VANISHING ASYMPTOTIC MASLOV INDEX FOR CONFORMAL SYMPLECTIC FLOWS

j.w. A. Florio
& M.-C. ARNAUD

M closed (connex + compact) n-manifold

(T^*M, ω) where $\omega = -d\lambda$ ← Liouville 1-form, $\lambda = pdq$ in local coordinates

A diffeo $\phi : T^*M \circlearrowleft$ is conformally symplectic if $\exists \alpha > 0$ s.t. $\phi^*\omega = \alpha\omega$.

$\left.\begin{array}{l} H : \mathbb{R}\times T^*M \longrightarrow \mathbb{R} \text{ Ham} \\ \beta : \mathbb{R} \longrightarrow \mathbb{R} \ C^0 \end{array}\right) \rightsquigarrow$ conf. Ham v.f X_t^H $i_{X_t^H}\omega = dH_t + \beta(t)\lambda$ $\rightsquigarrow$ conf. Ham. flow $(\phi_{s,t}^H)$ $\phi_t^*\omega = \alpha_t\omega \ \forall t$.

Interest of conformal symplectic dynamics

- celestial mechanics (extended KAM theory, Calleja-Celletti-de la Llave)
- deal with discounted HJ equation (Davini-Fathi-Iturriaga-Zavidovique)
- extended (Aubry-)Mather theory (Sorrentino-Maro)

Why do we look for points of vanishing asymptotic Maslov index?
A motivation from the Tonelli case:

<u>Thm</u>. Let $H : T^*M \to \mathbb{R}$ be a Tonelli Ham, ϕ_t^H the associated flow.
Then

Contreras-Gambaudo-Iturriaga-Paternain: $MI_\infty(x, \phi^H) = 0$ for a.e $x \in T^*M$
$\Longleftrightarrow$ ϕ^H has no conjugated points
$\Longleftrightarrow$ ϕ^H is C^0-integrable ($\exists\, C^0$ foliation made of invariant Lag. graphs)
Arcostanzo-Arnaud-Bolle-Zavidovique

(In fact for Tonelli dynamics, Maslov index vanishes along minimizing orbits)

A large set of points with vanishing Maslov index points is an indicator of the dynamics being "not too wild". We spot a few ones:

Theorem 1 (AFR) Let $H : \mathbb{R} \times T^*M \to \mathbb{R}$ be Tonelli and $\beta : \mathbb{R} \to \mathbb{R}$ be C^0, if ϕ^H is the associated conf. Ham. flow, $\forall\, \mathcal{L} \subset T^*M$ Lag. subm. Ham. isotopic to a Lag. graph, there exists $x \in \mathcal{L}$ s.t. $\boxed{MI_\infty(x, \phi^H) = 0.}$

(Rk: this extends a result of Florio's PHD for twist maps on the annulus)

(I) Asymptotic Maslov index

Maslov index was first introduced for smooth (Arnold) paths of Lagrangian subspaces $t \mapsto L_t$ in a symplectic linear space ($\simeq (\mathbb{R}^n \times \mathbb{R}^n, \omega)$) w.r.t. another Lagrangian subspace V (that could be $\{0\} \times \mathbb{R}^n$ or $\mathbb{R}^n \times \{0\}$)

Denoted by $MI(L_t, V)_{t \in [0,T]}$, it is the algebraic number of intersections with the Maslov singular cycle, hence counts more or less the # of times when the Lag. is not transverse to V.

In this talk V will be the vertical foliation all along.

Prop: • $\tau \mapsto (L_t^\tau)_{t \in S^1}$ is an isotopy of Lag. loops, then $\tau \mapsto MI(L_t^\tau)_{t \in S^1}$ is constant.

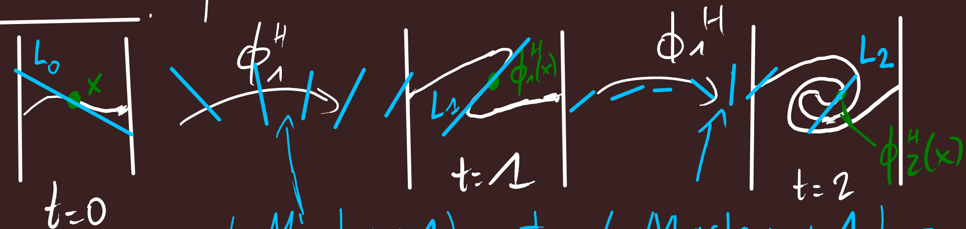
• if η is a form on T^*M, and $\gamma(t) \in M$, $\forall t \in I$, $MI(T_{\gamma(t)}\, gr\, \eta)_{t \in I} = 0$.

DYNAMICAL MASLOV INDEX

if $(\phi_t)_{t\in I}$ is an isotopy of diffeos preserving Lag. submflds (for ex. (conf.)symp. diffeos)
and (x,L) is a Lagrangian subspace at point x,

$$DMI(x, L, \phi_t)_{t\in I} := MI(D_x\phi_t(L))_{t\in I}$$

Toy model: pendulum



$(\text{Maslov} + 1) + (\text{Maslov} + 1) \Longrightarrow MI(L_t, \text{Vert.})_{t\in[0,2]} = 2$

Prop: $\exists\, C(n),\ \forall(\phi_t)_{t\in I},\ \forall (x,L_1),(x,L_2)$ Lag. spaces,

$$|DMI(x, L_1, \phi_t)_{t\in I} - DMI(x, L_2, \phi_t)_{t\in I}| \leq C(n)$$

Hence if it exists, $\lim\limits_{t\to+\infty} \dfrac{DMI(x,L,\phi_s)_{s\in[0,t]}}{t}$ does not depend on L,
and is called ASYMPTOTIC MASLOV INDEX for the flow (ϕ_t) at the point x:

$$MI_\infty(x, \phi_t)_{t\geq 0} := \lim_{t\to+\infty} \frac{DMI(x,L,\phi_s)_{s\in[0,t]}}{t} = \lim_{t\to+\infty} \frac{MI(D_x\phi_s(L))_{s\in[0,t]}}{t}$$

(Ruelle, 85)

$\longrightarrow$ can be extended to invariant measures on T^*M thanks to the independance wrt to L.

$\longrightarrow$ see Contreras - Gambaudo - Iturriaga - Paternain
Abbondandolo - Figalli
Bialy - Polterovitch

and many others...

II. MASLOV INDEX ALONG A GRAPH SELECTOR

Theorem: if $\mathcal{L} = \{(q, \frac{\partial S}{\partial q}) \mid \frac{\partial S}{\partial \xi} = 0\} \subset T^*M$ is a Lag. submfld generated by a GFQI $S: M \times \mathbb{R}^k \to \mathbb{R}$,

$\exists\, u : M \to \mathbb{R}$ Lipschitz function and $U \subset M$ an open set of full measure s.t.

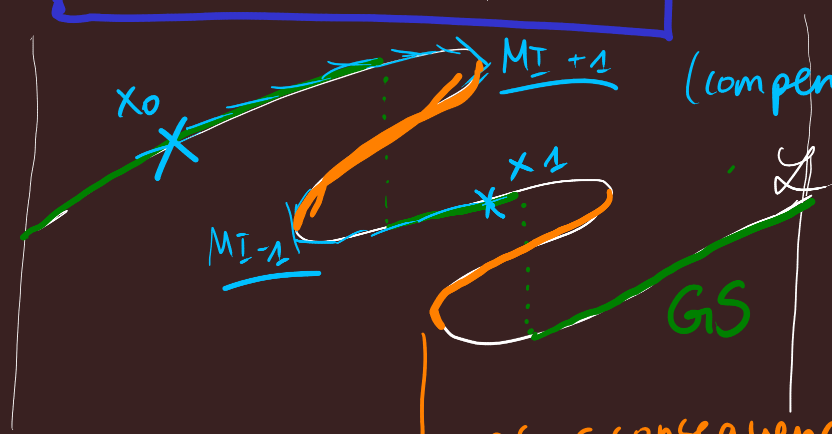
- $u|_U$ is C^1 and $GS := \mathrm{gr}\, du|_U \subset \mathcal{L}$ "graph selector"
- $\forall q \in U$, $S(q, \cdot)$ is a Morse function with critical points having distinct critical values.

Rk: in particular $\forall q \in U$, $\exists! \xi$ s.t. $\begin{cases} \frac{\partial S}{\partial q}(q,\xi) = du(q) \\ \frac{\partial S}{\partial \xi}(q,\xi) = 0 \end{cases}$, and $u(q) = S(q,\xi)$.

LEMMA with these notations, if $x_0, x_1 \in GS$ and $\gamma(t) \in \mathcal{L}$ is a path joining them,

then $\boxed{MI(T_{\gamma(t)}\mathcal{L})_{t \in [0,1]} = 0}$

Example.



↳ as a consequence orange part cannot belong to the graph selector

Proof with two ingredients

* CLAIM (almost Viterbo, 87): in this situation (since $T^*_{x_i}\mathcal{L} \pitchfork V$),

$$MI(T_{\gamma(t)}\mathcal{L}) = \frac{\partial^2 S}{\partial \xi^2}(q_1,\xi_1) - \frac{\partial^2 S}{\partial \xi^2}(q_0,\xi_0),$$

where $(q_i,\xi_i) \in \underline{\underline{U}} \times \mathbb{R}^k$ are uniquely defined by $\begin{cases} x_i = \left(q_i, \frac{\partial S}{\partial q}(q_i,\xi_i)\right), \\ \frac{\partial S}{\partial \xi}(q_i,\xi_i) = 0 \end{cases}$ $i \in \{0,1\}$.

$*$ if $S_q := S(q,\cdot)$, $S_q^c := \{S_q \leq c\}$, m is the index of the qf S_q at ∞, we have since $S_q^{-\infty} \subset S_q^{u(q)-\varepsilon} \subset S_q^{u(q)+\varepsilon} \subset S_q^{+\infty}$ an exact sequence:

$$\begin{tikzcd} H^m(S_q^{u(q)+\varepsilon}, S_q^{u(q)-\varepsilon}) \arrow[r, "j_1^*"] & H^m(S_q^{u(q)+\varepsilon}, S_q^{-\infty}) \arrow[r, "j_2^*"] \arrow[dr, phantom, "\circlearrowleft"] & H^m(S_q^{u(q)-\varepsilon}, S_q^{-\infty}) \\ & & H^m(S_q^{+\infty}, S_q^{-\infty}) \arrow[ul, "i^*_{u(q)+\varepsilon} \neq 0"] \arrow[u, "i^*_{u(q)-\varepsilon} = 0"'] \end{tikzcd}$$

and furthermore $\longrightarrow$

where j_1, j_2 are the natural relative identities and i_c^* the usual morphism defining minmax:

$$u(q) = \inf\{c \in \mathbb{R} \,/\, i_c^* \neq 0\}.$$

By commutativity of the right part

$$\{0\} \neq \operatorname{Im} i^*_{u(q)+\varepsilon} \subset \ker j_2^* \underset{(\text{exact})}{=} \operatorname{Im} j_1^*$$

and as a consequence

$$H^m(S_q^{u(q)-\varepsilon}, S_q^{u(q)+\varepsilon}) \neq 0,$$ hence since S_q is Morse $(q \in \underline{\underline{U}})$, $\frac{\partial^2 S}{\partial \xi^2}(q,\xi)$ has index m. (Morse theory).

III) APPLICATIONS TO CONFORMAL SYMPLECTIC DYNAMICS

no convexity needed!

Theorem 2 (AFR) Let $\mathcal{L} \subset T^*M$ be a Lag. graph and (ϕ_s) be an isotopy of confo. symp. diffeos with $\phi_0 = id_{T^*M}$

Then there exist

- $u_t : M \longrightarrow \mathbb{R}$ Lipschitz function
- η_t 1-form on M
- U_t an open set of full measure on which u_t is diff.

s.t

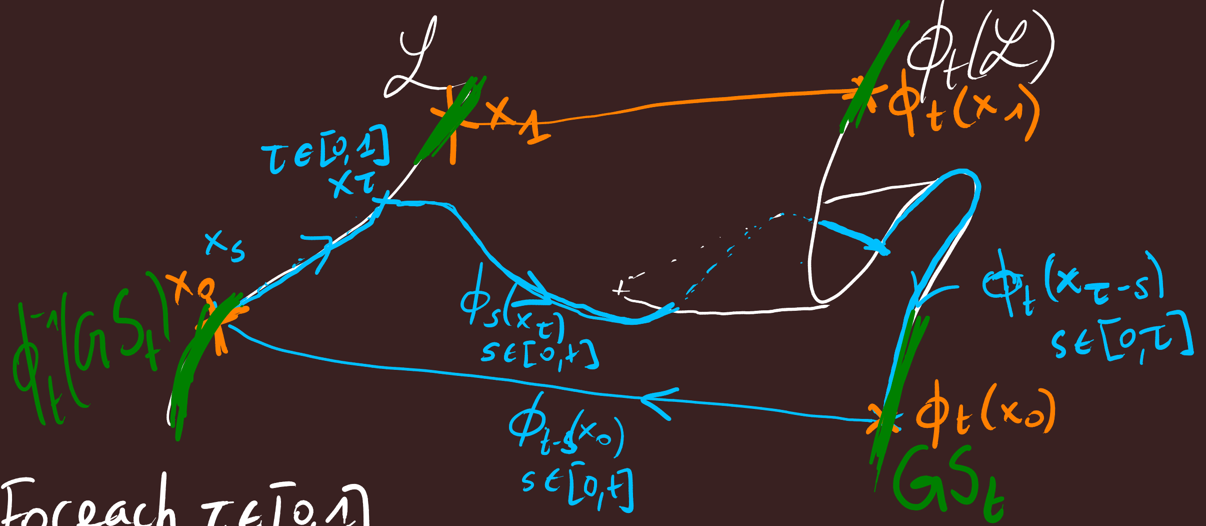
- $GS_t := gr(\eta_t + du_t|_{U_t}) \subset \phi_t(\mathcal{L})$ "graph selector"
- $\forall x \in \phi_t^{-1}(GS_t)$, $DMI(T_x\mathcal{L}, \phi_s)_{s\in[0,t]} = 0$.

Sketch of proof: ① existence of GS_t. (...)

② The **LEMMA** still holds for $GS_t = gr(\eta_t + du_t|_{U_t})$ (modification up to a vertical transformation)

③ $\eta_t := DMI(T_x\mathcal{L}, \phi_s)_{s\in[0,t]}$ does not depend on the choice of $x \in \phi_t^{-1}(GS_t) \subset \mathcal{L}$:

Let $(x_0, x_1) \in \phi_t^{-1}(GS_t)$, $x_\cdot$ a path in $\mathcal{L}$ joining them.



For each $\tau \in [0,1]$
we follow with the indicated arrows the blue path, with Lag. subspace corresponding to the tangent space of $\phi_\cdot(\mathcal{L})$.

$\Longrightarrow$ isotopy of loops $(L^\tau_s)_{s \in S^1}$, $\tau \in [0,1]$.

Hence $\tau \mapsto MI(L^\tau_s)_{s \in S^1}$ is constant,
and since $MI(L^0_s)_{s \in S^1} = 0$,

$$0 = MI(L^1_s)_{s \in S^1} = \cancel{MI(T_{x_s}\mathcal{L})_{s\in[0,1]}} \quad \text{because } \mathcal{L} \text{ is a graph}$$

$$+ DMI(T_{x_1}\mathcal{L}, \phi_s)_{s\in[0,t]} - DMI(T_{x_0}\mathcal{L}, \phi_s)_{s\in[0,t]}$$

$$+ \cancel{MI(T_{\phi_t(x_{1-s})}\phi_t(\mathcal{L}))_{s\in[0,1]}} \quad \text{because } \phi_t(x_1), \phi_t(x_0) \in GS_t, \text{ by the } \underline{\text{LEMMA}}$$

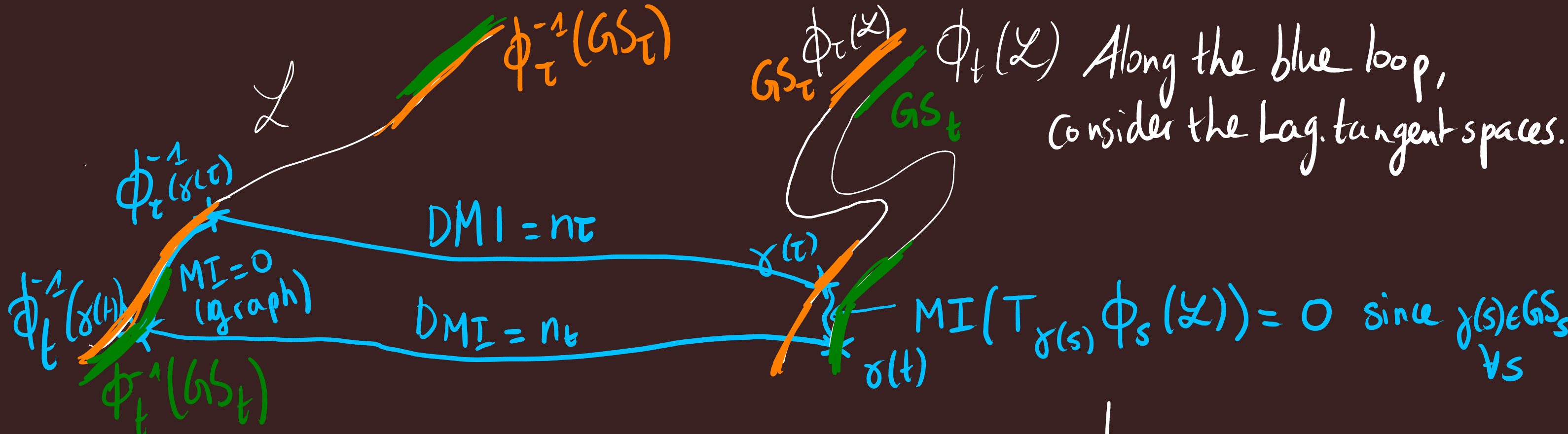
it follows:

$$DMI(T_{x_1}\mathcal{L}, \phi_s)_{s\in[0,t]} = DMI(T_{x_2}\mathcal{L}, \phi_s)_{s\in[0,t]}.$$

④ $t \mapsto n_t$ is locally constant:
* by continuity of $t \mapsto u_t(q)$ and definition of U_t,

$(\ldots)$, $\forall t$, $\exists \gamma :]t-\delta, t+\delta[\longrightarrow T^*M$ a C^0 path
s.t. $\gamma(s) \subset GS_s$ $\forall s \in]t-\delta, t+\delta[$.

* let $\tau \in]t-\delta, t+\delta[$ and show that $n_\tau = n_t$:



* since $n_0 = 0$, the proof is over!

Proof of Thm 1: it relies on Thm 2 composed with the two following facts.

Fact 1. A Tonelli confo. symp. flow "twists the vertical" (to be defined in local charts) and consequently $\boxed{DMI(L, \phi^H)_I \leq 0}$ $\forall I \subset \mathbb{R}$, $\forall L$ Lag. subspace.

Fact 2. There exists a C^0 analogous to MI, called angular Maslov index (αMI), s.t:

- $|\alpha MI - MI| \leq n$
- $(t,x) \longmapsto D\alpha MI(T_x\mathcal{L}, \phi_s)_{s\in[0,t]} := \alpha MI(D_x\phi_s(T_x\mathcal{L}))_{s\in[0,t]}$ is continuous.

Thanks!