, x_{15}, x_{45}) & = & 0 \\ D(0,1,2,3,5)(c_{12}, c_{13}, c_{25}, c_{23}, x_{15}, x_{35}) & = & 0 \\ D(0,1,3,5,6)(c_{13}, c_{36}, c_{56}, x_{15}, x_{16}, x_{35}) & = & 0 \end{cases}$$



- Upper bound
  - Mixed volume: 56
- Lower Bound?

Sampling



Number of assembly modes

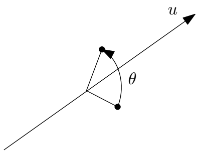
Maximal number of assembly modes

bars	3	5	7	9	11	13	15	17
upper	2	4	8	24	56	136	344	880
lower	2	4	8	24	56	136	344	860

- [Bartzosa, Emiris, Legerský, Tsigaridas 2021]
- Started in 2002 with Borcea
- Bartzosa, Borcea, Emiris, Legerský, M., Streinu, Capco, Gallet, Grasegger, Koutschan, Lubbes, Schicho, Tsigaridas, . . .

Rotations matrix in 3D

$$R = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$



$$R^T R = I$$

- Action of R is a rotation by θ around an axe u
- Set of rotation has dim 3

Euler matrix

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R_y(\varphi) = \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix}$$

$$R_z(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R = R_z(\psi)R_y(\varphi)R_x(\theta) \quad \text{or} \quad R = R_z(\psi)R_y(\varphi)R_z(\theta)$$

Quaternions matrix

$$q_i^2 + q_j^2 + q_k^2 + q_r^2 = 1$$

$$R = \begin{pmatrix} 1 - 2(q_j^2 + q_k^2) & 2(q_i q_j - q_k q_r) & 2(q_i q_k + q_j q_r) \\ 2(q_i q_j + q_k q_r) & 1 - 2(q_i^2 + q_k^2) & 2(q_j q_k - q_i q_r) \\ 2(q_i q_k - q_j q_r) & 2(q_j q_k + q_i q_r) & 1 - 2(q_i^2 + q_j^2) \end{pmatrix}$$

Rotation of axe: (q_i, q_j, q_k)
angle: $2\arccos(q_r)$

Anti-symmetric matrix - Exponential map

In 2D

$$\theta \Rightarrow \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

In 3D

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

$$R = e^A$$

Rotation of axe: (x, y, z)
angle: $\|(x, y, z)\|_2$

Anti-symmetric matrix - Cayley transform

In 2D

$$t = \tan\left(\frac{\theta}{2}\right) \Rightarrow \begin{pmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{pmatrix}$$

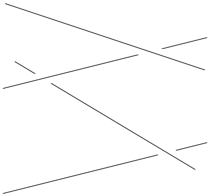
In 3D

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

$$R = (I - A)(I + A)^{-1}$$

Rotation of axe: (x, y, z)
angle: $2\arctan(\|(x, y, z)\|_2)$

Plücker coordinates



How many lines intersect 4 given lines?

Plücker coordinates

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad P_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad M = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \\ 1 & 1 \end{pmatrix}$$

Definition

The Plücker coordinates of the line $(P_1 P_2)$ are the 6 minors of M

$$(d_x, d_y, d_z, m_x, m_y, m_z) = (P_2 - P_1, OP_1 \times OP_2)$$

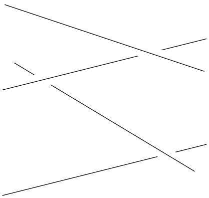
- $(d, m) \in \mathbf{P}_5$ is on the Klein quadric

$$d \cdot m = 0$$

- Lines (d, m) and (d', m') intersect implies

$$d \cdot m' + m \cdot d' = 0$$

Plücker coordinates



$0, 1, 2$ or infinitely many lines cross the 4 given lines

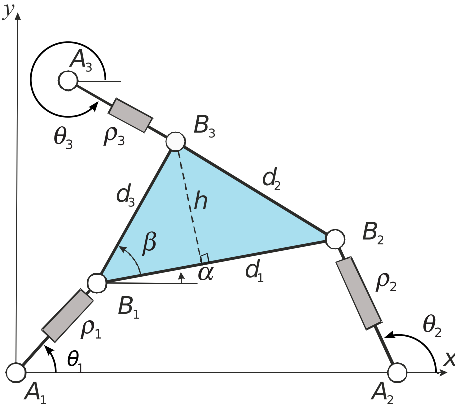
Singularity modeling

$$f_1(q,x),\dots,f_m(q,x)$$

$$B = \begin{array}{c} \\ f_1 \\ \vdots \\ f_m \end{array} \begin{array}{c} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_m} \\ \left(\begin{array}{ccc} & & \\ & \frac{\partial f_i}{\partial x_j} & \\ & & \end{array} \right) \end{array}$$

- $\det(B)$ multi-linear in its columns/rows
- $\det(B)$ of deg 1 in x_j leads to a parametrization of the singularities
- Simplifies the analysis of the singularities, as in the 3-RPR [Coste 2012]

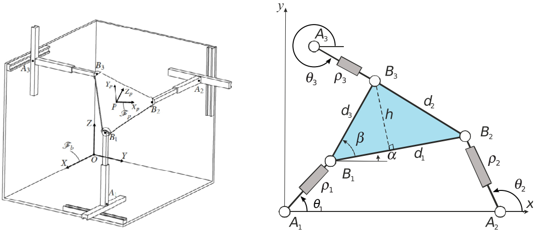
Plan parallel robot 3-RPR



- 3 degrees of freedom
- $d_1, d_2, d_3, A_1, A_2, A_3$ fixed
- Joint variables: r_1, r_2, r_3
- Pose variables: α, B_{1x}, B_{1y}

Demo

Singularity of parallel manipulator with Plücker vectors



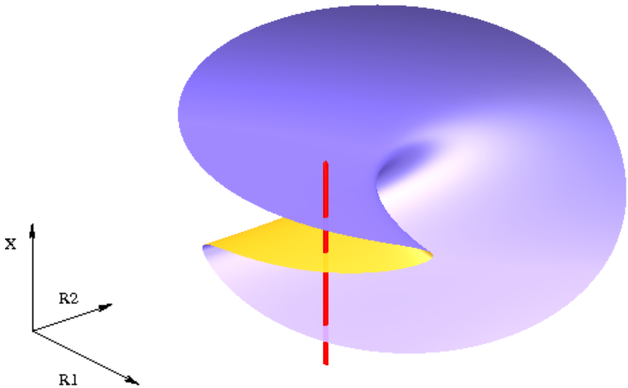
Remark

If the leg-platorm joints are spherical, the rows of the inverse kinematic Jacobian matrix $A^{-1}B$ will involve the Plücker coordinates of lines associated to the legs.

$\Rightarrow$ singularities can be interpreted geometrically

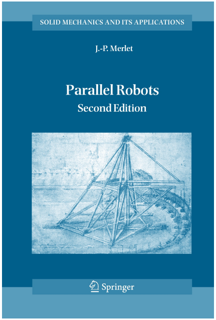
Cuspidal configuration

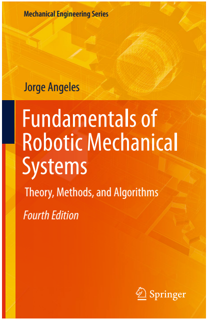
- Cuspidal point: point of order $\geqslant 3$
- Characterization: A cuspidal robot has at least one cuspidal point



Demo

Further reading





À suivre

Solving systems

1. With initial point
   - Newton
2. Without initial point
   - Symbolic approaches
   - Numerical approaches