Non-trivial homotopy in the space of Legendrians isotopic to the zero section in a jet-1 bundle.

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Work in progress - joint with Y. Eliashberg

Talk overview

- Definitions, pseudo-isotopy spaces and statement of result.
- Legendrian knots and pseudo-isotopies (the injectivity theorem).
- The map from pseudo-isotopy spaces into spaces of Legendrians.
- Sketch of proof (using doubling of generating function and h-cobordism spaces).

jet-1 bundle and space of legendrians

Let M be a smooth compact manifold possibly with boundary, corners, etc. Let $J^1M = T^*M \times \mathbb{R}$ be the jet-1 bundle with its canonical contact form

$$\alpha = pdq - dz$$

where pdq is the Liouville 1-form on T^*M and $z \in \mathbb{R}$.

Let $\mathcal{L}eg(M)$ denote the space of smooth Legendrians $L \subset J^1(M)$ satisfying

- L equals the zero section in a neighborhood of the boundary.
- L is isotopic through such to the zero section.

So $\mathcal{L}eg(M)$ is connected.

The reason we only focus on this particular component is that we will consider $\Omega\mathcal{L}eg(M)$.

Here ΩX denotes the based loop space of X - i.e. paths starting and ending at the base-point.

Note that $\pi_i(\Omega X) \cong \pi_{i+1}(X)$ for $i \geq 0$.

Pseudo-isotopy groups

With M as before and $I = [0,1]$ we define the *pseudo-isotopy group* $\mathcal{P}(M)$ as the *space* of functions

$$f : M \times I \to I$$

such that

- f equals the projection to I in a neighborhood of the boundary of $M \times I$ $(\partial M \times I \cup M \times \{0,1\})$.
- f has no critical points.

Non-trivial fact: The homotopy groups of $\mathcal{P}(M)$ only depends on the homotopy type of M in stable range. Here "stable range" means (Hatcher-Igusa) up to roughly dimension $\frac{n-4}{3}$, where $n = \dim M$.
This stable part can be partially computed using Waldhausen's algebraic K-theory of spaces.

For $k \geq 34$ we have:

n	$\pi_n(\mathcal{P}(D^k))$
0	0
1	$\mathbb{Z}/2$
2	0
3	$\mathbb{Z}$
4	0
5	$\mathbb{Z}/2$
6	0
7	$\mathbb{Z} \oplus \mathbb{Z}/2$
8	$\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2$
9	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
10	$\mathbb{Z}/4 \oplus K_{12}(\mathbb{Z})$

Even more non-trivial for M with homotopy.

From: Blumberg, Mandell and Rognes's calculations of $K(\mathbb{S})$.

Extension by "0"

If $i : N \subset M$ is a codimension 0 smooth embedding we have inclusions

$$i_! : \mathcal{P}(N) \to \mathcal{P}(M), \qquad i_! : \mathcal{L}eg(N) \to \mathcal{L}eg(M).$$

given by extending trivially. In the case of $\mathcal{L}eg(M)$ we extend by the zero-section. In the case of $\mathcal{P}(M)$ we extend $N \times I \subset M \times I \to I$ by the projection to I.

Note that when the inclusion $N \subset M$ is p-connected then $i_! : \mathcal{P}(N) \to \mathcal{P}(M)$ is $\min(\frac{n-7}{2}, \frac{n-4}{3}, p-2)$-connected (follows from Waldhausen's algebraic K-theory of spaces and Hatcher-Igusa stability).
Example: The inclusion $i : D^k \subset S^k$ for $k \geq 34$ shows that the previous list of homotopy groups is also valid for spheres of high dimensions.

We don't know much about the spaces $\mathcal{L}eg(M)$ in high dimensions (and the maps $\mathcal{L}eg(N) \to \mathcal{L}eg(M)$).

There are results by Sullivan and Sabloff (building on work by Kalman) about non-trivial π_1 using generating functions in a way similar to what I will describe.

However, as I understand these - they are not about the component containing the zero section.

Statement of result ($n = \dim M = \dim N$)

Theorem (Eliashberg, K)

There are maps $F_N : \mathcal{P}(N) \to \Omega\mathcal{L}eg(N \times I)$ with the following properties:

a) *For any co-dimension 0 embedding $i : M \subset N$ the maps F_N and F_M are compatible with the trivial extensions.*

b) *The maps F_N are split injective on homotopy groups up to dimension $\min(\frac{n-7}{2}, \frac{n-4}{3})$.*

Corollary

Let $X \subset \mathcal{P}(M)$ be an k-connected, k-dimensional CW approximation for $k \leq \frac{n-12}{4}$. Then there is a map

$$X \to \Omega\mathcal{L}eg(M)$$

split injective on homotopy groups.

$$\begin{array}{ccc} \mathcal{P}(N) & \xrightarrow{i_!} & \mathcal{P}(M) \\ \downarrow F_N & & \downarrow F_M \\ \Omega\mathcal{L}eg(N \times I) & \xrightarrow{\Omega i_!} & \Omega\mathcal{L}eg(M \times I) \end{array}$$

The splitting in b) is done by maps:

$$\begin{array}{ccc} \mathcal{P}(M) & \xrightarrow{F_N} & \Omega\mathcal{L}eg(M \times I) \\ \downarrow s & & \downarrow G_{M \times I} \\ \mathcal{P}_\infty(M) & \xrightarrow{=} & \mathcal{P}_\infty(M \times I) \end{array}$$

Stable range = connectivity of s.

So the previous list of 10 homotopy groups injects as a summand in $\pi_{*+1}(\mathcal{L}eg(S^k))$ for $k > 52$.

Sketch of proof of Corollary given the theorem

Let $n = 4k + 1$. Let $Y \to M$ be a k-connected, k dimensional CW approximation of M. One may replace Y with a smooth manifold of dimension $2k = \frac{n-1}{2}$ and assume $Y \subset M$.

Since Y is less than half the dimension it has a trivial factor in its normal bundle. Hence we can extend to a co-dimension 0 embedding $Y' \times I \subset M$ with $Y \simeq Y'$. Now consider the diagram

$$\begin{array}{ccccc}
\mathcal{P}(Y') & \xrightarrow{\ s\ } & \mathcal{P}(Y' \times I) & \xrightarrow{\ i_!\ } & \mathcal{P}(M) \\
\downarrow{\scriptstyle F_N} & & & & \downarrow{\scriptstyle s} \\
\Omega\mathcal{L}eg(Y' \times I) & \xrightarrow{\ \Omega i_!\ } & \Omega\mathcal{L}eg(M) & \xrightarrow{\ G_M\ } & \mathcal{P}_\infty(M)
\end{array}$$

- s is an equivalence in stable range due to Hatcher-Igusa.
- $i_!$ is an equivalence in stable range due to Waldhausen (K-theory).

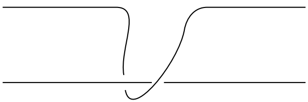
The stable range $\frac{n-12}{4}$ is from Waldhausens range.

The stabilization is heuristically defined by identifying I with $\mathbb{R}$ and either adding x^2 or $-x^2$ in the new coordinates. Similar to stabilizations in stable Morse theory (where you usually have critical points).

Legendrian knotting from pseudo-isotopies

Let $\pi_I : M \times I \to I$ be the projection. We let $\mathcal{L}ag_0(M)$ denote the space of Lagrangian submanifolds L in $T^*(M \times I) \setminus M \times I$ which:

- close to the boundary coincide with $d\pi_I$.
- are disjoint from the zero-section.



We have $\mathcal{P}(M) \subset \mathcal{L}ag_0(M)$ by $f \mapsto \text{graph}(df)$.

Injectivty theorem (Gromov, Eliashberg - 98)

in stable range this map is split injective on homotopy groups.

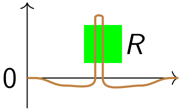
Note in particular that $\pi_0(\mathcal{P}(M)$ is the White-head group of M.

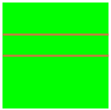
This thus creates knotting in the standard sense when M has non-trivial Whitehead group.

An example of this is when $\pi_1(M) = \mathbb{Z}/5$, like some Lens space.

More Legendrian knotting from pseudo-isotopies

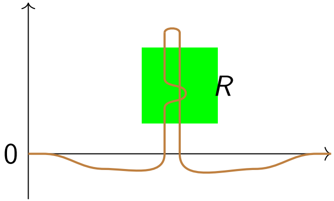
Consider the exact Lagrangian $l_0 \subset T^*I$ (lifting to a Legendrian in $\mathcal{L}eg(I)$):





Rotating and re-scaling the green box R we can identify the two lines in the green box with 0 and $d\pi_I$. Consider the Lagrangian $M \times l_0 \subset T^*(M \times I)$, and define a map $F'_M : \mathcal{P}(M) \to \mathcal{L}eg(M \times I)$ by sending $f \in \mathcal{P}(M)$ to the exact Lagrangian given by $M \times l_0$ except the part in $T^*M \times R$ is replaced by the zero-section and the graph of df.

The resulting **exact Lagrangian** can be visualized as:



Initially we had hoped this could produce a counter example to the nearby Lagrangian conjecture.

A null homotopy as Legendrians and one as Lagrangians (providing a loop)

1) The convex isotopy (from df to $d\pi_I$ in the green part) defines a homotopy through Legendrians from $F_M(f)$ to $M \times I_0$.

2) Give M a Riemannian metric (inducing one on $M \times I$). Consider the normalized gradient flow φ_t of any $f \in \mathcal{P}(M)$. The map φ_1 maps $M \times \{0\} \to M \times \{1\}$. In fact φ_t defines a diffeomorphism

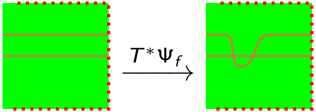
$$\Psi_f : M \times I \cong M \times I$$

by $(x, t) \mapsto \varphi_t(x, 0)$. This is the identity in a neighborhood of $M \times \{0\}$ and $\partial M \times I$. It is on product form $\psi \times \mathrm{Id}_I$ in a neighborhood of $M \times \{1\}$. Also $f = \Psi_f^{-1} \circ \pi_I$

Hence we get a map from $\mathcal{P}(M)$ into the space of such diffeomorphism. This is a homotopy equivalence.

(the inverse homotopy equivalence send such a diffeomorphism to the function given by composing with π_I.)

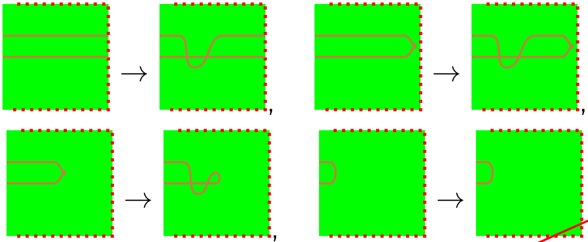
Replacing $d\pi_I$ with df (and leaving the 0-section where it is) can alternatively be described by composing with $T^*\Psi_f$.



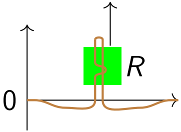
This symplectomorphism is only defined locally close to the zero-section.

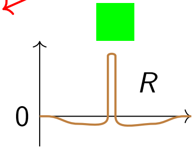
A null homotopy as Lagrangians and the defined loop

Even though the symplectomorphism $T^*\Psi_f$ is only defined close to the two copies the fold pulls through:



So sliding this symplectomorphism of the fold we get a **embedded Lagrangian** isotopy to $M \times I_0$:





So, for each $f \in \mathcal{P}(M)$ we define a loop in $\mathcal{L}eg(M \times I)$ based at $M \times I_0$ by:

- First convexly interpolating form $M \times I_0$ to $F'_M(f)$,
- Then we "pull of $T^*\Psi_f$" to get back to $M \times I_0$,

By applying a Hamiltonian symplectomorphism to $T^*(M \times I)$, we can assume that the loop is based at the zero-section $M \times I$. This defines

$$F_M : \mathcal{P}(M) \to \Omega\mathcal{L}eg(M \times I)$$

which is natural by construction.

Generating functions quadratic at infinity

The result from the previous slides is that we have constructed the map

$$F_M : \mathcal{P}(M) \to \Omega\mathcal{L}eg(M \times I).$$

However, detecting that this is non-trivial (splits on low homotopy groups) takes a bit of work.

For this we consider generating functions for a Legendrian $L \subset J^1(X)$. A generating function quadratic at infinity (g.f.q.i.) is a g.f.

$$f : X \times \mathbb{R}^{2k} \to \mathbb{R}$$

equaling the standard quadratic form

$$Q(q, x, y) = \|x\|^2 - \|y\|^2$$

for $x, y \in \mathbb{R}^k$ outside a compact set (and near the boundary $\partial X \times \mathbb{R}^{2k}$).

Q is itself a generating function for the zero-section. As this is the start of the loops we can use the result (follows from work by Chekanov and Eliashberg-Gromov) that one can transport g.f.q.i. along paths (requires increasing k and stabilizing) to define such along the entire loop.

That is we have

$$G_f^t : M \times I \times \mathbb{R}^{2k} \to \mathbb{R}$$

for each Legendrian $F_M(f)(t)$ with $f \in \mathcal{P}(M)$ and $t \in I$, with $G_f^0 = Q$.

Note that G_f^1 also generates the zero-section and is q.i., but its "probably" not Q.

Doubling

For a generating function $G : X \times \mathbb{R}^{2k} \to \mathbb{R}$ (generating $L \subset J^1(X)$) we consider the *double*:

$$D(G) = G \times_X -G : X \times \mathbb{R}^{2k} \times \mathbb{R}^{2k}.$$

This has critical points (q, x, y) where x and y are in the fiber-wise critical locus (diffeomorphic to L) and mapped to the same point in $T_q^* X$ under the Lagrangian projection of L.

The critical value is the time it takes the Reeb flow to flow x to y. Hence the critical locus is:

- A copy of L with critical value 0,
- two points for each self-$\cap$ with critical value $\pm$ the length of the Reeb chord.

This means that there is an interval $(0, \varepsilon]$ of regular values.

As the double of Q has $W_0 := D(Q)^{-1}(\varepsilon) \cong X \times S^{2k-1} \times \mathbb{R}^{2k}$, all generating functions that are "isotopic to Q" will have its ε-level-set isotopic to W_0.

It follows that we can define a map

$$D : \mathcal{GF} \to \{\text{sub-manifolds iso to } W_0\}$$

where $\mathcal{GF}$ denotes the space of g.f.q.i. isotopic to Q (hence generates a Legendrian in $\mathcal{L}eg(X)$).

Doubling

When a g.f. G generates an embedded Lagrangian all positive values of the double $D(G)$ are regular, and as the double can be bumped of to be $D(Q)$ outside a compact set, one may canonically isotopy the level-set to W_0. Indeed, flow far out with $\nabla D(G_f^1)$ and back in again with $\nabla D(Q)$. Hence we can assume that the map

$$D : \mathcal{GF} \to \{\text{sub-manifolds iso to } W_0\}$$

is constantly W_0 on the sub-space that generates embedded Lagrangians.

It follows that even though our family of generating functions G_f^t for $f \in \mathcal{P}(M)$ and $t \in I$ we considered earlier is not a loop. We can assume that the image of D makes it a loop again. We thus have a map

$$\Omega\mathcal{L}eg(M \times I) \to \Omega\{\text{sub-manifolds iso to } W_0\}$$

where $W_0 = M \times I \times S^{2k-1} \times \mathbb{R}^{2k} \subset M \times I \times \mathbb{R}^{4k}$.

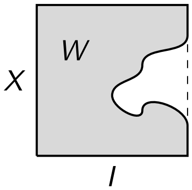
The rest of the proof sketch is understanding such sub-manifolds and relating them back to the pseudo-isotopies.

Note that stabilizing increases k, and as spheres of high dimensions are contractible these will essentially go away in the (co)limit.

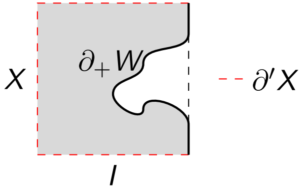
h-cobordism spaces

To make things slightly more concrete we use Waldhausen's space $\mathcal{H}(X)$ of h-cobordisms from X. This is the space of co-dimension 0 submanifolds $W \subset X \times I$ such that:

- W co-incides with $X \times I$ near $\partial' X = \partial X \times I \cup X \times \{0\}$.
- The inclusion $X \times \{0\} \subset W$ is a homotopy equivalence.



Note that W is completely determined by its "outgoing" boundary $\partial_+ W = \partial W \setminus \partial' M$:



$\Omega\mathcal{H}(X) \simeq \mathcal{P}(X)$

Lemma

$\mathcal{P}(X) \simeq \Omega\mathcal{H}(X)$

Proof: Consider the space $C(X)$ of *collars* $X \times I \to X \times I$ equal to the identity in a neighborhood of $\partial' X$. $C(X)$ is contractible.

The image of such a collar is an h-cobordism so we have a map

$$C(X) \to \mathcal{H}(X)$$

this is a Serre-fibration and the fiber over the base-point is "$\mathcal{P}(X)$" (diffeomorphism version). □

Note that the proof shows that: given a map $h : K \to \mathcal{H}(X)$. A null-homotopy of h is essentially the same as "collaring" the entire family of cobordisms continuously in h. .

Indeed, this is the same as a lift of the map to $C(X)$.

Such a collaring is essentially the same as picking functions with no-critical points.

$\Omega\,\mathcal{H}(X) \simeq \mathcal{P}(X)$ (function version)

Let's elaborate on the last statements. Let $\mathcal{H}'(X)$ denote the space of pairs (W, f) where $W \in \mathcal{H}(X)$ and $f : W \to I$ is a smooth function such that:

- it is the projection π_I close to $\partial' X$,
- it sends $\partial_+ W$ to 1 regularly.

As the choice of f is contractible the forgetful map

$$\mathcal{H}'(X) \to \mathcal{H}(X)$$

is a homotopy equivalence. The subspace $C'(X) \subset \mathcal{H}'(X)$ where f has no critical points is contractible. The composition $C'(X) \to \mathcal{H}(X)$ is a Serre-fibration with fiber $\mathcal{P}(X)$.

$\mathcal{H}'(X)$ is convenient since the homotopy equivalence $\Omega\,\mathcal{H}(X) \to \mathcal{P}(X)$ can be made more explicit.

Indeed, $\Omega\,\mathcal{H}(X) \simeq \Omega\,\mathcal{H}'(X) \simeq \Omega'\,\mathcal{H}'(X)$ where the latter space denotes paths in $\mathcal{H}'(X)$ starting and ending in $C'(X)$.

The homotopy equivalence $\mathcal{P}(X) \to \Omega'\,\mathcal{H}'(X)$ can be represented simply by keeping the h-cobordism constantly trivial $(X \times I)$ and take the path given by convexly interpolating to π_I.

The end of the proof

The part about the doubling being constant on embedded Lagrangian can now be rephrased. Indeed, the doubling creates a manifold, but also a function on the part where the double takes the values $[\varepsilon, \infty)$ which for embedded Lagrangians has no critical points.

This essentially means that the map creates paths in $\Omega' \mathcal{H}'(M \times I \times S^{2k-1} \times \mathbb{R}^{2k})$, and here it is easy to "recognize" the image of $\mathcal{P}(M)$. This makes it possible to recognize the composition

$$\mathcal{P}(M) \to \mathcal{L}eg(M \times I) \to \Omega' \mathcal{H}'(M \times I \times S^{2k-1} \times I^{2k})$$

as a stabilization (the sphere can be handled).

Thoughts:

- The Legendrian dgas are trivial, but one should be able to define a theory of parametrized dga's (with bases) to extract a lot of the same information. However, the map

$$K(\mathbb{S}) \to K(\mathbb{Z})$$

is a rational equivalence, but not an equivalence - and the dgas defined over $\mathbb{Z}$ would "only" see the image of this.