Curiosities and counterexamples in smooth convex optimization

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Learning and Optimization in Luminy

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Forewords:

We work in $\mathbb{R}^2$ (negative results).

$k \geq 2$ is an arbitrary smoothness index.

Functions on $\mathbb{R}^2$: convex, compact sublevel sets, C^k.

Exact line search gradient descent

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$$x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k).$$

$(x_k)_{k \in \mathbb{N}}$ converges toward a solution (Féjer monotonicity: for any solution x^*, $\|x_k - x^*\|$ is monotone).

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Contribution: does not converge in general,

- dimension 2
- f convex, C^k, coercive
- unique minimum along the trajectory (selection in the argmin).

Constructive proof

Construct f, C^k, with oscillating gradients around its minimum.

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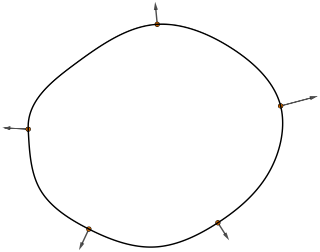
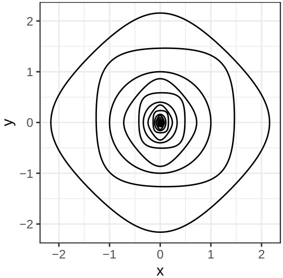
Main ideas:

- Gradient orthogonal to level sets.
- Interpolate a sequence of level sets.





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- **Kurdyka-Łojasiewicz:** convex function with no error bound (Bolte, Daniilidis, Ley, Mazet 2009).
- **Tikhonov regularization:** infinite length regularization path (Torralba 1996).
- **Newton flow:** non convergence ($\nabla^2 f$ PD outside $\arg\min f$)

$$\dot{x} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

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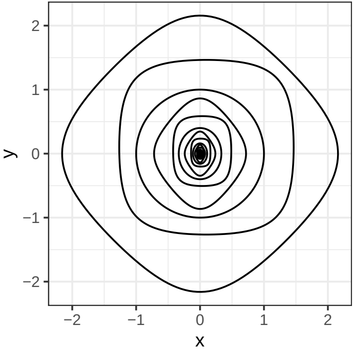
- **Frank-Wolfe algorithm:** non convergence (with C. Combettes).

$$x_{k+1} = (1 - \gamma_k) x_k + \gamma_k v_k, \qquad\qquad v_k \in \arg\min_{v \in \mathcal{C}} \langle \nabla f(x_k), v \rangle$$

Plan

Convex interpolation problem: Let $(T_i)_{i\in-\mathbb{N}}$ be convex compact with $T_{i-1} \subset \mathrm{int}(T_i) \neq \emptyset$ for $i \in -\mathbb{N}$. Construct f convex with T_i as a sublevel for all $i \in -\mathbb{N}$.



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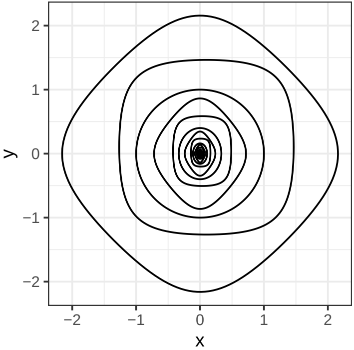

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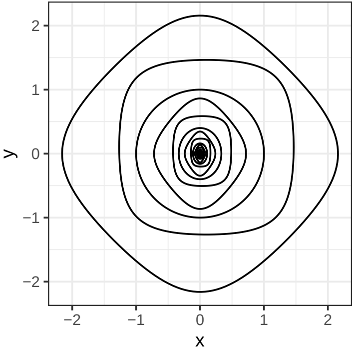

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Kannai-Torralba (77, 96): f exists (convex continuous).

Affine sublevel set interpolation with Minkowski sums

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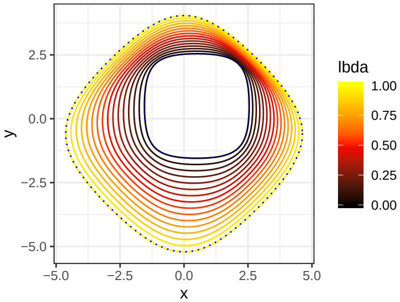

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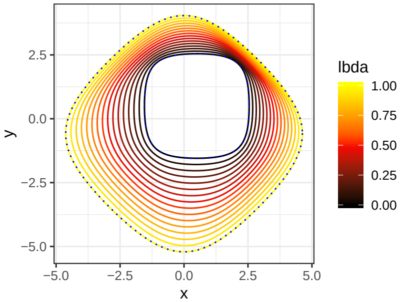

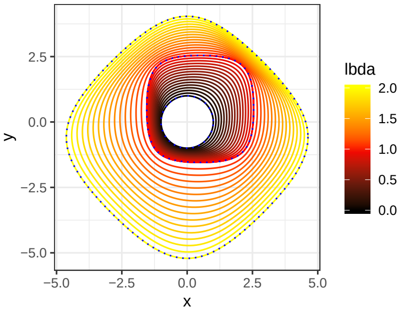

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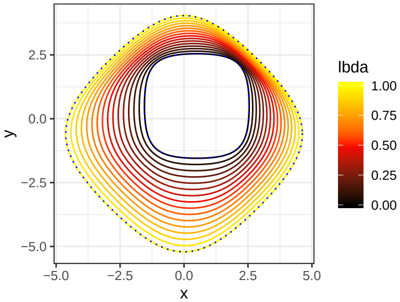

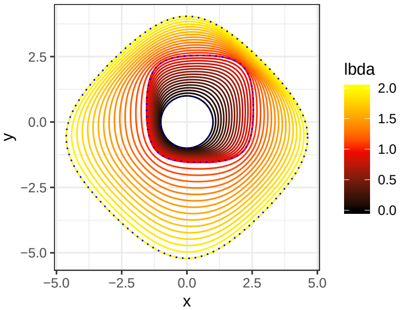

Kannai-Torralba (77, 96):

- Connect Minkowski interpolation.
- Choose function values to enforce convexity.

Add smoothness

Smooth convex interpolation problem:
Let $(T_i)_{i \in -\mathbb{N}}$ be convex compact with C^k boundary, such that $T_{i-1} \subset \mathrm{int}(T_i) \neq \emptyset$ for $i \in -\mathbb{N}$. Construct f convex C^k, such that, for all $i \in -\mathbb{N}$, T_i is a sublevel set of f.

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Minkowski addition and smoothness?

- There are $A, B \subset \mathbb{R}^p$ convex with C^∞ boundary, such that $A + B$ does not have C^2 boundary (Kiselman 1986)

Plan

Positive curvature

Schneider: Convex Bodies: The Brunn-Minkowski Theory. Section 2.5.

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Normal parametrization: inverses of n_A, $(A \in C^2_+)$. $c_A \colon S_{p-1} \to \partial A$.

$$c_A \colon n \mapsto \mathrm{argmax}_{x \in A} \langle n, x \rangle .$$

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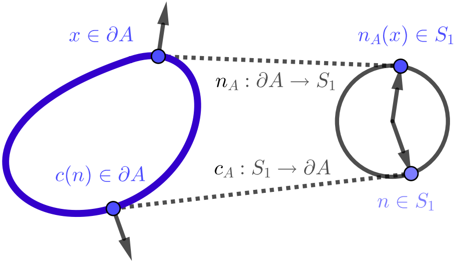

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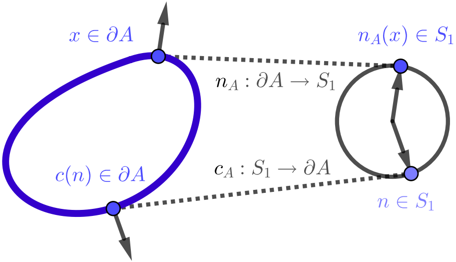
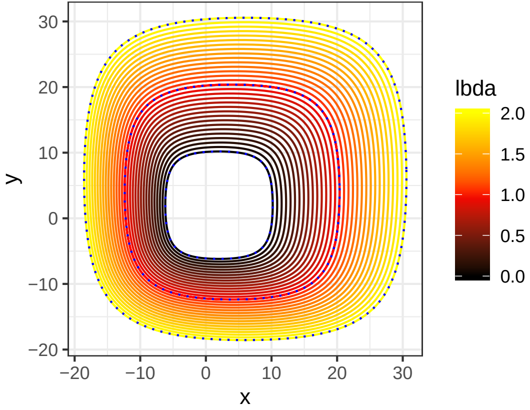


Get smoothness back: $A, B \in C^2_+$, C^k boundary, then $A + B \in C^2_+$ with C^k boundary

$$c_{A+B} = c_A + c_B$$

Smooth interpolation an easy example

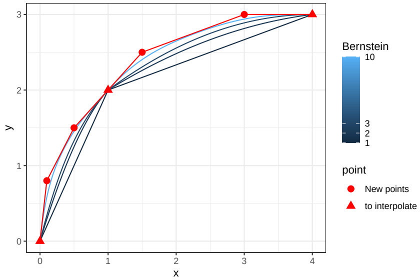
Homothetic sublevel sets.



Idea: Enforce a similar behavior for orders $0, 1, \ldots, k$.

Nonlinear interpolation: idea from Bézier curves and Bernstein interpolation

Preserves monotonicity, concavity, control derivatives up to order k at endpoints



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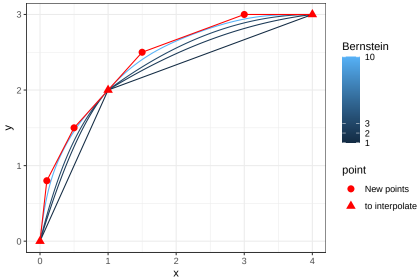

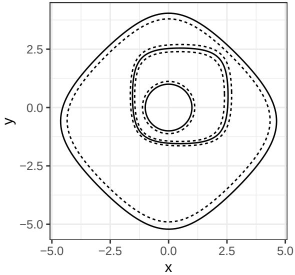

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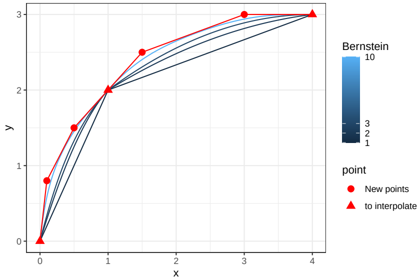

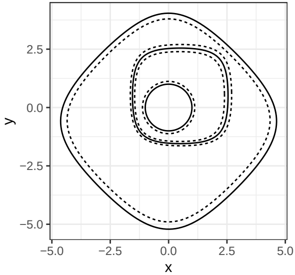

Conic interpolation:

$$T_\lambda = a(\lambda)\, T_0 + b(\lambda)\, T_1$$

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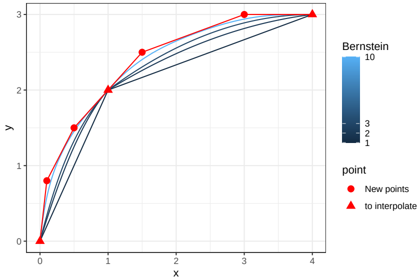

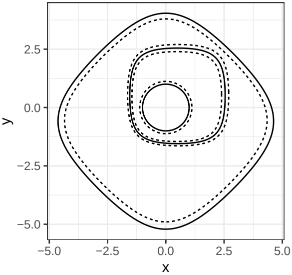

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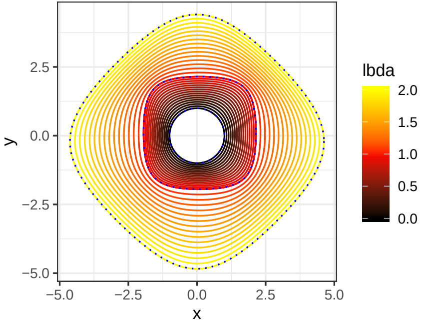
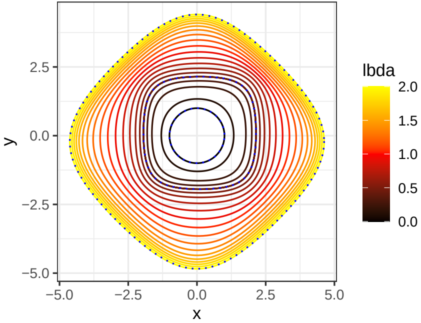
Explicit derivatives, C^k junction around boundaries:

- "normal" direction: C^1 control, variations of order $2, \dots, k$ vanish.
- "tangent" direction: C^k control.

Illustration





C^k convex interpolation

Theorem:
Let $(T_i)_{i\in\mathbb{Z}}$ be a sequence of C^2_+ and C^k subsets in $\mathbb{R}^2$, such that $T_i \subset \operatorname{int} T_{i+1} \neq \emptyset$ for all $i \in \mathbb{Z}$. Then there is C^k convex function f, such that each T_i is a sublevel set of f.

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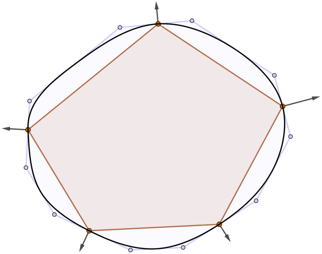

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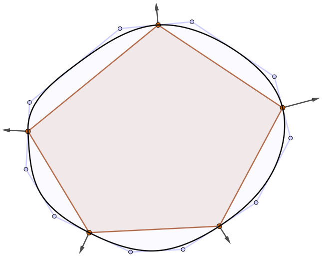

"Practical" counterexamples: polygonal skeleton sequence + normals at vertices.
$\rightarrow$ C^k convex interpolation: gradient proportional to normals at vertices.

Plan

1. Overview of the convex interpolation problem

2. Positive curvature and smooth convex interpolation

3. Construction of algorithmic counter examples

4. Conclusion

Exact line search: f convex C^k.

$$x_{k+1} \in \arg\min_x f(x), \qquad s.t. \quad x \in x_k - \mathbb{R}_+ \nabla f(x_k).$$

Optimality condition: $\langle x_{k+1} - x_k, \nabla f(x_{k+1}) \rangle = 0$.

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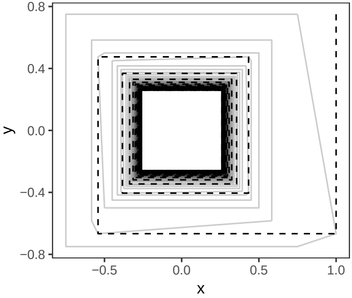

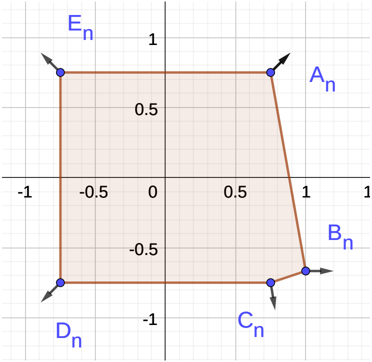

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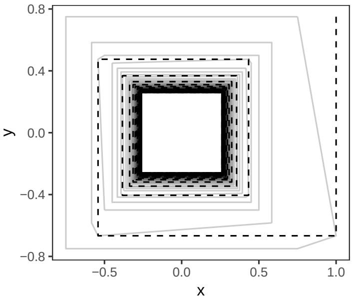

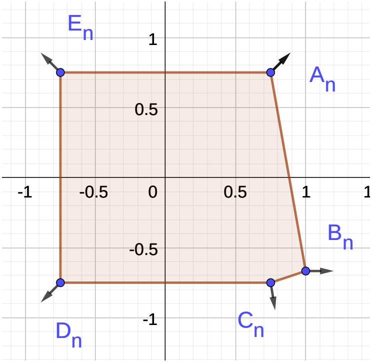

Alternating minimization / Gauss-Seidel: same counterexample.

$$b_{k+1} \in \arg\min_b f(a_k, b), \qquad\qquad a_{k+1} \in \arg\min_a f(a, b_{k+1})$$

Linear program: $\mathcal{C}$ convex compact, $c \in \mathbb{R}^2$, $\max_{x \in \mathcal{C}} \langle x, c \rangle$

Mirror descent / Bregman gradient (Nemirovski, Yudin)

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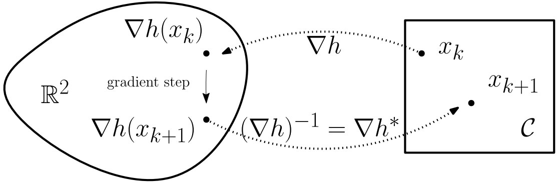

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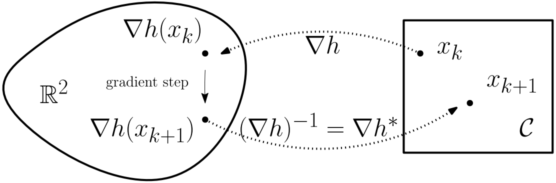

- $O(1/k)$ optimality gap (Bauschke et. al. 17)
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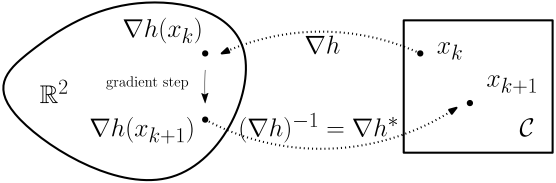

- $O(1/k)$ optimality gap (Bauschke et. al. 17)
- sequential convergence? (yes, under additional assumptions, no in general).

Absence of convergence for Bregman gradient algorithm

Algorithm: $x_{k+1} = \nabla h^*(\nabla h(x_k) + c)$

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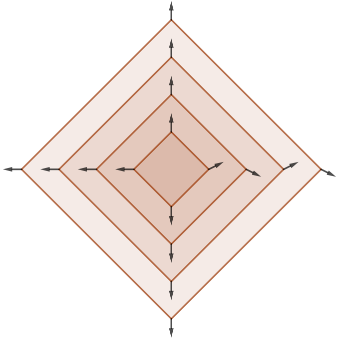

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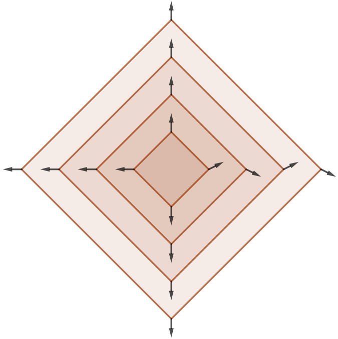

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Additional properties:

- h^* Lipschitz.
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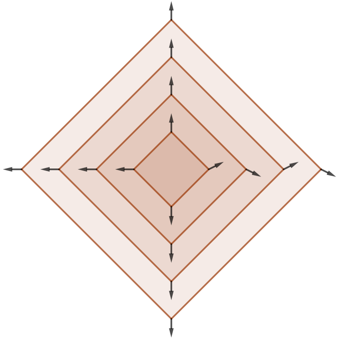

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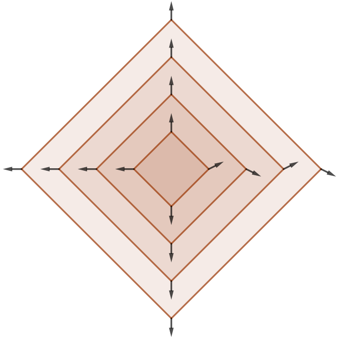

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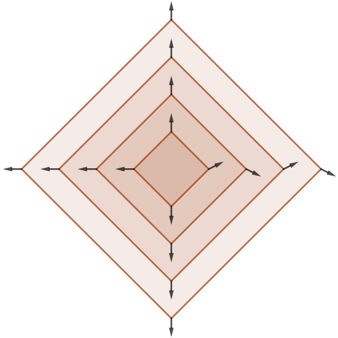

Remark: Possibly related phenomenon,
$\rightarrow$ no acceleration *à la* Nesterov (Dragomir et. al. 20)

Conditional gradient / Frank-Wolfe algorithm (with C. Combettes)

Frank-Wolfe algorithm: $\min_{x \in \mathcal{C}} f(x)$, $\mathcal{C}$ compact convex, f convex, Lipschitz gradient.

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$$x_{k+1} = (1-\gamma_k)x_k + \gamma_k v_k, \qquad v_k \in \arg\min_{v\in\mathcal{C}} \langle \nabla f(x_k), v\rangle$$

$$\gamma_k = \arg\min_{z=(1-\gamma)x_k+\gamma v_k} f(x_k) + \langle \nabla f(x_k), z - x_k\rangle + \frac{L}{2}\|x_k - z\|^2$$

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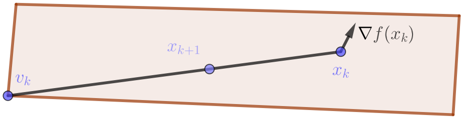

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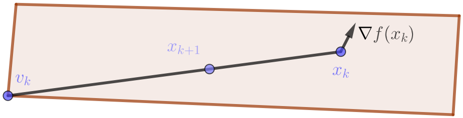
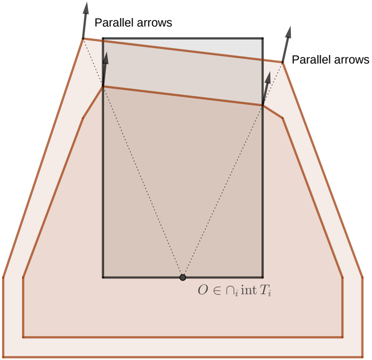


- Fixed points = minimizers.
- $O(1/k)$ optimality gap.
- Many step size variants.
- Sequential convergence?

Controling gradients beyond polygonal skeleton vertices



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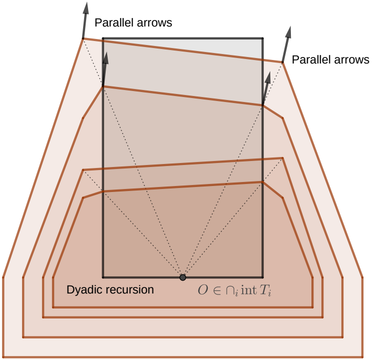

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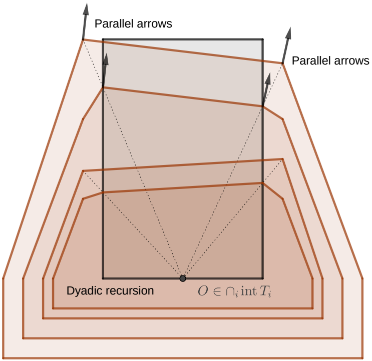
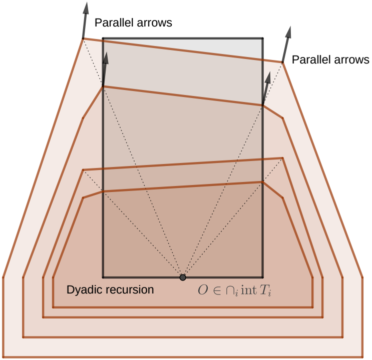


- **Conic interpolation:** alignment $\rightarrow$ control gradient directions on segments.

Controling gradients beyond polygonal skeleton vertices



- **Conic interpolation:** alignment $\to$ control gradient directions on segments.
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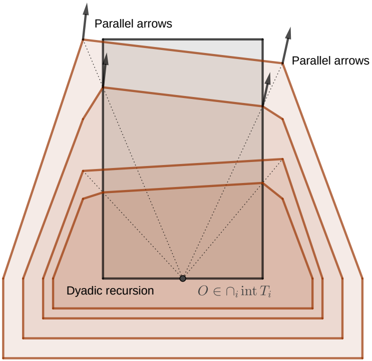

- **Conic interpolation:** alignment $\rightarrow$ control gradient directions on segments.
- **Linear oracle:** controled on a significative portion of the trajectory.
- **No convergence:** if $\gamma_k \rightarrow 0$, $\gamma_k < 1$ for all k, not summable (*e.g.* L slight over estimation).

Plan

Conclusion

Smooth convex interpolation in $\mathbb{R}^2$:

- Sublevel sets with positive curvature.
- Specification using polygonal skeleton.

Not detailed in the presentation:

- Additional counter-examples.
- Subtleties and computational aspects of the construction.

Next steps:

- More counter examples.
- C^∞ interpolation.
- Relax positive curvature.

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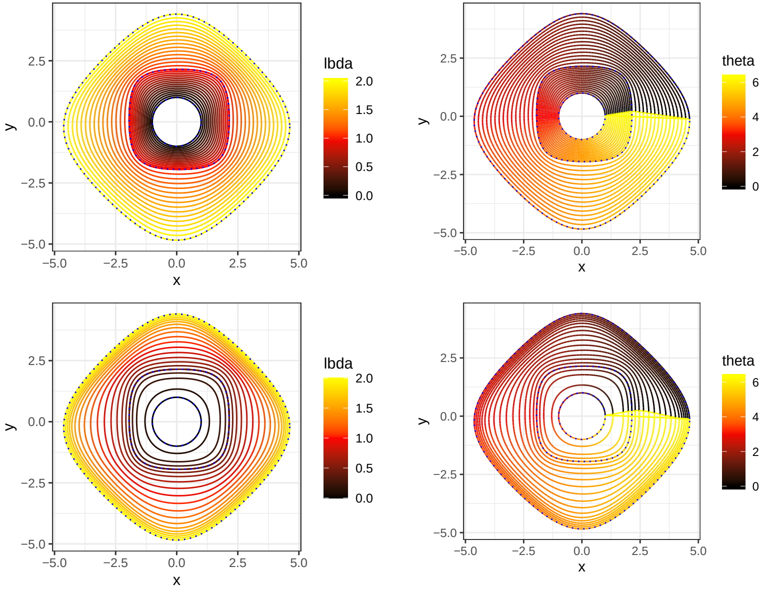
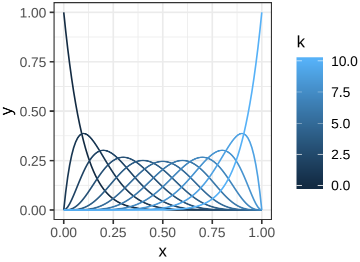
Thanks

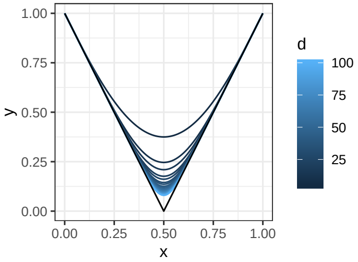
Illustration



Bernstein approximation

Bernstein polynomial and approximation of absolute value:





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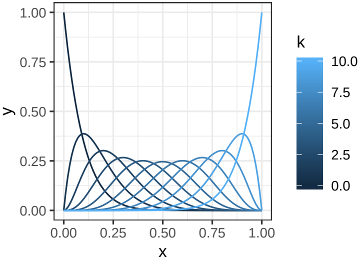

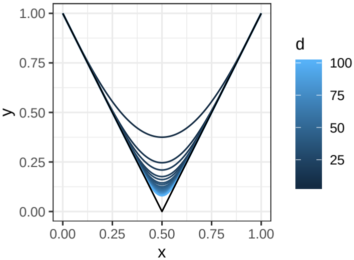

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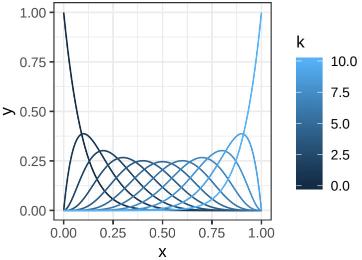

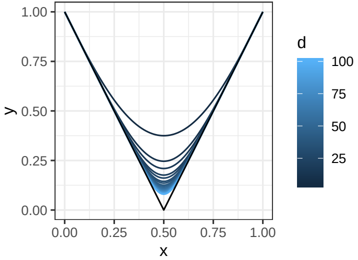

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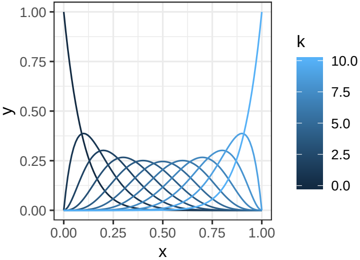

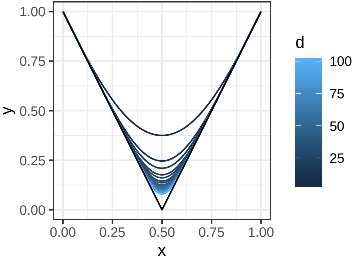

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Linear in f, preserves monotonicity, convexity, control derivatives at endpoints.