Regularity in Besov spaces of parabolic PDEs

Cornelia Schneider*

Conference

Geometry and Analysis on Non-Compact Manifolds

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*joint work with Stephan Dahlke, Philipps-University Marburg, Germany

Outline

Motivation: Adaptive algorithms

How to measure smoothness?
- Sobolev and Besov spaces
- Regularity results for elliptic PDEs

Regularity theory for parabolic PDEs
- Sobolev regularity of the heat equation
- Besov regularity of the heat equation
- $\longrightarrow$ General parabolic PDEs

Outlook
- Anisotropic Besov regularity of the heat equation
- Future Research

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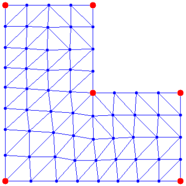

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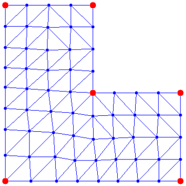

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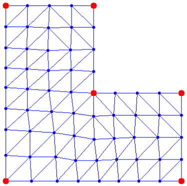

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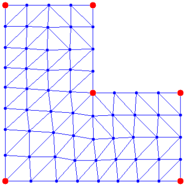

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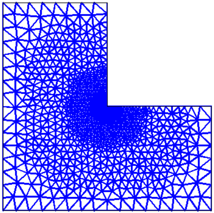

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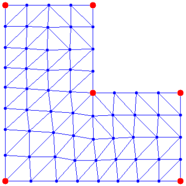

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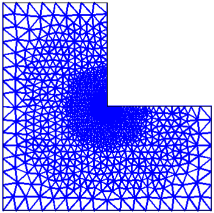

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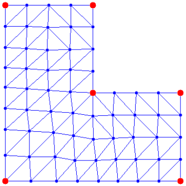

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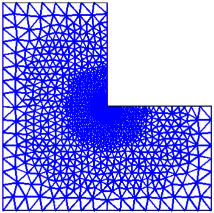

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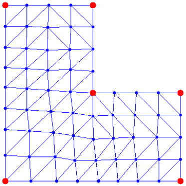

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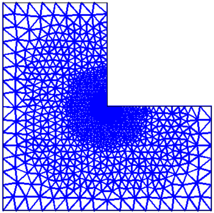

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TASK: Determine Sobolev and Besov regularity s and α and compare!

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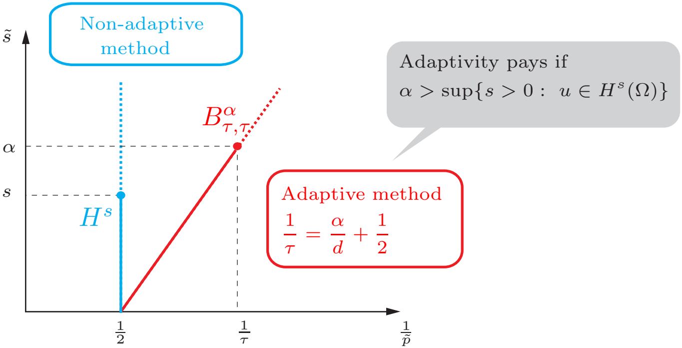

DeVore-Triebel diagram

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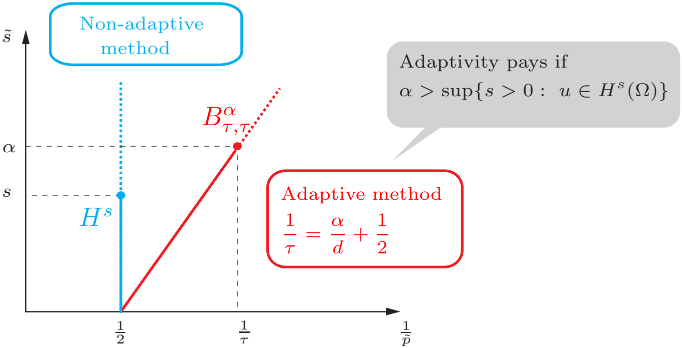

DeVore-Triebel diagram

Rule of thumb for elliptic PDEs:

- For smooth domains (and coefficients) there is no need for adaptivity.
- On Lipschitz domains adaptive methods are better.

Outline

Sobolev and Besov spaces

Besov spaces[†]: $0 < p, q \leq \infty$, $r > s > 0$

$$\|u\|_{B^s_{p,q}(\Omega)} := \|u\|_{L_p(\Omega)} + \left(\int_0^1 [t^{-s}\omega_r(u,t,\Omega)_p]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} < \infty$$

[†]classical definition (∼1959) goes back to O. V. Besov (*1933)

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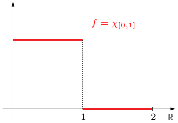

$\omega_1(f,t,\Omega) = t^{1/p} \quad (s < 1)$

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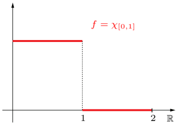

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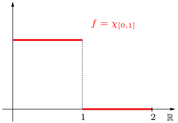

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Fractional Sobolev regularity for elliptic PDEs

Theorem ($H^{3/2}$-Theorem; Jerison, Kenig, 1995)

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and u be the solution of

$$\begin{aligned} \Delta u &= f \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

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Consider the following problem

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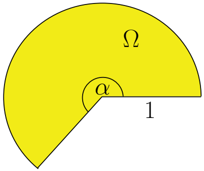

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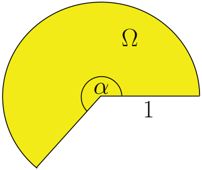

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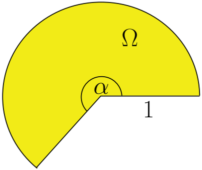

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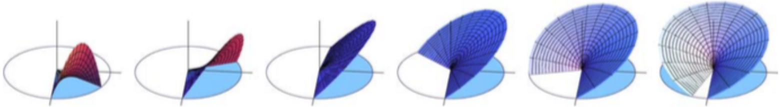

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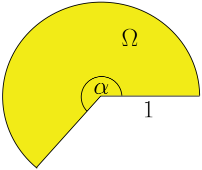

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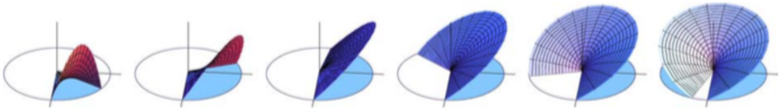

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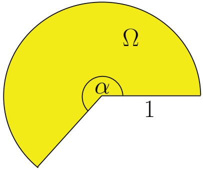

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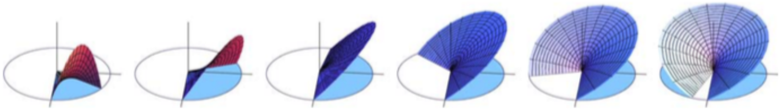

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Regularity theory for parabolic PDEs

- Sobolev regularity of the heat equation
- Besov regularity of the heat equation
- $\longrightarrow$ General parabolic PDEs

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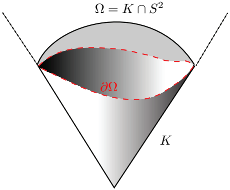
Sobolev regularity of the heat equation

Theorem (Dahlke, S., 2018)

Let $K \subset \mathbb{R}^3$ be a smooth cone and consider the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in} \quad \mathbb{R} \times K, \qquad u\big|_{\mathbb{R}\times\partial K} = 0.$$

Assume f belongs to some subspace of $L_2(\mathbb{R} \times K)$.



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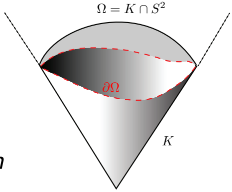
$$\varphi u \in L_2(\mathbb{R}, H^s(K)) \quad \text{for any} \quad s < \min\left(\frac{3}{2} + \lambda_1^+, 2\right),$$



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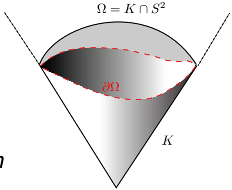

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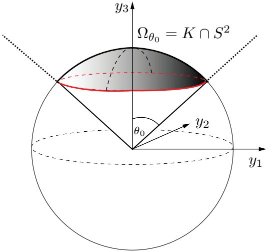
where $\lambda_1^+ := -\frac{1}{2} + \sqrt{\Lambda_1 + \frac{1}{4}}$ and Λ_1 is the first eigenvalue of the Dirichlet problem of the Laplace-Beltrami operator in Ω.

S. Dahlke and C. Schneider.
Describing the singular behaviour of parabolic equations on cones in fractional Sobolev spaces.
Int. J. Geomath. **9**, 293-315, 2018.

Example (Heat equation on spherical cap)



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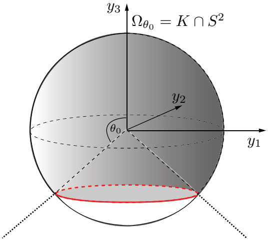

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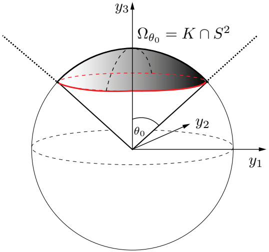

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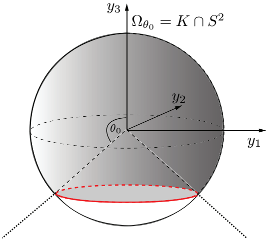

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θ_0	λ_1^+	θ_0	λ_1^+	θ_0	λ_1^+
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10°	13.2756	70°	1.4456	130°	0.5063
15°	8.6812	75°	1.3124	135°	0.4631
20°	6.3832	80°	1.1956	140°	0.4223
25°	5.0038	85°	1.0922	145°	0.3834
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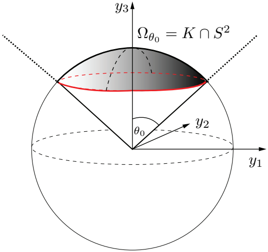

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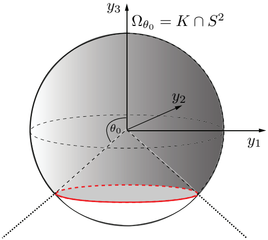

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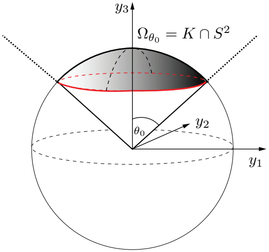

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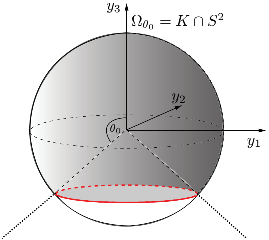

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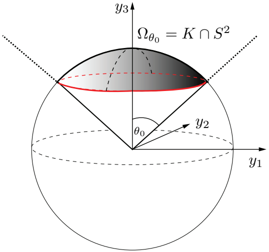

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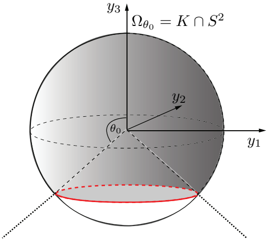

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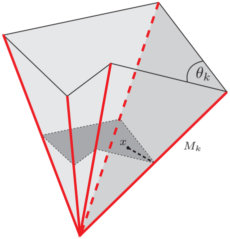
Besov regularity of heat equation

Theorem (Dahlke, S., 2018)



Let $K \subset \mathbb{R}^3$ be a polyhedral cone and consider

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } (0,T) \times K,$$
$$u(0,\cdot) = 0 \quad \text{on } K.$$

Assume f belongs to some subspace of $L_2((0,T) \times K)$ (at least differentiability $\gamma - 2$ with respect to variable x). Then subject to some further technical assumptions the truncated version of the solution u satisfies

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where $\frac{1}{2} < \frac{1}{\tau} < \frac{\alpha}{3} + \frac{1}{2}$.

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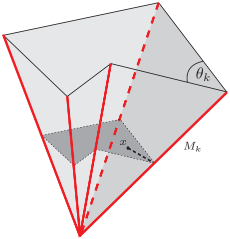

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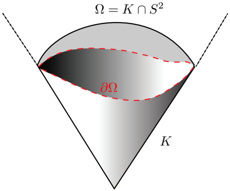

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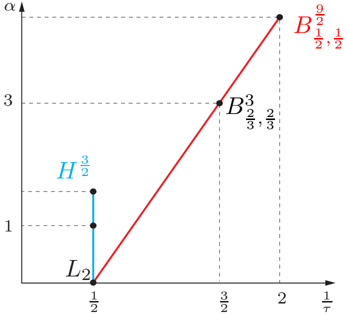

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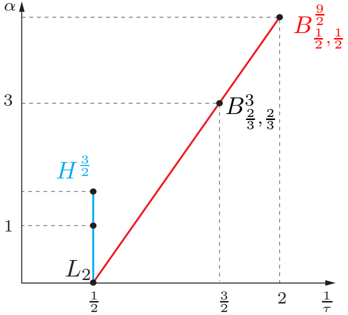

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Adaptivity pays off !

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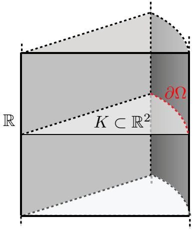

$\mathcal{K}=K\times\mathbb{R}$, $m=2<3=d$

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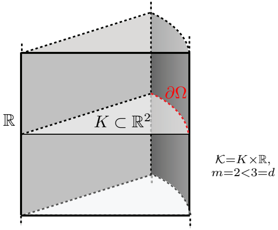

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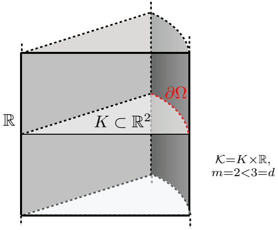

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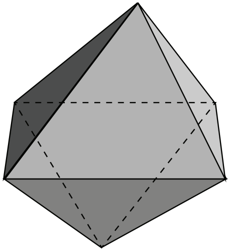

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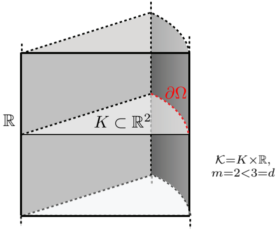

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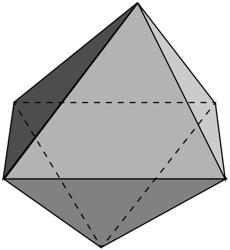

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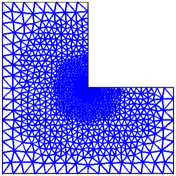
Outline

Why Anisotropic Spaces?

- Adaptive method: for elliptic problems



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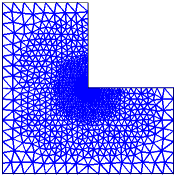

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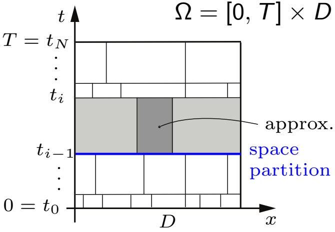

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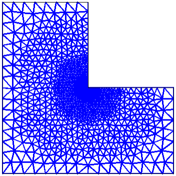

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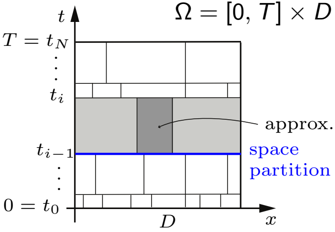

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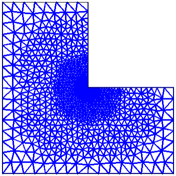

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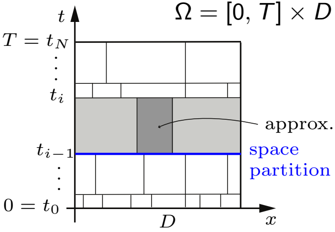

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M. Actis, P. Morin, and C. Schneider.
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Anisotropic Sobolev spaces : $1 < p < \infty$

$$\|u\|_{W_p^{1,2}(\Omega)} := \|u\|_{L_p(\Omega)} + \|\partial_t u\|_{L_p(\Omega)} + \sum_{i=1}^{d} \|\partial_{x_i} u\|_{L_p(\Omega)} + \sum_{i,j=1}^{d} \left\|\partial_{x_i x_j} u\right\|_{|L_p(\Omega)}$$

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and extend to general $s > 0$ via the action of the derivatives ∂_t and $\partial_{x_j x_i}$

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Let $1 < p < \infty$, $s > 0$, $\alpha > 0$, *and put* $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$. *Then*

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	[DS22]	[AG12]
$d = 3$	$\alpha < \frac{8}{3}$	$\alpha < \frac{3}{2}$
$d = 2$	$\alpha < 3$	$\alpha < 1$

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Outlook: Future Research

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- Regularity of parabolic PDEs in generalized dominating mixed smoothness spaces *(with Prof. Stephan Dahlke, Marburg)*
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- SPDEs and regularity in (weighted) Sobolev spaces *(with Dr. Petru Cioica-Licht, Essen; JProf. Markus Weimar, Bochum)*
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Thank you for your attention!