Value Function in Optimal Control on Wasserstein Spaces

H. Frankowska

CNRS IMJ-PRG and Sorbonne Université

helene.frankowska@imj-prg.fr

Abstract

Mean field control problems refer broadly to situations in which a centralised policy-maker emits a control signal at the macroscopic level, in order to stir microscopic agents towards a desired goal. Such models have led to optimal control problems stated on the so-called Wasserstein spaces. Various results of the classical control theory of ODEs adapt well to the Wasserstein spaces framework because flows of solutions to controlled ODEs generate solutions to the controlled continuity equation

$$\partial_t \mu(t) + \mathrm{div}_x \big(v(t, \mu(t), u(t))\mu(t)\big) = 0 \tag{1}$$

In particular, the depiction of optimality conditions in the form of variants of the Maximum Principle (PMP) has attracted a lot of attention.

Let $\mathcal{P}_c(\mathbb{R}^d)$ denote the set of Borel probability measures on $\mathbb{R}^d$ having a compact support. I will discuss various properties and applications of the value function $\mathcal{V} : [0,T] \times \mathcal{P}_c(\mathbb{R}^d) \to \mathbb{R}$ associated to a Mayer mean-field optimal control problem: Let U be a given compact metric space and consider the set of admissible controls $\mathcal{U} := \{u : [0,T] \to U$ is Lebesgue-measurable$\}$. Let the time-evolution of measure-curves be generated by the *controlled non-local velocity field* $v : [0,T] \times \mathcal{P}_c(\mathbb{R}^d) \times U \times \mathbb{R}^d \to \mathbb{R}^d$ and $\varphi : \mathcal{P}_c(\mathbb{R}^d) \to \mathbb{R}$ be a final cost. The value function of the Mayer type problem is defined by: for any $(\tau, \mu_\tau) \in [0,T] \times \mathcal{P}_c(\mathbb{R}^d)$

$$\mathcal{V}(\tau, \mu_\tau) := \inf_{u(\cdot)\in\mathcal{U}} \{\varphi(\mu(T)) \mid \mu \text{ solves (1) on } [\tau, T] \text{ with } \mu(\tau) = \mu_\tau\}$$

In particular, the following sensitivity relation holds true :

for $\tau = 0$, $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$ and for every minimiser $(\mu^*(\cdot), u^*(\cdot))$ of the above optimisation problem, there exists a state-costate curve of measures $\nu^* : [0,T] \to \mathcal{P}_c(\mathbb{R}^{2d})$ satisfying (PMP) such that for the associated Hamiltonian $H : [0,T] \times \mathcal{P}_c(\mathbb{R}^{2d}) \times U \to \mathbb{R}$ it holds

$$\Big(H(t, \nu^*(t), u^*(t))\,,\, -\bar{\nu}^*(t)\Big) \in \partial^+ \mathcal{V}(t, \mu^*(t)) \text{ a.e. in } [0,T],$$
$$-\bar{\nu}^*(t) \in \partial^+_\mu \mathcal{V}(t, \mu^*(t)) \text{ for all } t \in [0,T],$$

where $\bar{\nu}^*(t) \in L^\infty(\mathbb{R}^d, \mathbb{R}^d; \mu^*(t))$ denotes the barycentric projection of the state-costate curve $\nu^*(t)$ onto $\mu^*(t)$ and ∂^+ stands for the Dini superdifferential. This allows to deduce some further smoothness of the value function, in vein of [1].

[1] Cannarsa P. & Frankowska H. (1991) *Some characterizations of optimal trajectories in control theory*, SIAM J. Control Optim.

[2] Bonnet B. & Frankowska H. (2022) *Sensitivity analysis of the value function of mean-field optimal control problems and applications*, Journal de Mathématiques Pures et Appliquées

[3] Bonnet B. & Frankowska H. (2021) *Necessary optimality conditions for optimal control problems in Wasserstein spaces*, Applied Mathematics and Optimization