Knots, minimal surfaces and J-holomorphic curves

Joel Fine

Let K be a link in the 3-sphere, viewed as the ideal boundary of hyperbolic 4-space H^4. I will explain that the number of minimal surfaces in H^4 with ideal boundary K is a link invariant. I.e. the number of minimal surfaces doesn't change under isotopies of K. To define the count, the link must be generic in a certain sense and the minimal surfaces are counted with an appropriate sign. These counts actually give a family of link invariants, indexed by the genus of the filling and a second integer describing the extrinsic topology of how the surface sits in H^4. These invariants can also be seen as Gromov–Witten invariants counting J-holomorphic curves in the twistor space Z of H^4. Whilst Gromov–Witten theory suggests the general scheme for constructing the minimal surface link-invariants, there are substantial differences in how this must be carried out in this situation. These are due to the fact that the geometry of both H^4 and Z becomes singular at infinity, and so the J-holomorphic curve equation is degenerate, rather than elliptic, at the boundary. This means that both the Fredholm and compactness arguments involve completely new features, in some places more complicated and in others simpler, when compared with the usual story