Symplectic Landau-Ginzburg models and their Fukaya categories

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Plan:
1. Symplectic geometry of Landau-Ginzburg models
2. Fukaya categories, examples
3. Functors & homological mirror symmetry

(based on ideas of Seidel, Abouzaid, Ganatra, Sylvan, Hanlon, Jeffs, ...).

essentially the same picture, in a different language: Nadler, Ganatra-Pardon-Shende, Gammage, ...
and also: Kontsevich, Soibelman, Kapranov, ...

⚠ many of the statements below are only proved in settings more specific than stated.

- **Symplectic Landau-Ginzburg model** := $W : Y \to \mathbb{C}$ with $\begin{cases} (Y,\omega) \text{ symplectic, convex at infinity} \\ F_t = W^{-1}(t) \text{ are symplectic submanifolds} \\ \quad \text{(outside of critical locus)} \end{cases}$

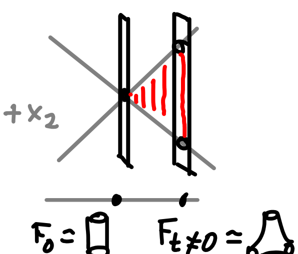
for example Y Kähler (complete) (quasiprojective), W holomorphic (regular function)

- Symplectic parallel transport along horizontal distribution $\mathcal{H} = (\ker dW)^{\perp\omega} = \mathrm{span}(X_{\mathrm{Re}\,W}, X_{\mathrm{Im}\,W})$ over a path $\gamma : [0,1] \to \mathbb{C}$ avoiding critval(W) gives symplectomorphisms $F_{\gamma(0)} \xrightarrow{\sim} F_{\gamma(1)}$
- transporting a Lagrangian $\ell \subset F_{\gamma(0)}$ gives a <u>fibered Lagrangian</u> $L \subset Y$ over γ.
- $\ell \subset F_{t_0}$ is a Lagrangian <u>vanishing cycle</u> for path $\gamma : t_0 \rightsquigarrow$ critval if parallel transport collapses ℓ entirely into crit(W); the fibered Lagrangian is then called a <u>thimble</u>.

<u>Ex</u>: <u>Lefschetz fibrations</u> : crit(W) isolated nondegenerate (local model: $\mathbb{C}^n \xrightarrow{\sum z_i^2} \mathbb{C}$) sing. fibers have ordinary double points; vanishing cycle = Lagrangian $S^{n-1} \subset F$.

<u>Remark</u>: • we include "critical points at ∞" in critval(W), e.g. $(\mathbb{C}^*)^2$
$\downarrow W = x_1 + x_2$
$\mathbb{C}$

• assume critval(W) finite
but crit(W) not assumed isolated, or even proper.



The objects of the Fukaya category $\mathcal{F}(Y,W)$ are admissible Lagrangians

i.e. properly embedded (or immersed) $L \hookrightarrow Y$ disjoint from the stop $Y_{-\infty} = \{\mathrm{Re}\, W \ll 0\}$

(or other top. equivt. subset near $W^{-1}(-\infty)$).

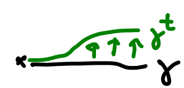
- with good control over holomorphic curves (maximum principle at ∂)
- unobstructed (no holom. discs with ∂ in L, or cancel by bounding cochain)
- equipped with spin structure, local system, grading, ...

Morphisms: $\hom(L,L') = \varinjlim CF(L^t, L')$, L^t = push L by a positive Hamiltonian

isotopy: $\begin{cases} \to \text{ increase } \arg(W) \text{ on ends of } L, \text{ without crossing the stop.} \\ \to \text{ if } L \text{ isn't fiberwise proper, fiberwise wrapping.} \end{cases}$

There are many competing definitions – 3 main flavors:

(1) (Seidel, Abouzaid, ...): assume L is fibered & fiberwise proper outside a compact subset

(e.g. thimbles in Lefschetz fibrations: L^t fibers over γ^t) 

(Seidel)

(This suffices if the fibration is loc. trivial at infinity,
then fiberwise proper objects generate and fiberwise wrapping isn't needed).

(2) When Y is exact Liouville (eg. affine variety), can use Sylvan's partially wrapped Fukaya category or view $Y \setminus Y_{-\infty}$ as a sector (Ganatra-Pardon-Shende)

ie: assume L is conical at infinity & modify the Hamiltonian perturb^{ns} of wrapped Floer theory to avoid crossing the stop.

- This is the most versatile, but the non-exact case hasn't been developed yet
- Note: we don't consider arbitrary sectors - the stops of LG models are swappable
- Wrapping is usually NOT consistent with the fibration W: L^t not necess. fibered at ∞.

(3) When Y is toric, can use monomial admissibility (Hanlon, Abouzaid-A.)

ie: consider a finite collection of distinguished functions (eg. toric monomials) z^ν st.
$\{z^\nu\}: Y \to \mathbb{C}^N$ is proper, and cover $Y \setminus \text{compact} = \bigcup U_\nu$ (eg. $U_\nu = \{|z^\nu| >> 1\}$)
+ require L to be admissible, ie. $\arg(z^\nu_{|L \cap U_\nu}) = $ locally constant at ∞.
Hamiltonian perturbations: $L^t =$ increase value of $\arg(z^\nu)$ at ∞.

- This makes the structures we discuss below most apparent, but a good definition for general Y isn't available yet.

★ In all cases, the picture becomes richest if we allow ourselves to split W into a sum of two terms – a "main" term w_0, and an "auxiliary" term w_F defining stops on the fibers of w_0. I.e., we decompose the stop of (Y, W) into two components, and explore the geometry of Y relative to the part of the stop which corresponds to w_0.

We change notation and consider F = fiber of w_0, rather than the whole fiber/stop.

★ Equivalently: consider Landau-Ginzburg models / sectors w/ swappable stops whose fibers are themselves L-G. models / sectors (F, w_F).

Requirement: parallel transport between fibers of w_0 preserves the fiberwise stop & admissibility of Lagrangians in F (eg. preserve $\arg(w_F)$ at $|w_F| \to \infty$).

Expectation: $\mathcal{F}(Y, w_0 + w_F)$ can be defined either directly for $W = w_0 + w_F$ or "in stages", requiring L to be fibered for w_0 and fiberwise admissible in (F, w_F), yielding equivalent categories.

This is what monomial admissibility achieves – in toric case so far.

5)

The fiberwise wrapped Fukaya category of $(Y, w_0 + w_F)$ (Y toric, w_0 monomial) (Abouzaid-A.)

<u>Objects</u>: properly embedded Lagrangians $L \subset Y$ (+ extra data: spin str., grading, ...) which are unobstructed + monomially admissible: (cf. A. Hanlon's thesis)

1) for $|w_0| \gg 1$, $\arg(w_{0|L})$ is loc. constant $\in (-\frac{\pi}{2}, \frac{\pi}{2})$ (ie. $w_{0|L} \in$ union of radial arcs)

2) inside the fibers F_t of w_0 (again toric!), impose admissibility w.r.t. a collection of toric monomials z^ν (incl. all terms in w_F), ie. $\arg(z^\nu) =$ loc. constant over subsets U_ν.

<u>Note</u>:
- monomial admissibility gives control over discs in Floer products via maximum principle
- use a specific toric Kähler form for which $\{\log w_0, \log z^\nu\}$ Poisson-commute in U_ν.

L admissible $\rightsquigarrow$ flow L^t (Ham. isotopic to L; admissible)

The flow increases the values of $\arg(w_0)$ and $\arg(z^\nu)_{|U_\nu}$ at ∞.

(within $\arg \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for w_0 and terms of w_F: <u>STOP</u> at $-\infty$; else $\arg \uparrow \infty$: <u>WRAP</u>)

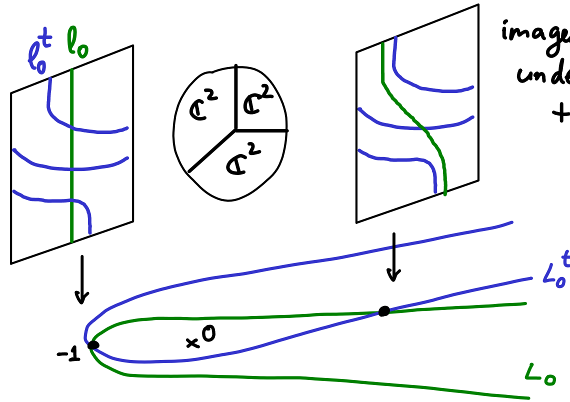
<u>Define</u> $\hom(L_0, L_1) := \varinjlim_{t\to\infty} CF^*(L_0^t, L_1)$ under natural continuation maps.

<u>Conj</u>: can make a similar definition outside of toric setting & the resulting "two-stage category" (whose objects are fibered Lagrangians wrt w_0, rather than $w_0 + w_F$), agrees with other versions.

Example: $(\mathbb{C}^3, w_0 = -z_1 z_2 z_3)$ (mirror of [pair of pants] $= \{(x_1,x_2) \in (\mathbb{C}^*)^2 \mid 1 + x_1 + x_2 = 0\}$).

L_0 = parallel transport $\ell_0 = (\mathbb{R}_+)^2 \subset (\mathbb{C}^*)^2 \simeq w_0^{-1}(-1)$ along U-shaped arc.

$w_0^{-1}(-1) \simeq (\mathbb{C}^*)^2$
$\cup$
$\ell_0 \simeq (\mathbb{R}_+)^2$



images of ℓ_0 under monodromy + wrapping Ham. $\quad \arg(z_i) = f_i(\log|z|)$

$\arg(z_i) = 0$ wherever $|z_i| \gg \min_j |z_j|$

$\downarrow$ wrapping

t (here $t \to +\infty$)

$$\hom(L_0, L_0) \simeq \underset{\mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}]}{\underset{\wr\wr}{CW^*(\ell_0,\ell_0)}} \oplus CW^*(\ell_0,\ell_0)[-1]$$

∂ = multiplication by $1 + x_1 + x_2$ (Abouzaid-A.)

$$H^* \hom(L_0, L_0) \underset{\text{(ring iso.)}}{\simeq} \mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}]/(1 + x_1 + x_2) \simeq \hom(\mathcal{O}, \mathcal{O}) \text{ in } D^b\mathrm{Coh}(\text{[pair of pants]}) \quad \checkmark$$

Example: $(\mathbb{C}^3, w_0 + w_F = -z_1 z_2 z_3 + q(z_1 + z_2 + z_3))$; $(F_t, w_F) \simeq ((\mathbb{C}^*)^2, z_1 + z_2 + \frac{t}{z_1 z_2})$

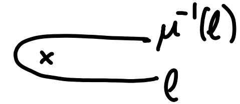
then $\mathcal{F}(F_t, w_F) \simeq D^b(\mathbb{P}^2)$ and $\mathcal{F}(\mathbb{C}^3, w_0 + w_F) \simeq D^b(\{(x_0 : x_1 : x_2) \mid x_0 + x_1 + x_2 = 0\})$ (ie. $\mathbb{P}^1 \subset \mathbb{P}^2$).

Abouzaid-A.: this approach leads to HMS for hypersurfaces (& complete intersections) in $(\mathbb{C}^*)^n$ & toric var's.

The Fukaya categories of (Y, w_0+w_F) and $(F=w_0^{-1}(t), w_F)$ are related by functors

monodromy of w_0 around origin $\quad \mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma \quad$ wrap once past the stop $w_0 \to -\infty$. (spherical functor)

$\cup \ell$ = parallel transport $\ell \subset F$ (admissible) along U-shape



$=: \cup \ell$

$\cap L$ = ends of $L \subset Y$ at $w_0 \to \infty$ (actually $\in \mathrm{Tw}\,\mathcal{F}(F, w_F)$ if $w_0|_L$ has more than one end).

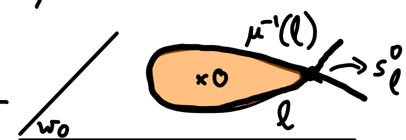
+ exact triangle of functors on $\mathcal{F}(F, w_F)$:

$$\boxed{\begin{tikzcd} \mu^{-1} \arrow[rr, "s"] & & id \arrow[dl] \\ & \cap\cup \arrow[ul] & \end{tikzcd}}$$

(Abouzaid-Ganatra, Sylvan).

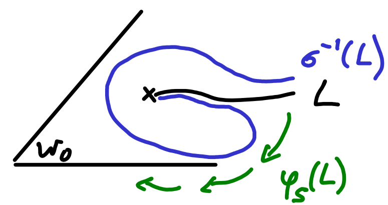
where s = section-counting natural transformation from μ^{-1} to id [Seidel]

$\forall \ell \subset F$, $\quad s_\ell^0 \in CF^0(\mu^{-1}(\ell), \ell)$ counts holom. sections of w_0 over



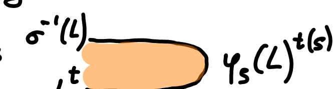
+ exact triangle of functors on $\mathcal{F}(Y, w_0+w_F)$:

$$\boxed{\begin{tikzcd} id \arrow[rr, "\kappa"] & & \sigma^{-1} \arrow[dl] \\ & \cup\cap \arrow[ul] & \end{tikzcd}}$$



κ = continuation element for non-positive Hamiltonian isotopy $id \longrightarrow \sigma^{-1}$

i.e. κ_L^0 counts holom. discs



monodromy of w_0 around origin

$$\mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma$$

wrap once past the stop $w_0 \to -\infty$. (spherical functor)

$\cup\ell$ = parallel transport $\ell \subset F$ (admissible) along U-shape

$=: \cup\ell$

$\cap L$ = ends of $L \subset Y$ at $w_0 \to \infty$ (actually $\in Tw\, \mathcal{F}(F, w_F)$ if $w_0|_L$ has more than one end).

On $\mathcal{F}(F, w_F)$, $\cap\cup \simeq \mathrm{Cone}(\mu^{-1} \xrightarrow{s} \mathrm{id})$

On $\mathcal{F}(Y, w_0+w_F)$, $\cup\cap \simeq \mathrm{Cone}(\mathrm{id} \xrightarrow{\kappa} \sigma^{-1})$

HMS for log CY/Fano (X, D_0), $D_0 + D' = -K_X$

$\rightsquigarrow$ $(Y, W = w_0 + w_F)$ mirror to X

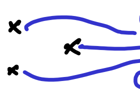
$(F = w_0^{-1}(t), w_F)$ mirror to D_0

$$\begin{tikzcd} \mu \circlearrowright \mathcal{F}(F, w_F) \arrow[r, shift left, "\cup"] \arrow[d, phantom, "\wr"] & \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma \arrow[l, shift left, "\cap"] \arrow[d, phantom, "\wr"] \\ \mathrm{Perf}(D_0) \arrow[r, shift left, "i_*"] & \mathrm{Perf}(X) \arrow[l, shift left, "i^*"] \end{tikzcd}$$

HMS for D_0 | HMS for X

Ex: $\mathcal{F}\left((\mathbb{C}^*)^2, z_1 + z_2 + \frac{q}{z_1 z_2}\right) \rightleftarrows \mathcal{F}\Big($  $\Big)$

mirror to $\mathbb{P}^2$



mirror to $x_0 x_1 x_2 = 0$

HMS for hypersurface $H \subset V$ [AAK]

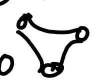
$\rightsquigarrow$ $(Y, W = w_0 + w_F)$ mirror to H

$(F = w_0^{-1}(t), w_F)$ mirror to V

$$\begin{tikzcd} \mu \circlearrowright \mathcal{F}(F, w_F) \arrow[r, shift left, "\cup"] \arrow[d, phantom, "\wr"] & \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma \arrow[l, shift left, "\cap"] \arrow[d, phantom, "\wr"] \\ \mathrm{Perf}(V) \arrow[r, shift left, "i^*"] & \mathrm{Perf}(H) \arrow[l, shift left, "i_*"] \end{tikzcd}$$

HMS for V | HMS for H

Ex: $\mathcal{F}(\mathbb{C}^3, -xyz) \rightleftarrows \mathcal{W}((\mathbb{C}^*)^2)$

mirror to  $1 + x_1 + x_2 = 0$

mirror to $(\mathbb{C}^*)^2$

• In this language, the above calculation for mirror (Y, w_0+w_F) of hypersurface $H\subset V$ is:

$$\hom_Y(\cup\ell,\cup\ell') \simeq \hom_F(\ell, \cap\cup\ell') \simeq \mathrm{Cone}\big(\hom_F(\ell,\mu^{-1}(\ell')) \xrightarrow{S} \hom_F(\ell,\ell')\big)$$

& homological mirror symmetry is proved by matching this with $D^bCoh(H) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D^bCoh(V)$

$$\hom_H(i^*\mathcal{L}, i^*\mathcal{L}') \simeq \hom_V(\mathcal{L}, i_*i^*\mathcal{L}') \simeq \mathrm{Cone}\big(\hom_V(\mathcal{L},\mathcal{L}'\otimes \mathcal{O}(-H)) \xrightarrow{f} \hom_V(\mathcal{L},\mathcal{L}')\big)$$

[Abouzaid-A.].

• The two functors $\cup$, $\cap$ are defined differently, but play symmetric roles in the spherical functor package. Conj: Fiber and total space can be swapped around by stabilization:

$$(Y,\ W = w_0 + w_F),\quad F = w_0^{-1}(t)\ \ (t \gg 0) \ \rightsquigarrow\ \big(\tilde{Y} = Y \times \underset{\substack{\rotatebox{90}{$\in$}\\ z}}{\mathbb{C}},\ \tilde{W} = z(1-t^{-1}w_0) + w_F\big)$$

• "A-model Knörrer periodicity" (M. Jeffs) $\Rightarrow$ $\mathcal{F}(Y\times\mathbb{C}, \tilde{W}) \simeq \mathcal{F}(F, w_F)$

• the levels of $\tilde{w}_0 := z$ are $\simeq (Y, w_0+w_F)$ by considering thimbles for $z(1-t^{-1}w_0)$ (Morse-Bott along $F\times\{0\}$)

• Jeffs' result is for splitting $\tilde{W} = \underbrace{z(1-t^{-1}w_0)}_{\text{main}} + w_F$, while $\tilde{F} \simeq (Y, w_0+w_F)$ is for $\tilde{W} = \underbrace{z}_{\text{main}} + \underbrace{(-t^{-1}zw_0 + w_F)}_{\text{fiberwise}}$.

• $(\widetilde{\cap,\cup})$ not quite $\sim (\cup,\cap)$: left vs. right adjoint, vs.

Localizations / quotients

monodromy of w_0 around origin $\quad \mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0 + w_F) \circlearrowleft \sigma \quad$ wrap once past the stop $w_0 \to -\infty$. $\quad$ (spherical functor)

$\cup \ell$ = parallel transport $\ell \subset F$ (admissible) along U-shape, $\quad \cap L$ = ends of $L \subset Y$ at $w_0 \to \infty$

On $\mathcal{F}(F, w_F)$, $\cap\cup \simeq \mathrm{Cone}(\mu^{-1} \xrightarrow{S} \mathrm{id})$ $\qquad$ On $\mathcal{F}(Y, w_0+w_F)$, $\cup\cap \simeq \mathrm{Cone}(\mathrm{id} \xrightarrow{K} \sigma^{-1})$

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1 Localizing $\mathcal{F}(Y, w_0 + w_F)$ wrt $\mathrm{id} \xrightarrow{K} \sigma^{-1}$ $\iff$ quotient by $\mathrm{Im}(\cup)$

(Abouzaid-Seidel, Sylvan, GPS) $\iff$ stop removal at w_0

yields $\mathcal{F}(Y, w_F)$ (if no fiberwise stop, $\mathcal{W}(Y)$ – the fully wrapped Fukaya category)

In HMS for log CY / Fano, with fiber mirror to $D_0 \subset X$, $D_0 + D' = -K_X$,

σ^{-1} corresponds to $-\otimes \mathcal{O}(D_0)$ and $\cup$ to i_*; localization gives mirror to $X - D_0$

2 Localizing $\mathcal{F}(F, w_F)$ wrt $\mu^{-1} \xrightarrow{S} \mathrm{id}$ $\iff$ quotient by $\mathrm{Im}(\cap)$ (vanishing cycles)

when $\mathrm{critval}(w_0) = \{0\}$, can be interpreted as the Fukaya cat. of the singular fiber F_0.

Eg. $\mathcal{W}(\{xy=0\} \subset \mathbb{C}^2) = \mathcal{W}(T^*S^1)/\langle S^1 \rangle \simeq \mathrm{Perf}($  $)$, $\mathcal{W}(\{xyz=0\}) \simeq \mathrm{Perf}((\mathbb{C}^*)^2 - \{1 + x_1 + x_2 = 0\})$. 2d pants

Thm (Jeffs): (Knörrer periodicity) $\boxed{\mathcal{W}(F_0) \simeq \mathcal{F}(Y \times \mathbb{C}, z w_0)}$ $\quad$ (expect $\mathcal{F}(F_0, w_F) \simeq \mathcal{F}(Y \times \mathbb{C}, z w_0 + w_F)$)

Families of LG models and monodromy

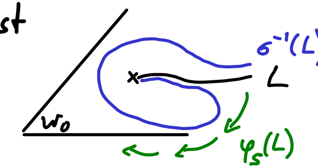
(Y, W_t) s.t. topology of the stop doesn't change with t $\Rightarrow$ $\mathcal{F}(Y, W_t)$ indep^t of t.

For a loop $(W_t)_{t \in S^1}$, get a <u>monodromy functor</u> $\phi \in \mathrm{Auteq}\, \mathcal{F}(Y, W)$.

\+ if loop is "positive", Floer continuation maps give a natural transformation $id \to \phi$.

<u>Main example</u>: $(Y, W = e^{i\theta} w_0 + w_F)$ $\Rightarrow$ the monodromy functor is σ^{-1}, with $id \xrightarrow{K} \sigma^{-1}$.

(wrap once clockwise past the stop of w_0)



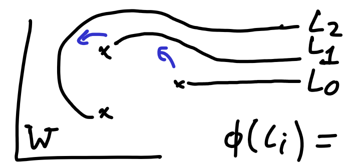
<u>Ex</u> (Hanlon) V toric var. $\longleftrightarrow$ $(\mathbb{C}^*)^n$, $W = \sum q_\nu z^\nu$

toric divisors D_ν

rays of fan $= \nu \in \mathbb{Z}^n$

monodromy for $e^{i\theta} q_\nu$ is $\longleftrightarrow$ $- \otimes \mathcal{O}(D_\nu)$.

<u>Ex</u>: for $\mathbb{P}^n$, $W = z_1 + \dots + z_n + \dfrac{e^{i\theta} q}{z_1 \cdots z_n}$ $\Rightarrow$



$\phi(L_i) = L_{i+1}$.

"Seidel's philosophy in upgraded form": (sectors / LG models instead of symplectic manifolds):

In a fibration $(Y, w_0 + w_F) \xrightarrow{w_0} \mathbb{C}$ whose fibers are LG-models / sectors (F, w_F):

$\to$ monodromy of w_0 takes values in $\mathrm{Aut}(F, w_F) \rightsquigarrow \mathrm{Auteq}\, \mathcal{F}(F, w_F)$ (functors)

$\to$ $w_0|_{\mathbb{C}^*}$ is a cobordism between loops around $0/\infty$ $\rightsquigarrow$ natural transformations