Optimal multiple change-point detection and Localization

Nicolas Verzelen

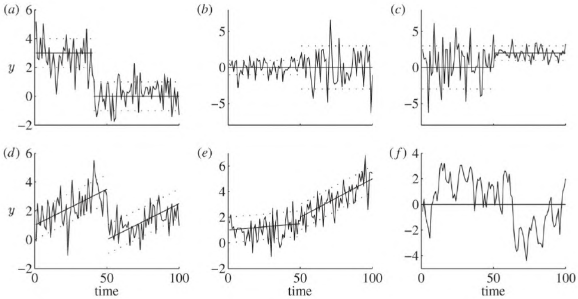
Joint works with A. Carpentier, M. Fromont, M. Lerasle, E. Pilliat, and P. Reynaud-Bouret

https://arxiv.org/abs/2010.11470
https://arxiv.org/abs/2011.07818

MMS Luminy - December 15th

Offline Change-point Analysis

General problem of detecting changes in distribution of a time series



Beaulieu et al.('12)

Old Problem [Wald, 1945] but still vivid.

See [Niu et al., 2016] and [Truong et al., 2020] for recent surveys.

(Sub)-Gaussian univariate mean change-point Model

Data : Time series $\mathbf{Y} \in \mathbb{R}^n$

$$Y_i = \theta_i + \epsilon_i, \qquad \text{where } \epsilon_i \overset{ind.}{\sim} \mathcal{SG}(1) \ ,$$

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We leave aside possible time dependencies

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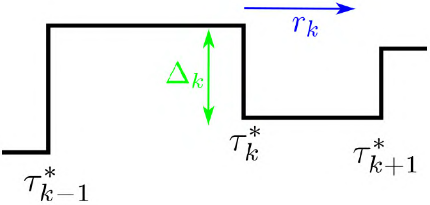
Notation : change-point vector $\boldsymbol{\tau}^*$
$1 < \tau_1^* < \ldots < \tau_K^* \le n$
s.t. θ is constant over $[\tau_k^*, \tau_{k+1}^*)$.

Height $\Delta_k = \theta_{\tau_k^*} - \theta_{\tau_k^*-1}$

Radius $r_k = \frac{(\tau_{k+1}^* - \tau_k^*)(\tau_k^* - \tau_{k-1}^*)}{\tau_{k+1}^* - \tau_{k-1}^*}$

$$\asymp (\tau_{k+1}^* - \tau_k^*) \wedge (\tau_k^* - \tau_{k-1}^*).$$



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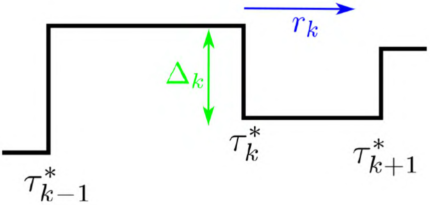

Definition of the Energy of τ_k^*

The **Square Energy** of τ_k^* is $\qquad E_k^2 = r_k \Delta_k^2$

l_2 distance between $\boldsymbol{\theta}$ and best approximation by a piece-wise constant vector on $\boldsymbol{\tau}^{(-k)} = (\tau_1^*, \ldots, \tau_{k-1}^*, \tau_{k+1}^*, \ldots)$.

Two mathematical perspectives on change-point Detection

- Denoising/Estimation : Estimating $\boldsymbol{\theta}$ (in l_2 norm).
- Clustering : Recover the change-points $\boldsymbol{\tau}^*$; partition of $[n]$ into segments.

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Denoising perspective :
Minimax-Optimal rates (for $K \geq 2$) $$K\left[1 + \log\left(\frac{n}{K}\right)\right]$$

achieved e.g. by penalized least-squares [Birgé and Massart, 2001]

Quadratic computational complexity by dynamic programming.

Change-point detection as a clustering problem

Several lines of literature :

- At Most One Change-point (AMOC) $[K \leq 1]$. **Least-square** estimator detects $\widehat{K} = 1$ if $E_1 \gg \sqrt{\log\log(n)}$ and $|\widehat{\tau}_1 - \tau_1^*| = O(\Delta_1^{-2})$ [Csorgo and Horváth, 1997].

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  Binary segmentation [Scott and Knott, 1974] = iterative bisection.
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Theorem (Typical modern result. *sloppy version* ; [Wang et al., 2020, Fryzlewicz, 2018, Kovács et al., 2020]**)**

If $\min_k E_k^2 \gtrsim \log(n)$*, then whp* $\widehat{K} = K$ *and*

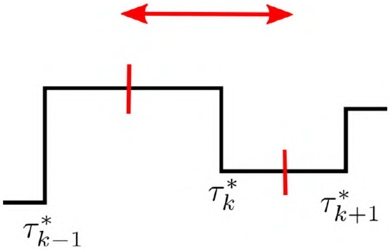
$$d_H(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*) = \inf_{k=1,\ldots,K} |\widehat{\tau}_k - \tau_k^*| \lesssim \frac{\log(n)}{\min_k \Delta_k^2}$$

Surprisingly, the tightest known results [Frick et al., 2014] are a bit older.

Objectives

Two sub-problems

Change-Point Detection
= Detecting the existence of the change-point



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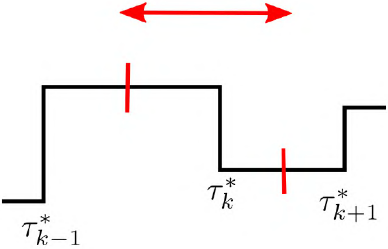
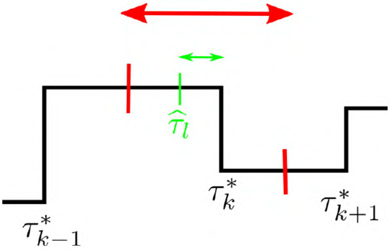


Change-Point Localization
= small estimation error
$d_{H,1}(\widehat{\boldsymbol{\tau}}, \tau^*_k) = \min_l |\widehat{\tau}_l - \tau^*_k|$



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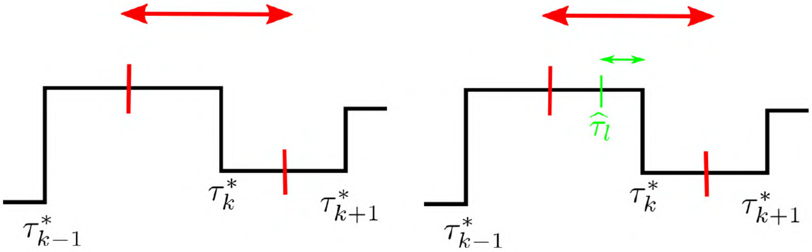
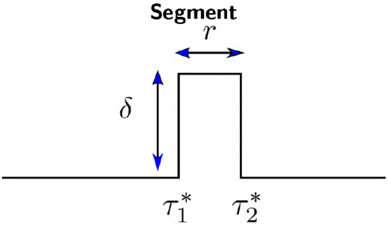


some questions

- What is the energy requirement for detection ?
- How is the transition between detection and localization ?
- Is penalized least-square optimal ? For which penalty ?

Simpler problem : testing $\boldsymbol{\theta} = 0$ versus

$$\boldsymbol{\theta} \in \Theta[r, \delta] = \left\{ \boldsymbol{\theta} \in \mathbb{R}^n : \exists \tau \in \{n/4, n/4+r, n/4+2r, \ldots, 3n/4\} \text{ such that } \theta_i = \delta \mathbb{1}_{i \in [\tau, \tau+r)} \right\} .$$



Gaussian Change-point Detection

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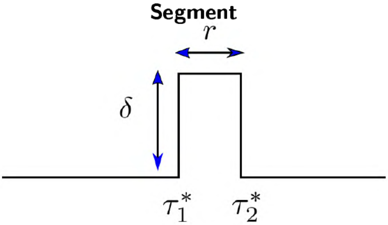

$\lfloor \frac{n}{2r} \rfloor$ possible positions

For each τ, sufficient statistic

$$Z_\tau = r^{-1/2} \sum_{i=\tau}^{\tau+r-1} Y_i \sim \begin{cases} \mathcal{N}(0,1) \\ \mathcal{N}(r^{1/2}\delta, 1) \end{cases}$$

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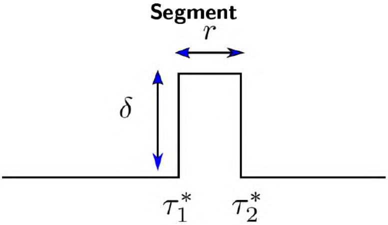

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Proposition (Segment Detection ≈ [Arias-Castro et al., 2011])

Fix ξ in $(0,1)$. If $\delta\sqrt{r} \leq \sqrt{2(1-\xi)\log[n/(2r)]}$, then for all tests T
$\mathbb{P}_0[T=1] + \sup_{\boldsymbol{\theta} \in \Theta[\delta,r]} \mathbb{P}_{\boldsymbol{\theta}}[T=0] \geq 1 - c\left(\frac{r}{n}\right)^{c'\xi^2}$.

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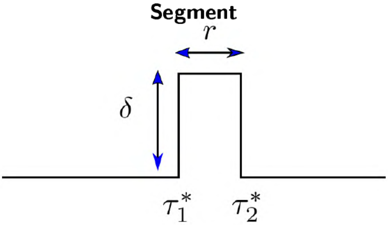

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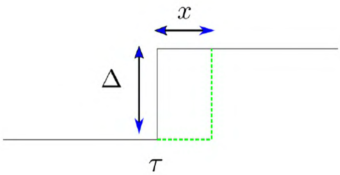
$\kappa > 1$; $q > 0$.

Definition

τ_k^* is a (κ, q)-**high-energy** change-point if $\mathbf{E}_k(\boldsymbol{\theta}) > \kappa\sqrt{2\log\left(\frac{n}{r_k}\right) + q}$.

Simplified setting : one change-point ; known means $\boldsymbol{\mu} = (\mu_1, \mu_2)$;
two possible positions for τ^* : τ or $\tau + x$.



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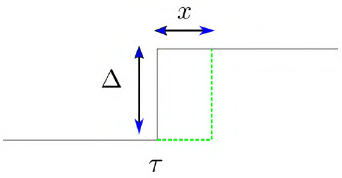

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Write $\Delta = \mu_2 - \mu_1$. *For any* x *in* $[1/2, n/2 - 1 - 2\Delta^{-2})$,

$$\inf_{\widehat{\tau}} \sup_{\tau^* \in \{2,\dots,n\}} \mathbb{P}_{\boldsymbol{\theta}(\tau^*, \boldsymbol{\mu})} \left(|\widehat{\tau} - \tau^*| \geq 2\Delta^{-2} + x \right) \gtrsim e^{-cx\Delta^2} ,$$

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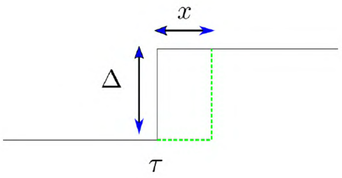

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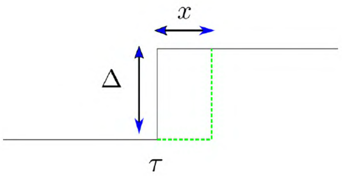

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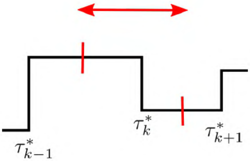
Desiderata for a suitable change-point procedure

Under an event $\mathcal{A}$ of high (*to be discussed*) probability .

(NoSp). No spurious change-point is detected :

$$\begin{cases} \left|\{\widetilde{\boldsymbol{\tau}}\} \cap \left(\frac{\tau^*_{k-1}+\tau^*_k}{2}, \frac{\tau^*_k+\tau^*_{k+1}}{2}\right]\right| \leq 1 \ , \text{ for all } k \text{ in } \{2,\ldots,K-1\} \ ; \\ \left|\{\widetilde{\boldsymbol{\tau}}\} \cap \left[2, \frac{\tau^*_1+\tau^*_2}{2}\right]\right| \leq 1 \ ; \ \left|\{\widetilde{\boldsymbol{\tau}}\} \cap \left(\frac{\tau^*_{K-1}+\tau^*_K}{2}, n\right]\right| \leq 1 \ . \end{cases}$$



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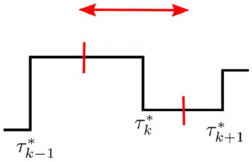

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(Loc[κ, q]). High-energy change-points are localized at the optimal rate.
Any (κ, q)-high-energy change-point τ^*_k satisfies

$$\mathbb{P}\left(d_{H,1}(\widetilde{\boldsymbol{\tau}}, \tau^*_k)\mathbb{1}_{\mathcal{A}} \geq cx\Delta_k^{-2}\right) \lesssim e^{-x}, \quad \forall x \geq 1 \ .$$

Penalized least-square estimator

$\boldsymbol{\tau}$= vector of tentative change-points
$\boldsymbol{\Pi}_{\boldsymbol{\tau}}$ = projector onto the space of piece-wise constant vectors with changes at $\boldsymbol{\tau}$

$$\widehat{\tau} = \arg\min_{\tau} \mathrm{Cr}_0(\mathbf{Y}, \boldsymbol{\tau}) = \arg\min_{\boldsymbol{\tau}} \|\mathbf{Y} - \boldsymbol{\Pi}_{\boldsymbol{\tau}}\mathbf{Y}\|^2 + L \ \mathrm{pen}_0(\boldsymbol{\tau}, q) \ ,$$

Multi-scale penalty $\mathrm{pen}_0(\boldsymbol{\tau}, q) = q|\boldsymbol{\tau}| + 2\sum_{k=1}^{|\boldsymbol{\tau}|+1} \log\left(\frac{n}{\tau_k - \tau_{k-1}}\right).$

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Remarks :

- Additive Penalty $\rightsquigarrow$ dynamic programming (and its refinements [Killick et al., 2012])
- Over-penalizes **small** segments.
- Highly differs from complexity penalties $\mathrm{pen}_0(\boldsymbol{\tau}, q) = (|\boldsymbol{\tau}| + 1)(1 + \log(n/|\boldsymbol{\tau}|))$.

Connection between CUSUM and Least-square penalty

Definition (CUSUM Statistic)

For $\mathbf{t} = (t_1, t_2, t_3)$, $\mathbf{C}(\mathbf{Y}, \mathbf{t}) = \left[\overline{\mathbf{Y}}_{[t_2, t_3)} - \overline{\mathbf{Y}}_{[t_1, t_2)}\right] \sqrt{\frac{(t_2 - t_1)(t_3 - t_2)}{t_3 - t_1}}$

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Lemma (deletion of a change-point)

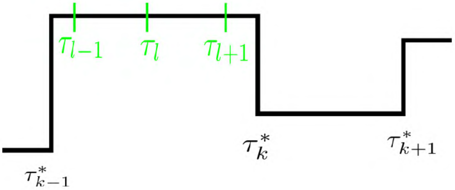
$\boldsymbol{\tau}^{(-l)} = (\tau_1, \ldots, \tau_{l-1}, \tau_{l+1}, \ldots)$

$$\|\mathbf{Y} - \mathbf{\Pi}_{\boldsymbol{\tau}}\mathbf{Y}\|^2 - \|\mathbf{Y} - \mathbf{\Pi}_{\boldsymbol{\tau}^{(-l)}}\mathbf{Y}\|^2 = -\mathbf{C}^2[\mathbf{Y}, (\tau_{l-1}, \tau_l, \tau_{l+1})] \ .$$

$$\begin{aligned}\mathrm{Cr}_0(\mathbf{Y}, \boldsymbol{\tau}) - \mathrm{Cr}_0(\mathbf{Y}, \boldsymbol{\tau}^{(-l)}) \quad = \quad & -\mathbf{C}^2\big(\mathbf{Y}, (\tau_{l-1}, \tau_l, \tau_{l+1})\big) \\ & + L\left[2\log\left(\frac{n(\tau_{l+1} - \tau_{l-1})}{(\tau_{l+1} - \tau_l)(\tau_l - \tau_{l-1})}\right) + q\right] \ .\end{aligned}$$

Consider $\boldsymbol{\tau}$ such that
$\boldsymbol{\theta}$ is constant on $[\tau_{l-1}, \tau_{l+1})$

Goal : show that $\boldsymbol{\tau} \neq \widehat{\boldsymbol{\tau}}$?



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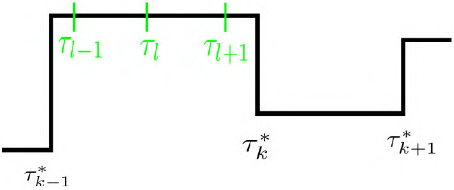

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$\boldsymbol{\tau} \neq \widehat{\boldsymbol{\tau}}$ as long as $\mathbf{C}^2(\boldsymbol{\epsilon}, (\tau_{l-1}, \tau_l, \tau_{l+1}))$ small enough.

Local Optimality and uniform Control of the CUSUM

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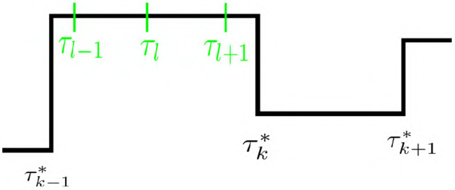

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Local Optimality $\rightsquigarrow$ Uniform bound for the CUSUM

Lemma (Multi-scale chaining ; in the spirit of [Dumbgen and Spokoiny, 2001])

$$\mathcal{A}_q = \left\{ |\mathbf{C}(\boldsymbol{\epsilon}, \mathbf{t})| \le 2\sqrt{2\log\left(\frac{n(t_3 - t_1)}{(t_3 - t_2)(t_2 - t_1)}\right) + q}, \quad \forall \mathbf{t} = (t_1, t_2, t_3) \right\} .$$

We have $\mathbb{P}[\mathcal{A}_q] \ge 1 - ce^{-c'q}$.

First Analysis of Penalized Least-square

Proposition (V. et al. ('20))

For any L and q large enough, under $\mathcal{A}_q$, the penalized least-square estimator $\widehat{\tau}$ satisfies

(a) **(NoSp)** *No Spurious Jump is detected.*

(b) **(Detec[κ_L, q])** *All (κ_L, q)-high-energy change-points τ_k^* are detected*

$$d_{H,1}(\widehat{\boldsymbol{\tau}}, \tau_k^*) \leq \min\left\{\frac{\tau_{k+1}^* - \tau_k^*}{2}, \frac{\tau_k^* - \tau_{k-1}^*}{2}, \kappa_L \frac{\log\left(n\Delta_k^2\right) + q}{\Delta_k^2}\right\}$$

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[Frick et al., 2014] require $\min_k |\Delta_k|^2 \min_k |\tau_{k+1}^* - \tau_k^*| \gtrsim \log\left(\frac{n}{\min_k |\tau_{k+1}^* - \tau_k^*|}\right)$

Discussion :

- We allow arbitrarily low-energy jumps.
- Local condition for high energy.
- Dependency in q is optimal with respect to the probability $1 - ce^{-c'q}$
- Complexity-based penalties are highly suboptimal.

Localization **(Loc)** by Penalized Least-squares

Proposition (V. et al. ('20))

Fix any L and q large enough. For any (κ_L, q)-high-energy change-point τ_k^*, we have

$$\mathbb{P}\left(d_{H,1}(\widehat{\boldsymbol{\tau}}, \tau_k^*)\mathbb{1}_{\mathcal{A}_q} \geq cx\Delta_k^{-2}\right) \lesssim e^{-x} \quad \forall x \geq 1 \ .$$

Remarks :

- Recovers the optimal subexponential rate of order Δ_k^{-2} for a specific change-point
- **Regional** to **Local** phenomenon :
  Detection= High-Energy **Localization** only depends on Δ_k !

Haussdorff and Wasserstein Loss

If $|\widehat{\boldsymbol{\tau}}| = |\boldsymbol{\tau}^*|$,

$$d_W(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*) = \sum_{k=1}^{K} |\widehat{\tau}_k - \tau_k^*|$$

$$d_H(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*) = \max_{k=1}^{K} |\widehat{\tau}_k - \tau_k^*|$$

Corollary

Assuming that all change-points have high-energy, we deduce

$$\mathbb{E}\left[d_W\left(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*\right)\mathbb{1}_{\mathcal{A}_q}\right] \lesssim \sum_{k=1}^{K}\left(e^{-c''\Delta_k^2} \wedge \frac{1}{\Delta_k^2}\right),$$

$$\mathbb{E}\left[d_H\left(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*\right)\mathbb{1}_{\mathcal{A}_q}\right] \lesssim \max_{k\in\{1,\dots,K\}}\left(Ke^{-c''\Delta_k^2} \wedge \frac{\log K}{\Delta_k^2}\right).$$

Remark : Haussdorff and Wasserstein rates are minimax optimal.

First Step : Detection

CUSUM Statistics $\mathbf{C}[\mathbf{Y}, \mathbf{t}]$ higher than $\sqrt{2\log\left(\frac{n(t_3-t_1)}{(t_3-t_2)(t_2-t_1)}\right)} + \zeta_{1-\alpha}$

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Change-point Detection = Aggregation of Multiple Local Tests

2 Caveats :

- **Too many tests** $n^3/6 \rightsquigarrow$ symmetric intervals + smaller grid $(n \log(n))$
- **Tests** are not always self consistent
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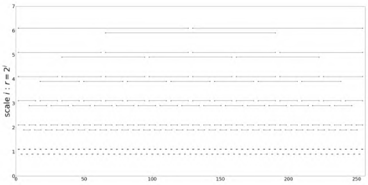

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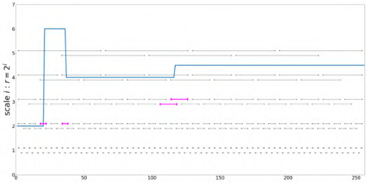

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... but, $\widehat{\tau}_{ag}$ does not seem to achieve **Loc** at least in worst-case.

Second Step : Localization

For each estimated **change-point** $\widehat{\tau}_k$, we re-estimate the change-point position :

$\rightsquigarrow$ least-square estimator inside a CI $I_{\widehat{\tau}_k}$ of size $\widehat{r}_k$ based on data in a larger interval of size $2\widehat{r}_k - 1$.

$$\widetilde{\tau}_k \in \underset{\tau' \in I_{\widehat{\tau}_k}}{\arg\min} \|\mathbf{Y} - \mathbf{\Pi}_{\tau'} \mathbf{Y}^{(\widehat{\tau}_k, 2\widehat{r}_k - 1)}\|^2 \quad .$$

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Proposition

The refitted estimator $\widetilde{\boldsymbol{\tau}}$ *satisfies, on an even* $\mathcal{B}_\alpha$, *of probability higher than* $1-\alpha$, **(NoSp)**, **(Detec)**, *and* **(Loc)**.

Remark :

- Similar to penalized least-square estimator.
- Computational complexity $O(n \log(n))$

Summary

Wrap-up :

- Regional to Local phenomenon.
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When $K = 1$, $\log$ conditions are replaced by $\log\log$ conditions.

Possible Extensions/ Open Questions :

- Heavier **tail** distribution, **time dependencies** :
  $\rightsquigarrow$ uniform control of the CUSUM (e.g.[Cho and Kirch, 2019])
- **Exact constant** for detection ?

High-Dimensional Setting

Gaussian Multivariate Change-point Model

$$Y_i = \boldsymbol{\theta}_i + \epsilon_i, \qquad \text{where } \boldsymbol{\theta}_i \in \mathbb{R}^p \text{ and } \epsilon_i \overset{iid}{\sim} \mathcal{N}(0, \mathbf{I}_p).$$

Objective : Detecting **times** $\tau_1^*, \ldots, \tau_K^*$ such that $\boldsymbol{\theta}_{\tau_k^*} \neq \boldsymbol{\theta}_{\tau_k^*-1}$ (with the side information that the difference $\boldsymbol{\theta}_{\tau_k^*} - \boldsymbol{\theta}_{\tau_k^*-1}$ is possibly sparse)

[Wang and Samworth, 2018, Enikeeva and Harchaoui, 2019, Liu et al., 2019]

General analysis of the bottom-up algorithm

We are given :

- A grid $\mathcal{G}$ of (l, r) (positions, radius) corresponding to $(l - r, l + r)$.
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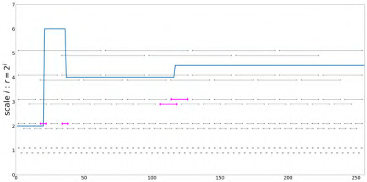

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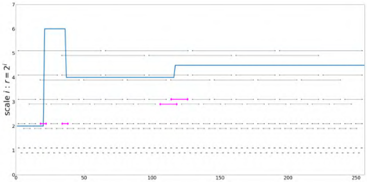

Proposition

If $FWER(\mathcal{T}) \le \alpha$, then $\widehat{\tau}_{ag}$ satisfies $(\mathbf{NoSp})$ with probability higher than $1-\alpha$. All change points $\boldsymbol{\tau}_k^*$ detected by a local test (up to some margin), are detected by $\widehat{\boldsymbol{\tau}}_{ag}$.

Generic Schemes :

- Introducing a sensible notion of energy
- Optimal testing with respect to that energy.

Energy of a Change-Point

$$E_k^2 = r_k \frac{\|\boldsymbol{\theta}_{\tau_k^*} - \boldsymbol{\theta}_{\tau_k^*-1}\|^2}{\sigma^2}$$

Local Homogeneity Tests on $[l-r, l+r)$
1st Simplification : two-sample tests over data in $[l-r, l)$ versus $[l, l+r)$.

2nd Simplification : (possibly-sparse) signal detection test with multivariate CUSUM statistics

$$\mathbf{C}_{l,r} = \left[\overline{\mathbf{Y}}_{[l,l+r)} - \overline{\mathbf{Y}}_{[l-r,l)}\right] \frac{\sqrt{2r}}{\sigma} \sim \mathcal{N}\left[\left(\overline{\boldsymbol{\theta}}_{[l,l+r)} - \overline{\boldsymbol{\theta}}_{[l-r,l)}\right) \frac{\sqrt{2r}}{\sigma}, \mathbf{I}_p\right]$$

Energy and Optimal Tests for sparse high-dimensional data

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Old toy detection Problem : [Donoho and Jin, 2004, Collier et al., 2015]
$\rightsquigarrow$ Higher-Criticism + χ^2 type statistics (minimax optimal wrt sparsity s and p)

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No Sufficient : $\Omega[n\log(n)]$ tests are considered
$\rightsquigarrow$ one also needs optimal dependencies wrt Types I and II error probabilities :
e.g. variants of HC [Liu et al., 2019] or Pilliat et al.('20).

Optimal Detection

$\delta \in (0,1)$; s_k sparsity of change-point τ_k^*.

High-energy change-point

τ_k^* is a high-energy change-point if $E_k^2 \geq c\psi_{s,p,s_k,\delta}$ where

$$\psi_{s,p,s_k,\delta} = s_k \log\left(1 + \frac{\sqrt{p}}{s_k}\sqrt{\log\left(\frac{n}{r_k\delta}\right)}\right) + \log\left(\frac{n}{r_k\delta}\right) \quad .$$

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Theorem (Pilliat et al.('20))

With probability higher than $1-\delta$, $\widehat{\boldsymbol{\tau}}_{ag}$ *achieves* **(NoSp)** *and* **(Detects)** *all* **high-energy** *change-points* τ_k^* *with* $E_k^2 \geq c_+\psi_{s,p,s_k,\delta}$.

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With probability higher than $1-\delta$, $\widehat{\boldsymbol{\tau}}_{ag}$ *achieves* **(NoSp)** *and* **(Detects)** *all* **high-energy** *change-points* τ_k^* *with* $E_k^2 \geq c_+\psi_{s,p,s_k,\delta}$.

Conversely, no procedure achieving **(NoSp)** *is able to* **(Detect)** *high-energy change-points (up to a constant)* τ_k^* *with* $E_k^2 \geq c_-\psi_{s,p,s_k,\delta}$

Remark : For $K \leq 1$, see [Liu et al., 2019].

Main Message

Sloppy Conjecture

For general change-points models, optimal detection (almost) amounts to optimal multiple homogeneity testing

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Open Questions

Localization rates require model-specific techniques.

For (Sparse) High-dimensional change-points, there seem to exist several phase transitions from regional to local (*work in progress*)

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