Toric Eigenvalue Methods for Solving Sparse Polynomial Systems

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- Joint work with Simon Telen (MPI MiS). [arXiv:2006.10654]

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    - Relies on toric geometry.
    - Size of matrices controlled by regularity.
      In general $\rightarrow$ combinatorics of the Newton polytopes.
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- Algorithm works in presence of multiplicities.

Solving affine pol. systems [Lazard '81, Auzinger & Stetter '88]

- Consider $f_1, \dots, f_r \in S := \mathbb{C}[x_1, \dots, x_n]$ and $I := \langle f_1, \dots, f_r \rangle$.

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Eigenvalues of the multiplication maps

Given $g \in S$, consider $M_g : R \to R$, s.t. $h + I \mapsto g\, h + I$. Then,

$$\mathrm{CharPol}(M_g)(\lambda) = \prod_{i=1}^{\delta} (g(\zeta_i) - \lambda)^{\mu_i}.$$

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Note that, for $h, g \in S$, $M_h\, M_g = M_{h\,g} = M_g\, M_h$.

Solve system $\rightarrow$ simultaneous Schur triangulation (diagonalization) of $M_{x_1}, \ldots, M_{x_n}$.

Example - Computing multiplication maps

$$\begin{cases} f_1 := x^2 + 3\,xy - 5\,x - y + 2 \\ f_2 := x^2 + xy - 3\,x - 3\,y + 4 \end{cases}$$

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$$\mathrm{Sylv} : (g_1, g_2) \mapsto g_1\, f_1 + g_2\, f_2$$

$$S/I \cong \mathrm{coker}(\mathrm{Sylv})$$

	xy^2	y^2	x^2y	xy	y	x^3	x^2	x	1
f_1				3	-1		1	-5	2
$x\,f_1$			3	-1		1	-5	2	
$y\,f_1$	3	-1	1	-5	2				
f_2				1	-3		1	-3	4
$x\,f_2$			1	-3		1	-3	4	
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$x^2\,y$			1						
$x\,y$				1					
$1\,y$					1				

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$y\,f_2$	1	−3	1	−3	4				
$x^2\,y + I$							5/4	− 1/2	1/4
$x\,y + I$							− 1/4	3/2	− 1/4
$1\,y + I$							1/4	− 1/2	5/4

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The system has 2 different sols: $(-1, 2)$, and $(1, 1)$ with multiplicity 2.

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$$M_{\color{blue}y} : S/I \to S/I \cong \mathrm{coker}(\mathrm{Sylv})$$

	xy^2	y^2	x^2y	xy	y	x^3	x^2	x	1
f_1				3	−1		1	−5	2
$x\,f_1$			3	−1		1	−5	2	
$y\,f_1$	3	−1	1	−5	2				
f_2				1	−3		1	−3	4
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${\color{red}x^2}\,{\color{blue}y} + I$							5/4	− 1/2	1/4
${\color{red}x}\,{\color{blue}y} + I$							− 1/4	3/2	− 1/4
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$$\begin{bmatrix} 5/4 & -1/2 & 1/4 \\ -1/4 & 3/2 & -1/4 \\ 1/4 & -1/2 & 5/4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} {\color{blue}2} & & \\ & {\color{blue}1} & \\ & & {\color{blue}1} \end{bmatrix} \cdot \begin{bmatrix} 1/4 & -1/2 & 1/4 \\ -1/4 & 1/2 & 3/4 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

Problems with solutions at "near infinity"

$$\begin{cases} f_1 := x^2 + (1+\varepsilon)\,xy - (3+\varepsilon)\,x + (1-\varepsilon)\,y + \varepsilon \\ f_2 := x^2 - (1-\varepsilon)\,xy - (1+\varepsilon)\,x - (1+\varepsilon)\,y + 2 + \varepsilon \end{cases}$$

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yf_1	$\varepsilon+1$	$-\varepsilon+1$	1	$-\varepsilon-3$	ε				
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As $\varepsilon \to 0$,

- First solution "goes to infinity".

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- First solution "goes to infinity".
- We invert a nearly-singular matrix $\to$ Numerically bad!

Compactification over $\mathbb{P}^n$

$$\begin{cases} f_1 := x^2 + (1+\varepsilon)\,xy - (3+\varepsilon)\,x \quad + (1-\varepsilon)\,y \quad + \varepsilon \\ f_2 := x^2 - (1-\varepsilon)\,xy - (1+\varepsilon)\,x \quad - (1+\varepsilon)\,y \quad + (2+\varepsilon) \end{cases}$$

- Main issue $\rightarrow$ affine space is not compact.

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  - At certain degrees, quotient ring $\cong$ affine coord. ring of the points V.

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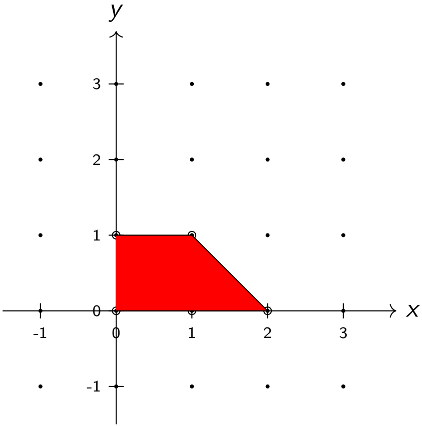
Warning: The homogenization might increase the dim. of our variety.

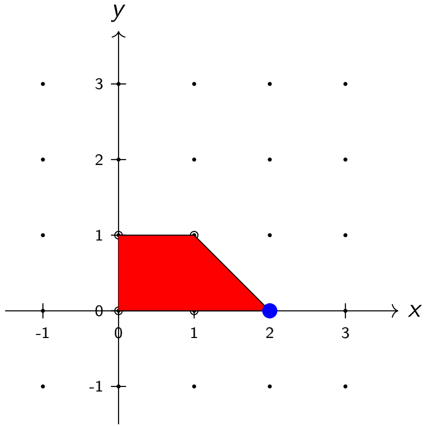
A better compactification: Projective toric varieties



$$\begin{cases} x^2 + xy - 3x + y + \varepsilon \\ x^2 - xy - x - y + 2 \end{cases}$$

$$\in \mathbb{C}[x, y]$$

A better compactification: Projective toric varieties



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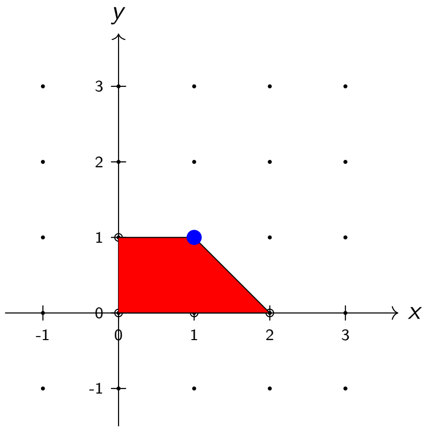

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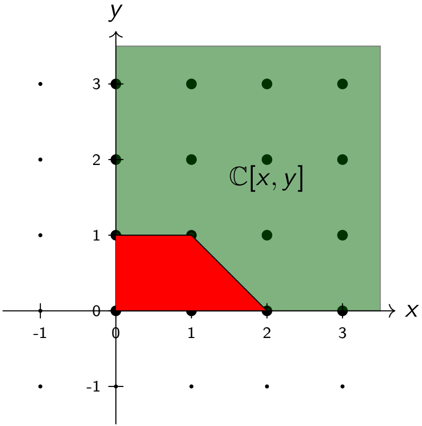

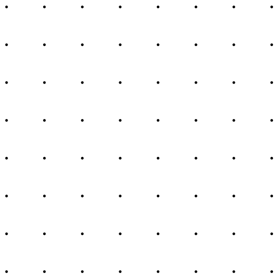

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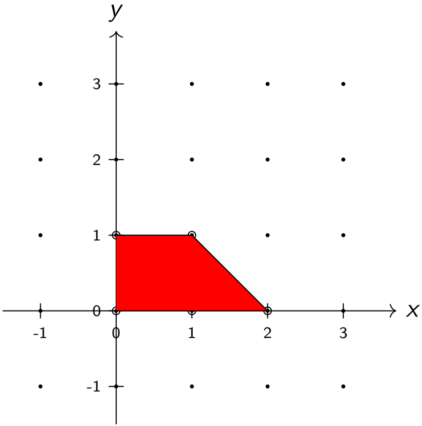

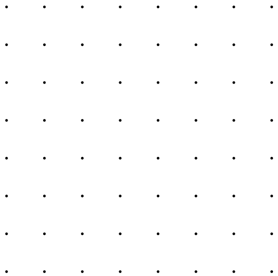

$$\begin{cases} x^2 + xy - 3\,x{\color{blue}z} + y{\color{blue}z} + \varepsilon {\color{blue}z}^2 \\ x^2 - xy - x{\color{blue}z} - y{\color{blue}z} + 2{\color{blue}z}^2 \end{cases}$$

$$\in \mathbb{C}[x, y, {\color{blue}z}]$$

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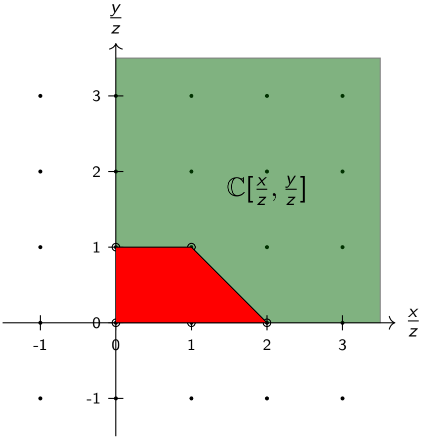

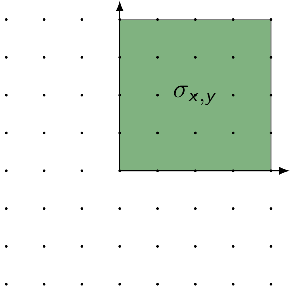

$$\begin{cases} \frac{x^2}{z^2} + \frac{x\,y}{z^2} - 3\,\frac{x}{z} + \frac{y}{z} + \varepsilon \\ \frac{x^2}{z^2} - \frac{x\,y}{z^2} - \frac{x}{z} - \frac{y}{z} + 2 \end{cases}$$

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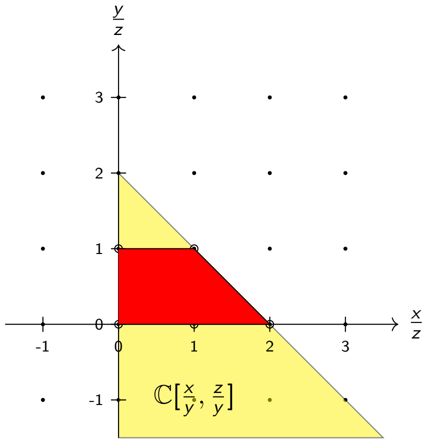

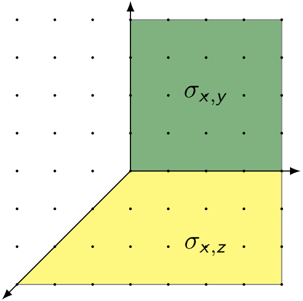

$$\begin{cases} \frac{x^2}{y^2} + \frac{x}{y} - 3\,\frac{x\,z}{y^2} + \frac{z}{y} + \varepsilon\,\frac{z^2}{y^2} \\ \frac{x^2}{y^2} - \frac{x}{y} - \frac{x\,z}{y^2} - \frac{z}{y} + 2\,\frac{z^2}{y^2} \end{cases}$$

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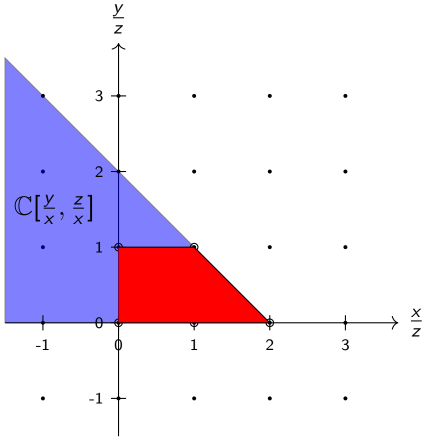

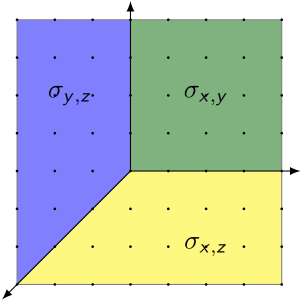

$$\begin{cases} 1 + \frac{y}{x} - 3\,\frac{z}{x} + \frac{y\,z}{x^2} + \varepsilon\,\frac{z^2}{x^2} \\ 1 - \frac{y}{x} - \frac{z}{x} - \frac{y\,z}{x^2} + 2\,\frac{z^2}{x^2} \end{cases}$$

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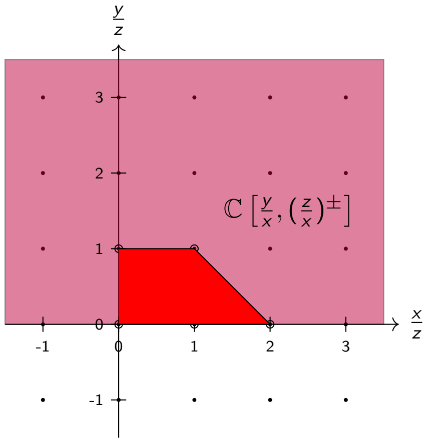

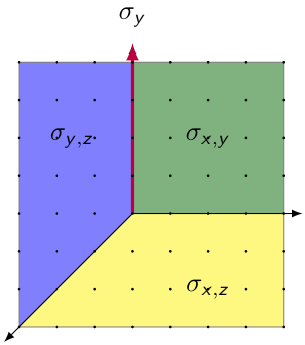

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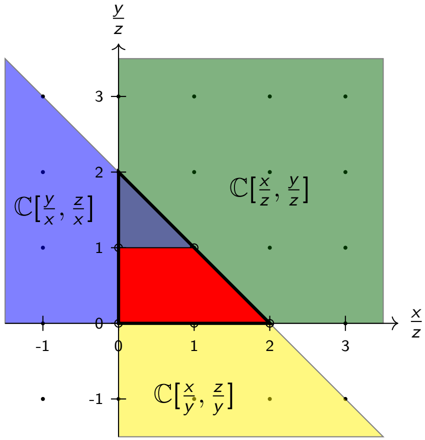

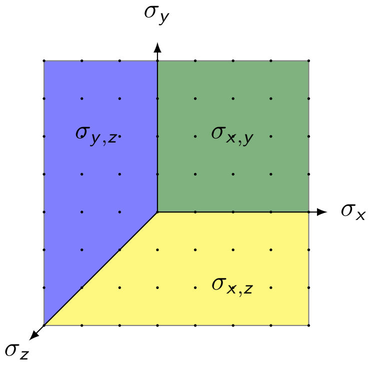

$$\begin{cases} x^2 + xy - 3\,xz + yz + \varepsilon z^2 \\ x^2 - xy - xz - yz + 2z^2 \end{cases}$$

Normal fan Σ of standard 2-dim simplex
Gluing the affine pieces $\to \mathbb{P}^2$.

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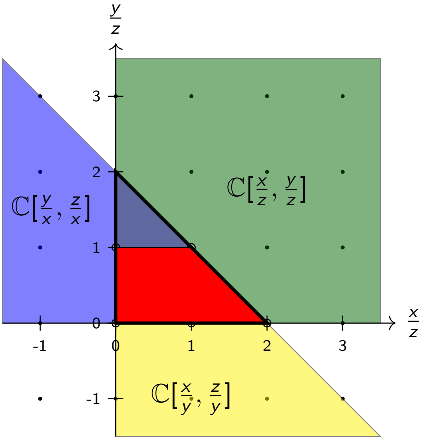

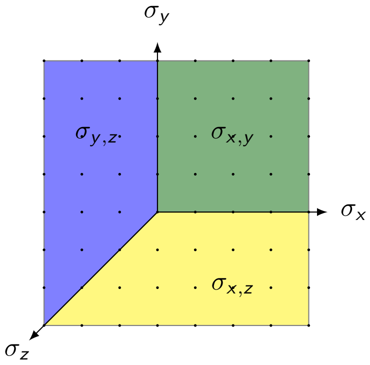

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But Newton Polytope is not a simplex...

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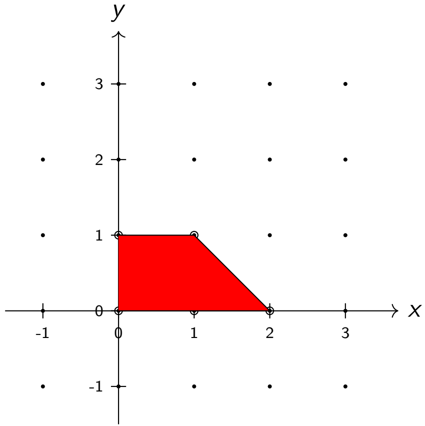

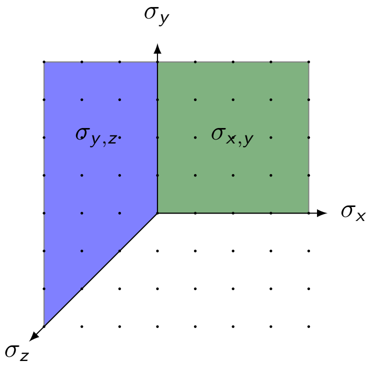

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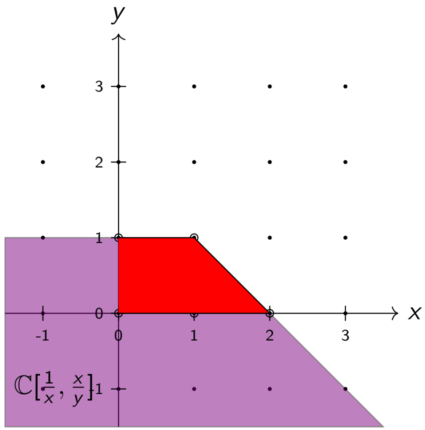

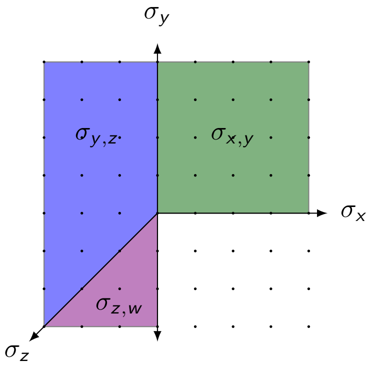

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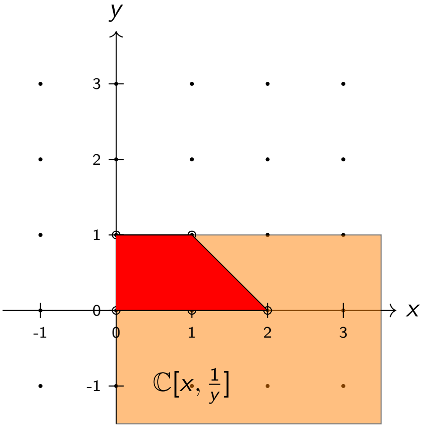

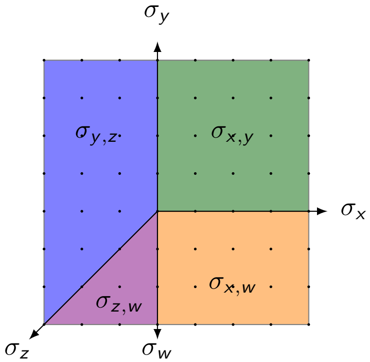

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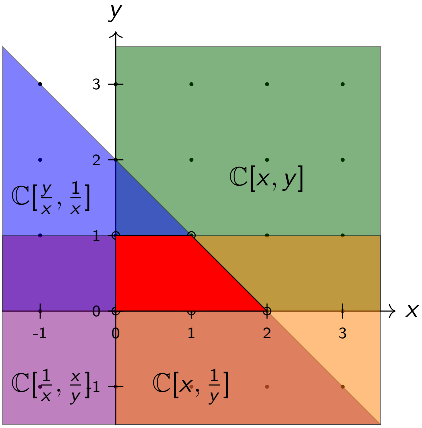

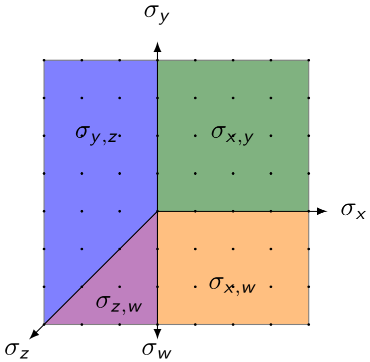

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Gluing the affine pieces $\rightarrow \mathcal{H}_1$.

But how we "homogenize"?

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- Hence, we work with $I := \langle f_1, \dots, f_r \rangle \subset S$ and its associated scheme $V_X(I)$.

$$V_{\mathbb{C}^n}(\langle \hat{f}_1, \dots, \hat{f}_r \rangle) \cap (\mathbb{C}^*)^n \hookrightarrow V_X(I).$$

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- Assuming 0-dim $V_X(I) \to$ we want coordinates of the points.

Solving using Eigenvalue Computations

- We introduce new notion of regularity of 0-dim subschemes $V_X(I)$.
- Main property, if $\alpha \in \mathrm{Reg}(S/I)$, then $(S/I)_\alpha \cong \mathbb{C}[V_X(I)]$.

Toric eigenvalue theorem [Thm 3.1; B.,Telen '21+]

- Consider a 0-dim subscheme of X, $V_X(I) = \{\zeta_1, \ldots, \zeta_\delta\}$,
  each ζ_i with multiplicity μ_i.
- Let α, $(\alpha + \alpha_0) \in \mathrm{Reg}(S/I)$.
- For each $g \in S_{\alpha_0}$, consider the multiplication map

$$M_g : (S/I)_\alpha \to (S/I)_{\alpha+\alpha_0}.$$

Then,

- For $h \in S_{\alpha_0}$ such that $V_X(I) \cap V_X(h) = \emptyset$, M_h is invertible.
- Moreover, for $g \in S_{\alpha_0}$,

$$\mathrm{CharPol}\big(M_h^{-1} \circ M_g\big)(\lambda) = \prod_i \left(\frac{g}{h}(\zeta_i) - \lambda\right)^{\mu_i}$$

It generalizes reduced version in [Telen '20]

Regularity for Complete Intersections

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Theorem [Thm 4.4; B., Telen '21+]

Let $I = \langle f_1, \ldots, f_n \rangle$ such that $f_i \in S_{\alpha_i}$, nef $\alpha_i \in \mathrm{Pic}(X)$, and $V_X(I)$ CI.
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- In practice, we compute with matrices of size

$$\left(\sum \mathrm{NewtonPolytope}(\hat{f}_i)\right) \cap \mathbb{Z}^n$$

as symbolic techniques like sparse resultants [Emiris, 96'] and Gröbner bases [B., Faugère, Tsigaridas '19].

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  - Multiprojective space $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$,
    $\beta = (n_1, \ldots, n_r)$.

Wrapping up

In the paper

- Symbolic-numerical algorithm
- Eigenvalue theorem in presence of multiplicities
- (Characterization of Eigenvectors)
- Regularity for complete intersections
  - Improvements for special cases:
    Unmixed, multihomogeneous, weighted homogeneous, (product of varieties, non-all variables present...)
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Thank you!

[arXiv:2006.10654]

Regularity for 0-dimensional subschemes

- Non-unique definition of CM regularity on multigraded setting, e.g., [Maclagan, Smith '03, Sidman, Van Tuyl '06, Botbol, Chardin '17]

Regularity

For zero-dimensional $Y := V_X(I)$ (with δ^+ points, counting multipl.):

$$\mathrm{Reg}(S/I) := \{\alpha \in \mathrm{Cl}(X) : \quad \begin{array}{l} \dim_{\mathbb{C}}((S/I)_\alpha) = \delta^+, \\ I_\alpha = (I : B^\infty)_\alpha, \text{ and} \\ \alpha \text{ is } V_X(I)\text{-basepoint free}\} \end{array}$$

(α is $V_X(I)$-basepoint free $\leftrightarrow$ $(\exists h \in S_\alpha)\, V_X(h) \cap Y = \emptyset$)

$$\alpha \in \mathrm{Reg}(S/I) \implies (S/I)_\alpha \cong \mathbb{C}[Y]$$ [Thm 4.1; B.,Telen '21+]

Regularity pair

Pair $(\alpha, \alpha_0) \in \mathrm{Cl}(X)^2$ such that

- α_0 is $V_X(I)$-basepoint free.
- $\alpha, \alpha + \alpha_0 \in \mathrm{Reg}(S/I)$