Simultaneous Matrix Diagonalization Problem

Rima Khouja

Advisor: Bernard Mourrain

Joint work with Jean-Claude Yakoubsohn

JNCF

March 1 - 5, 2021

Definition

- A pencil of matrices $M = [M_1, ..., M_s]$ is said to be simultaneously diagonalizable if there exists an invertible matrix E (called diagonalizer) such that $E^{-1}M_iE$ is a diagonal matrix for every $M_i \in M$.

Definition

- A pencil of matrices $M = [M_1, ..., M_s]$ is said to be simultaneously diagonalizable if there exists an invertible matrix E (called diagonalizer) such that $E^{-1}M_iE$ is a diagonal matrix for every $M_i \in M$.
- An important property which characterizes such sets, is that their matrices commute.

Definition

- A pencil of matrices $M = [M_1, ..., M_s]$ is said to be simultaneously diagonalizable if there exists an invertible matrix E (called diagonalizer) such that $E^{-1}M_iE$ is a diagonal matrix for every $M_i \in M$.
- An important property which characterizes such sets, is that their matrices commute.
- Contrarily, not every pencil of commuting matrices is simultaneously diagonalizable, example: let $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $B = \begin{pmatrix} \sigma & 1 & 0 \\ 0 & \sigma & 1 \\ 0 & 0 & \sigma \end{pmatrix}$, two Jordan blocks matrices. We have $AB = BA$, but $[A, B]$ is not simultaneously diagonalizable since a Jordan block matrix is not diagonalizable.

Motivation (simultaneous matrix diagonalization and tensor decomposition)

- $\mathcal{P}$ is a symmetric tensor of order d of dimension $n \in \mathcal{S}^d(\mathbb{C}^n)$:

$$\mathcal{P} = [v_{i_1 \dots i_d}] \in \mathbb{C}^{\overbrace{n \times \dots \times n}^{d \text{ times}}}$$

such that $v_{i_1 \dots i_d} = v_{i_{\sigma(1)} \dots i_{\sigma(d)}} \quad , \forall \sigma \in \mathcal{G}_d$

Motivation (simultaneous matrix diagonalization and tensor decomposition)

- $\mathcal{P}$ is a symmetric tensor of order d of dimension $n \in \mathcal{S}^d(\mathbb{C}^n)$:

$$\mathcal{P} = [v_{i_1 \dots i_d}] \in \mathbb{C}^{\overbrace{n \times \dots \times n}^{d \text{ times}}}$$

  such that $v_{i_1 \dots i_d} = v_{i_{\sigma(1)} \dots i_{\sigma(d)}} \quad , \forall \sigma \in \mathcal{G}_d$
- The symmetric tensor decomposition of $\mathcal{P}$ consists in writing it as a sum of rank one symmetric tensors

$$\mathcal{P} = \sum_{i=1}^{r} w_i v_i \otimes \dots \otimes v_i, \quad w_i \in \mathbb{C},\ v_i \in \mathbb{C}^n \qquad (1)$$

Motivation (simultaneous matrix diagonalization and tensor decomposition)

- $\mathcal{P}$ is a symmetric tensor of order d of dimension $n \in \mathcal{S}^d(\mathbb{C}^n)$:

$$\mathcal{P} = [v_{i_1 \dots i_d}] \in \mathbb{C}^{\overbrace{n \times \dots \times n}^{d \text{ times}}}$$

  such that $v_{i_1 \dots i_d} = v_{i_{\sigma(1)} \dots i_{\sigma(d)}} \quad, \forall \sigma \in \mathcal{G}_d$

- The symmetric tensor decomposition of $\mathcal{P}$ consists in writing it as a sum of rank one symmetric tensors

$$\mathcal{P} = \sum_{i=1}^{r} w_i v_i \otimes \dots \otimes v_i, \quad w_i \in \mathbb{C}, \ v_i \in \mathbb{C}^n \tag{1}$$

- By identification of $\mathcal{S}^d(\mathbb{C}^n)$ with $\mathbb{C}[\mathbf{x}]_d$, (1) is equivalent to write a homogeneous polynomial P (associated to $\mathcal{P}$) as a sum of linear forms to the d^{th} power:

$$P = \sum_{i=1}^{r} w_i (v_{i,1} x_1 + \dots + v_{i,n} x_n)^d := \sum_{i=1}^{r} w_i (v_i^t x)^d$$

- The symmetric rank of $\mathcal{P}$, denoted by $rank_s(\mathcal{P})$, is the smallest "r" such that the decomposition exists.

[1] R.Khouja, H.Khalil, and B.Mourrain, "A Riemannian Newton optimization framework for the symmetric tensor rank approximation problem" https://hal.archives-ouvertes.fr/hal-02494172.

- The symmetric rank of $\mathcal{P}$, denoted by $rank_s(\mathcal{P})$, is the smallest "r" such that the decomposition exists.
- In practice the input tensor $\mathcal{P}$ is known with some perturbations on its coefficients. For this reason, computing an approximate decomposition of low rank is usually more interesting than computing the exact symmetric decomposition of $\mathcal{P}$:

$$(STA) \quad \frac{1}{2} \min_{Q \in \sigma_r} ||Q - P||_d^2, \tag{2}$$

where $\sigma_r = \{Q \in \mathbb{C}[\mathbf{x}]_d \mid rank_s(Q) \leq r\}$.

[1] R.Khouja, H.Khalil, and B.Mourrain, "A Riemannian Newton optimization framework for the symmetric tensor rank approximation problem" https://hal.archives-ouvertes.fr/hal-02494172.

- The symmetric rank of $\mathcal{P}$, denoted by $rank_s(\mathcal{P})$, is the smallest "r" such that the decomposition exists.
- In practice the input tensor $\mathcal{P}$ is known with some perturbations on its coefficients. For this reason, computing an approximate decomposition of low rank is usually more interesting than computing the exact symmetric decomposition of $\mathcal{P}$:

$$(STA) \quad \frac{1}{2} \min_{Q \in \sigma_r} ||Q - P||_d^2, \tag{2}$$

where $\sigma_r = \{Q \in \mathbb{C}[\mathbf{x}]_d \mid rank_s(Q) \leq r\}$.

- We develop a Riemannian Newton iteration with trust-region scheme (RNS-TR) to solve localy (2)[1]. In such algorithm the choice of the **initial point** is important.

[1] R.Khouja, H.Khalil, and B.Mourrain, "A Riemannian Newton optimization framework for the symmetric tensor rank approximation problem" https://hal.archives-ouvertes.fr/hal-02494172.

For the initialization we use a direct algorithm[2] based on moment decomposition via SVD and eigenvector computation:

- **Input:** homogeneous polynomial $P \in \mathbb{C}[\mathbf{x}]_d$, $r \le r_g$ and $\iota \le \lfloor \frac{d-1}{2} \rfloor$, where ι denotes the interpolation degree.
  $d_1 := \lceil \frac{d+1}{2} \rceil$ and $d_2 := \lfloor \frac{d-1}{2} \rfloor$
- Compute the Hankel matrix $H_P^{d_1,d_2}$.
- Compute the singular value decomposition of $H_P^{d_1,d_2} = USV^*$.
- Let $M_i = S_r^{-1} U_r^* H_{x_i P}^{d_1,d_2} \bar{V}_r$, for $i = 1, \ldots, n$.
- Compute the eigenvectors ξ_j of $\sum_{i=1}^n l_i M_i$ for a random choice of $l_i \in [-1, 1]$, and for $j = 1, \ldots, r$ do the following:
  - Compute $v_{j,i}$ such that $M_i \xi_j = v_{j,i} \xi_j$ for $i = 1, \ldots, n$ and deduce the point $v_j := (v_{j,1}, \ldots, v_{j,n})$.
  - Compute $w_j = \frac{\langle (\xi_j^t x)^d, P \rangle_d}{\langle (\xi_j^t x)^d, (v_j^t x)^d \rangle_d}$.
- **Output:** $w_j \in \mathbb{C}^*, v_j \in \mathbb{C}^n$ for $j = 1, \ldots, r$.

[2] J. Harmouch, H. Khalil, and B. Mourrain, Structured low rank decomposition of multivariate Hankel matrices, Linear Algebra and its Applications, (2017).

- In this algorithm we diagonalize simultaneously a pencil of matrices $[M_1, \ldots, M_n]$.

[3] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas, Symmetric tensor decomposition, Linear Algebra and its Applications, (2009).

[4] V. Strassen, Rank and optimal computation of generic tensors, Linear Algebra and its Applications, (1983).

[5] L. De Lathauwer, A Link between the Canonical Decomposition in Multilinear Algebra and Simultaneous Matrix Diagonalization, SIMAX, (2006).

[6] P. Koiran, On tensor rank and commuting matrices, 2020.

- In this algorithm we diagonalize simultaneously a pencil of matrices $[M_1, \ldots, M_n]$.
- More generally, for symmetric tensors of sub-generic rank, similar approach based on simultaneous diagonalization of extensions of Hankel matrices is presented[3].

[3] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas, Symmetric tensor decomposition, Linear Algebra and its Applications, (2009).

[4] V. Strassen, Rank and optimal computation of generic tensors, Linear Algebra and its Applications, (1983).

[5] L. De Lathauwer, A Link between the Canonical Decomposition in Multilinear Algebra and Simultaneous Matrix Diagonalization, SIMAX, (2006).

[6] P. Koiran, On tensor rank and commuting matrices, 2020.

- In this algorithm we diagonalize simultaneously a pencil of matrices $[M_1, \ldots, M_n]$.
- More generally, for symmetric tensors of sub-generic rank, similar approach based on simultaneous diagonalization of extensions of Hankel matrices is presented[3].
- In the case of multilinear tensors, decomposition methods and bounds on tensor rank are obtained by constructing subspaces of matrices from tensors that satisfy various commutation properties[4] [5] [6].

[3] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas, Symmetric tensor decomposition, Linear Algebra and its Applications, (2009).

[4] V. Strassen, Rank and optimal computation of generic tensors, Linear Algebra and its Applications, (1983).

[5] L. De Lathauwer, A Link between the Canonical Decomposition in Multilinear Algebra and Simultaneous Matrix Diagonalization, SIMAX, (2006).

[6] P. Koiran, On tensor rank and commuting matrices, 2020.

The variety of simultaneous diagonalizable matrices

- Let $\mathfrak{D}_s$ be the variety of pencils of matrices $M = [M_1, \ldots, M_s]$ such that there exists matrices $E, F \in \mathbb{C}^{n\times n}$ invertibles with

$$M_k = F\ diag(\Sigma_{[:,k]})\ E^T = F\Sigma_{[k]}E^T$$

where $\Sigma = [\sigma_{i,j}] \in \mathbb{R}^{n\times s}$ and $\Sigma_{[:,k]}$ is the k^{th} column of Σ and $\Sigma_{[k]} = diag(\Sigma_{[:,k]})$. This variety is of dimension $2n^2 + n(s-1)$.

The variety of simultaneous diagonalizable matrices

- Let $\mathfrak{D}_s$ be the variety of pencils of matrices $M = [M_1, \ldots, M_s]$ such that there exists matrices $E, F \in \mathbb{C}^{n\times n}$ invertibles with
$$M_k = F\ diag(\Sigma_{[:,k]})\ E^T = F\Sigma_{[k]}E^T$$
where $\Sigma = [\sigma_{i,j}] \in \mathbb{R}^{n\times s}$ and $\Sigma_{[:,k]}$ is the k^{th} column of Σ and $\Sigma_{[k]} = diag(\Sigma_{[:,k]})$. This variety is of dimension $2n^2 + n(s-1)$.
- $M = [M_1, \ldots, M_s]$ is simultaneously diagonalizable iff $M' = [Id, M_1, \ldots, M_s] \in \mathfrak{D}_{s+1}$.

The variety of simultaneous diagonalizable matrices

- Let $\mathfrak{D}_s$ be the variety of pencils of matrices $M = [M_1, \ldots, M_s]$ such that there exists matrices $E, F \in \mathbb{C}^{n \times n}$ invertibles with
$$M_k = F\ diag(\Sigma_{[:,k]})\ E^T = F \Sigma_{[k]} E^T$$
where $\Sigma = [\sigma_{i,j}] \in \mathbb{R}^{n \times s}$ and $\Sigma_{[:,k]}$ is the k^{th} column of Σ and $\Sigma_{[k]} = diag(\Sigma_{[:,k]})$. This variety is of dimension $2n^2 + n(s-1)$.
- $M = [M_1, \ldots, M_s]$ is simultaneously diagonalizable iff $M' = [Id, M_1, \ldots, M_s] \in \mathfrak{D}_{s+1}$.
- A pencil M of matrices can be seen as a tensor $\mathbf{M} \in \mathfrak{T} = \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^s$ where the slice $\mathbf{M}_{[:,:,i]}$ is the matrix M_i.

The variety of simultaneous diagonalizable matrices

- Let $\mathfrak{D}_s$ be the variety of pencils of matrices $M = [M_1, \dots, M_s]$ such that there exists matrices $E, F \in \mathbb{C}^{n\times n}$ invertibles with
$$M_k = F\ diag(\Sigma_{[:,k]})\ E^T = F\Sigma_{[k]}E^T$$
where $\Sigma = [\sigma_{i,j}] \in \mathbb{R}^{n\times s}$ and $\Sigma_{[:,k]}$ is the k^{th} column of Σ and $\Sigma_{[k]} = diag(\Sigma_{[:,k]})$. This variety is of dimension $2n^2 + n(s-1)$.
- $M = [M_1, \dots, M_s]$ is simultaneously diagonalizable iff $M' = [Id, M_1, \dots, M_s] \in \mathfrak{D}_{s+1}$.
- A pencil M of matrices can be seen as a tensor $\mathbf{M} \in \mathfrak{T} = \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^s$ where the slice $\mathbf{M}_{[:,:,i]}$ is the matrix M_i.
- $M = [M_1, \dots, M_s] \in \mathfrak{D}_s$ iff:
$$\mathbf{M} = \sum_{k=1}^{n} F_k \otimes E_k \otimes \Sigma_{[k,:]} \text{ is of rank } \leq n,$$
where F_k (resp. E_k) is the k^{th} column of F, (resp. E) and $\Sigma_{[k,:]} = [\sigma_{k,1}, \dots, \sigma_{k,s}]$ is the k^{th} row of Σ.

Newton-type method for simultaneous matrix diagonalization

- Let us consider s simultaneously diagonalizable matrices $M_1, \cdots, M_s$ in $\mathbb{C}^{n\times n}$. We suppose that all the eigenvalues of M_i are simple. We aim to solve the following system of equations:

$$\begin{pmatrix} FE - I_n \\ FM_1E - \Sigma_1 \\ \vdots \\ FM_sE - \Sigma_s \end{pmatrix} = 0 \qquad (3)$$

Newton-type method for simultaneous matrix diagonalization

- Let us consider s simultaneously diagonalizable matrices $M_1, \cdots, M_s$ in $\mathbb{C}^{n\times n}$. We suppose that all the eigenvalues of M_i are simple. We aim to solve the following system of equations:

$$\begin{pmatrix} FE - I_n \\ FM_1E - \Sigma_1 \\ \vdots \\ FM_sE - \Sigma_s \end{pmatrix} = 0 \quad (3)$$

- The strategy to solve (3) be the following. From E_0, F_0 and $\Sigma_{1,0}$ close to a solution of $(FE - I_n, FM_1E - \Sigma_1) = 0$ we numerically solve this system.

Newton-type method for simultaneous matrix diagonalization

- Let us consider s simultaneously diagonalizable matrices $M_1, \cdots, M_s$ in $\mathbb{C}^{n\times n}$. We suppose that all the eigenvalues of M_i are simple. We aim to solve the following system of equations:

$$\begin{pmatrix} FE - I_n \\ FM_1E - \Sigma_1 \\ \vdots \\ FM_sE - \Sigma_s \end{pmatrix} = 0 \qquad (3)$$

- The strategy to solve (3) be the following. From E_0, F_0 and $\Sigma_{1,0}$ close to a solution of $(FE - I_n, FM_1E - \Sigma_1) = 0$ we numerically solve this system.
- Another strategy to solve (3), is to take a linear combination M of $M_1, \ldots, M_s$, and to solve $(FE - I_n, FME - \Sigma) = 0$.

Newton-type method for simultaneous matrix diagonalization

- Let us consider s simultaneously diagonalizable matrices $M_1, \cdots, M_s$ in $\mathbb{C}^{n\times n}$. We suppose that all the eigenvalues of M_i are simple. We aim to solve the following system of equations:

$$\begin{pmatrix} FE - I_n \\ FM_1E - \Sigma_1 \\ \vdots \\ FM_sE - \Sigma_s \end{pmatrix} = 0 \qquad (3)$$

- The strategy to solve (3) be the following. From E_0, F_0 and $\Sigma_{1,0}$ close to a solution of $(FE - I_n, FM_1E - \Sigma_1) = 0$ we numerically solve this system.
- Another strategy to solve (3), is to take a linear combination M of $M_1, \ldots, M_s$, and to solve $(FE - I_n, FME - \Sigma) = 0$.
- Next we deduce the diagonal matrices Σ_i by using the formula:

$$\Sigma_i = FM_iE$$

- We consider the numerical resolution of:

$$(FE - I_n, FME - \Sigma) = 0. \tag{4}$$

From an approximation (E_0, F_0, Σ_0) close enough to a solution of (4) we propose a sequence which converges quadratically towards a solution.[7]

[7] Similar approach in the case of one matrix diagonalization equivalently eigenproblem is considered in J. van der Hoeven, B. Mourrain, Efficient certication of numeric solutions to eigenproblems, MACIS 2017.

- We consider the numerical resolution of:

$$(FE - I_n, FME - \Sigma) = 0. \tag{4}$$

From an approximation (E_0, F_0, Σ_0) close enough to a solution of (4) we propose a sequence which converges quadratically towards a solution.[7]

- We use the max norm for vectors and the corresponding matrix norm given for a vector $v \in \mathbb{C}^n$ and a matrix $M \in \mathbb{C}^{n\times n}$ as follow:

$$\begin{aligned} \|v\| &= \max\{|v_1|, \ldots, |v_n|\} \\ \|M\| &= \max_{\|v\|=1} \|Mv\|. \end{aligned}$$

[7] Similar approach in the case of one matrix diagonalization equivalently eigenproblem is considered in J. van der Hoeven, B. Mourrain, Efficient certication of numeric solutions to eigenproblems, MACIS 2017.

- We consider the perturbations $E + EX$, $F + YF$, and $\Sigma + S$ respectively of E, F, and Σ. We get with $Z = FE - I_n$, and $\Delta = FME - \Sigma$:

$$\begin{aligned} &(F + YF)(E + EX) - I_n \\ &= Z + (Z + I_n)X + Y(Z + I_n) + Y(Z + I_n)X \end{aligned} \tag{5}$$

$$\begin{aligned} &(F + YF)M(E + EX) - \Sigma - S \\ &= FME - I_n + FMEX + YFME + YFMEX \\ &= \Delta - S + \Sigma X + Y\Sigma + \Delta X + Y\Delta + Y(\Delta + \Sigma)X \end{aligned} \tag{6}$$

- We consider the perturbations $E+EX$, $F+YF$, and $\Sigma+S$ respectively of E, F, and Σ. We get with $Z=FE-I_n$, and $\Delta = FME - \Sigma$:

$$\begin{aligned} &(F+YF)(E+EX)-I_n \\ &= Z+(Z+I_n)X+Y(Z+I_n)+Y(Z+I_n)X \end{aligned} \tag{5}$$

$$\begin{aligned} &(F+YF)M(E+EX)-\Sigma-S \\ &= FME-I_n+FMEX+YFME+YFMEX \\ &= \Delta - S+\Sigma X+Y\Sigma+\Delta X+Y\Delta+Y(\Delta+\Sigma)X \end{aligned} \tag{6}$$

- The Newton method consists in solving the linear system in (X,Y,S) obtained from (5), (6):

$$\begin{cases} Z+X+Y & =0 \\ \Delta - S+\Sigma X+Y\Sigma & =0 \end{cases}$$

Lemma

Let $\Sigma = diag(\sigma_1, \cdots, \sigma_n)$, $Z = (z_{i,j})$ *and* $\Delta = (\delta_{i,j})$ *be given matrices. Assume that* $\sigma_i \neq \sigma_j$ *for* $i \neq j$. *Let* S, X *and* Y *be matrices defined by*

$$\begin{aligned} S &= diag(\Delta - Z\Sigma) \\ x_{i,i} &= 0 \\ x_{i,j} &= \frac{-\delta_{i,j} + z_{i,j}\sigma_j}{\sigma_i - \sigma_j}, \qquad i \neq j \\ y_{i,i} &= -z_{i,i} \\ y_{i,j} &= \frac{\delta_{i,j} - z_{i,j}\sigma_i}{\sigma_i - \sigma_j}, \qquad i \neq j. \end{aligned}$$

Then we have

$$Z + X + Y = \Delta - S + \Sigma X + Y\Sigma = 0, \textit{ and } \|X\|, \|Y\| \leqslant \kappa\varepsilon(K+1),$$

where $\varepsilon = \max(\|Z\|, \|\Delta\|)$, $\kappa = \max\left(1, \max_{i\neq j} \frac{1}{|\sigma_i - \sigma_j|}\right)$ *and* $K = \max(1, \max_i |\sigma_i|)$.

Theorem

Let E_0*,* F_0 *and* Σ_0 *be given and define the sequences for* $i \geqslant 0$*,*

$$\begin{array}{rcl} Z_i &=& F_iE_i - I_n \\ \Delta_i &=& F_iME_i - \Sigma_i \\ S_i &=& diag(\Delta_i - Z_i\Sigma_i) \\ E_{i+1} &=& E_i(I_n + X_i) \\ F_{i+1} &=& (I_n + Y_i)F_i \\ \Sigma_{i+1} &=& \Sigma_i + S_i \end{array}$$

let $\varepsilon_0 = \max(\|Z_0\|, \|\Delta_0\|)$*,* $\kappa_0 = \max\left(1, \max_{i\neq j} \frac{1}{|\sigma_{0,i} - \sigma_{0,j}|}\right)$ *and* $K_0 = \max(1, \max_i |\sigma_{0,i}|)$*. Assume that* $u := \kappa_0^2(K_0+1)^3\varepsilon_0 \leqslant 0.136$*. Then the sequence* $(\Sigma_i, E_i, F_i)_{i\geqslant 0}$ *converges quadratically to a solution of* $(FE - I_n, FME - \Sigma) = 0$*. More precisely* E_0 *and* F_0 *are invertible and*

$$\|E_i - E_\infty\| \leqslant 0.61 \times 2^{1-2^{i+1}} \|E_0\| u; \ \|F_i - F_\infty\| \leqslant 0.61 \times 2^{1-2^{i+1}} \|F_0\| u.$$

Numerical results

We take $M \in \mathbb{C}^{n\times n}$ random diagonalizable matrix: $M = E\Sigma E^{-1}$. To apply the theorem, we take a perturbation of E, Σ and $F = E^{-1}$ as an initial point, such that this initial point verifies $\kappa_0^2(K_0+1)^3\varepsilon_0 \leqslant 0.136$.

Iteration	$n = 10$	$n = 50$	$n = 100$
1	0.01277	0.00706	0.00252
2	$5.41e-5$	$7.54e-8$	$4.1\ e-9$
3	$4.03e-11$	$4.7e-16$	$2.51\ e-18$
4	$1.31e-21$	$2.65e-32$	$1.65\ e-36$
5	$5.06e-42$	$1.78e-64$	$2.45\ e-75$
6	$1.96e-84$	$8.65e-129$	$3.41e-150$

$M_1 = \begin{pmatrix} 0.71761 & 0.39502 \\ 0.15013 & 0.41416 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0.28899 & 0.1828 \\ 0.06947 & 0.14857 \end{pmatrix}$,
$M_3 = \begin{pmatrix} 0.33737 & -0.44756 \\ -0.17009 & 0.68118 \end{pmatrix}$.
$M_i = E\Sigma_i E^{-1}$ for $i \in \{1,2,3\}$, where $E = \begin{pmatrix} -0.66918 & 0.94612 \\ 0.7431 & 0.32381 \end{pmatrix}$,
$\Sigma_1 = diag(0.27896, 0.8528)$, $\Sigma_2 = diag(0.086004, 0.35155)$,
$\Sigma_3 = (0.83436, 0.18419)$, $F = E^{-1}$.
First strategy: we consider $(EF - I_n, FM_1E - \Sigma_1) = 0$, from initial point $(E_0, F_0, \Sigma_{0,1})$ which verifies the condition in the Theorem we apply the Newton iteration:
iter1: 0.00516
iter2: 1.2.e-5
iter3: 9.4.e-11
iter4: 5.1.e-21
Solution:
$E_{sol} = \begin{pmatrix} -0.6676 & 0.9467 \\ 0.74133 & 0.32411 \end{pmatrix}$, $\Sigma_{sol,1} = diag(0.27896, 0.8528)$.

$\Sigma_{sol,2} = FM_2E = diag(0.086004, 0.35155)$,
$\Sigma_{sol,3} = FM_3E = diag(0.83436, 0.18419)$.

Second strategy: we take for example the linear combination $M = M_1 + M_2 + M_3$. We consider $(EF - I_n, FME - \Sigma) = 0$, from initial point (E_0, F_0, Σ_0) which verifies the condition in the Theorem we apply the Newton iteration:

iter1:0.00084

iter2:1.1.e-6

iter3: 6.5.e-12

iter4: 3.4.e-24

Solution:

$E_{sol} = \begin{pmatrix} -0.66912 & 0.94603 \\ 0.74302 & 0.32378 \end{pmatrix}$.

$\Sigma_{sol,1} = FM_1E = diag(0.27896, 0.8528)$,
$\Sigma_{sol,2} = FM_2E = diag(0.086004, 0.35155)$,
$\Sigma_{sol,3} = FM_3E = diag(0.83436, 0.18419)$.

We tested with another linear combinations and we found the same solutions.

What's next ?

- Let $M = [M_1, \ldots, M_s]$ be a pencil of matrices which is not simultaneously diagonalizable. The objective is to approximate locally M into a pencil of simultaneously diagonalizable matrices $M' = [M'_1, \ldots, M'_s]$.
- This problem is already considered when M_i are symmetric matrices by using Riemannian optimization techniques[8] [9]. We aim to investigate the general case when M_i are general square matrices and to developp an efficient algorithm.

[8] P. Absil, K. A. Gallivan, Joint diagonalization on the oblique manifold for independent component analysis, international conference on acoustics speech and signal processing, (2006).

[9] F. Bouchard, B. Afsari, J. Malick, and M. Congedo, Approximate joint diagonalization with Riemannian optimization on the general linear group, SIMAX, (2020).

Thank you for your attention !