The scaling limit of random planar maps with large faces

Armand Riera
joint work with Nicolas Curien and Grégory Miermont

CIRM: Random Geometry

Sommaire

I. Planar maps and scaling limits

Planar maps

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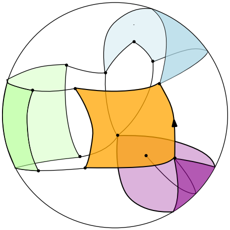

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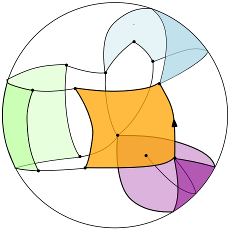

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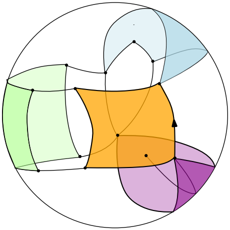

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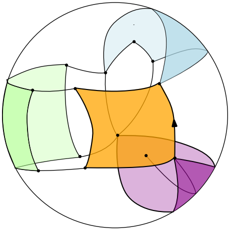

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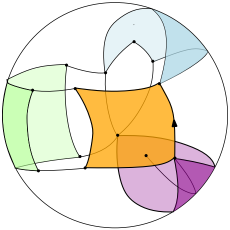

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Models without mass : without Ising or loops configurations (as in $\mathcal{O}(n)$ models).

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$$\left(\mathrm{m}, d_{\mathrm{m}}, \rho_{\mathrm{m}}\right),$$

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- Let $\mathbf{q} = (q_k)_{k \geqslant 1}$ be a non-zero sequence of non-negative numbers.
- The **q**-*Boltzmann measure* $w_{\mathbf{q}}$ is defined by

$$w_{\mathbf{q}}(\mathfrak{m}) := \prod_{f \in \mathrm{Faces}(\mathfrak{m})} q_{\mathrm{deg}(f)/2} \, ,$$

where $\mathfrak{m}$ is any finite bipartite map.

Scaling limit in the generic case

Theorem (Le Gall 2013, Miermont 2013)

Let m_n be a uniform quadrangulation of the sphere with n vertices. Then we have the following result :

$$(\mathfrak{m}_n, n^{-\frac{1}{4}} d_{\mathfrak{m}_n}, \rho_{\mathfrak{m}_n}) \xrightarrow{(d)} (\mathcal{S}, \Delta, \rho_*)$$

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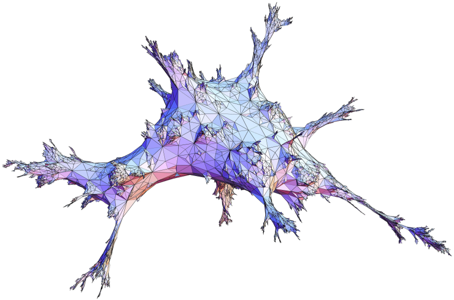

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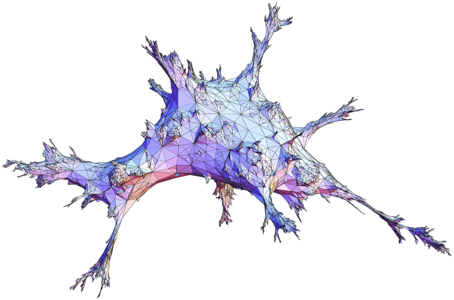

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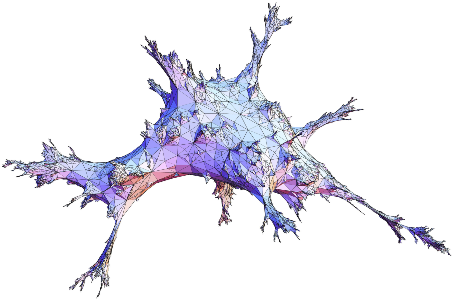

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Quantum gravity : Gwynne, Holden, Miller, Shefield, Sun ...

Scaling limit in the stable case

Stable maps :

$$q_k \sim c_{\mathbf{q}} \, \kappa_{\mathbf{q}}^k \, k^{-\alpha-\frac{1}{2}} \quad \text{as } k \to \infty,$$

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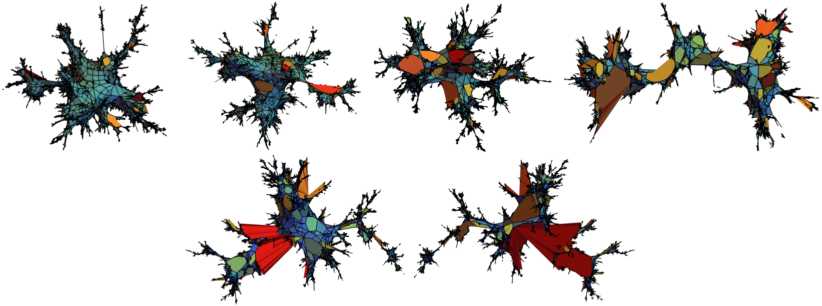

FIGURE – Simulations of large non-generic critical random Boltzmann planar maps of index $\alpha \in \{1.9, 1.8, 1.7, 1.6, 1.5, 1.4\}$ from top left to bottom right

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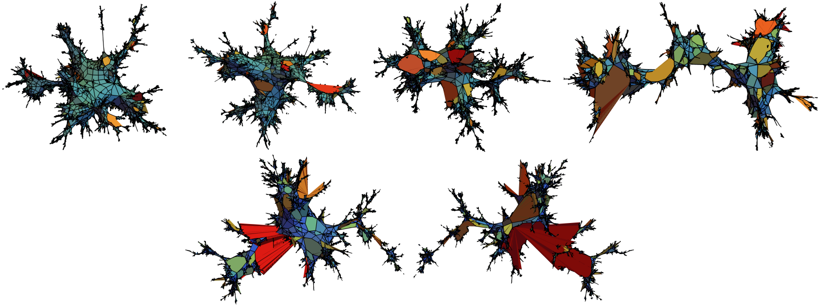

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Le Gall – Miermont (2009) : tightness.

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We have the following convergence in distribution for the Gromov–Hausdorff topology :

$$\left(\mathfrak{M}_n,\ n^{-\frac{1}{2\alpha}} \cdot \mathrm{d}_{\mathrm{gr}}^{\mathfrak{M}_n},\ \rho_n\right) \xrightarrow[n\to\infty]{(d)} (\mathcal{S}_\alpha, c_q \cdot \Delta_\alpha,\ \rho_*)\,,$$

where the random compact metric space $(\mathcal{S}_\alpha, \Delta_\alpha)$ is called the α-stable carpet if $\alpha \in [3/2, 2)$ and the α-stable gasket if $\alpha \in (1, 3/2)$.

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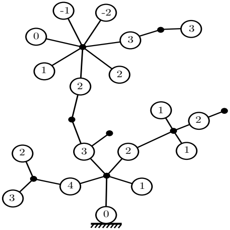
We also show that $\mathcal{S}_\alpha$ verifies the following properties :

	Faces	Topology
All α	Dimension 2	Planar
$\alpha \in [3/2, 2)$	Simple loops	Sierpinski carpet
$\alpha \in [1, 3/2)$	self and mutually intersecting	? ?

II. Construction of the scaling limit

Main discrete ingredient

Bouttier – Di Francesco – Guitter encoding :



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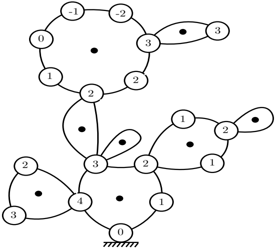

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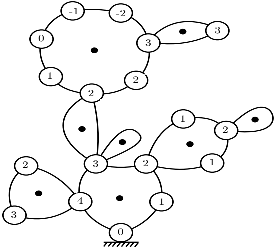

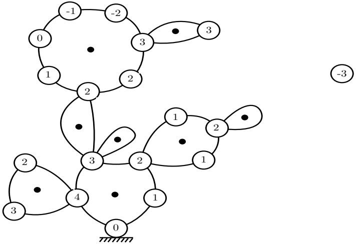

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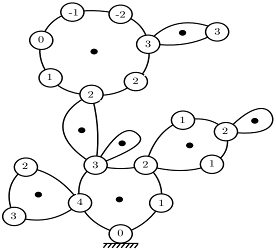

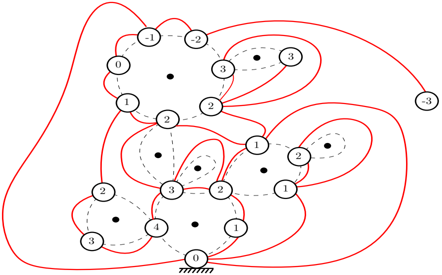

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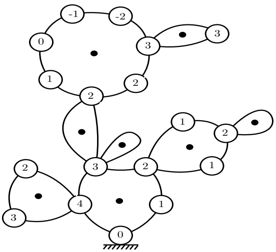
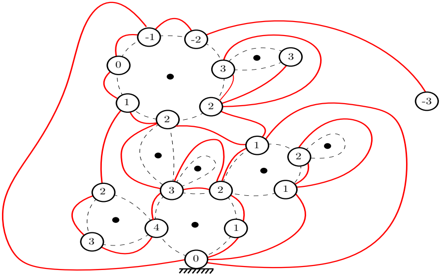
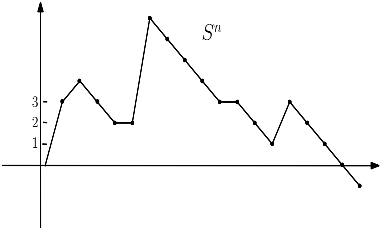
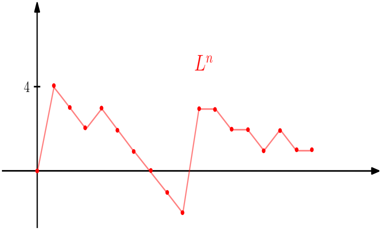




Lukasiewicz path

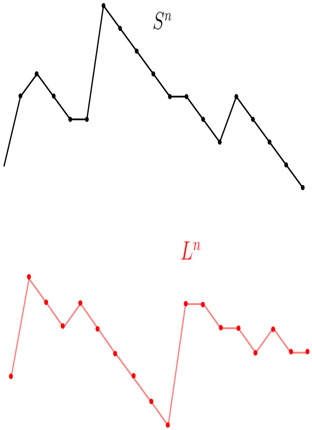
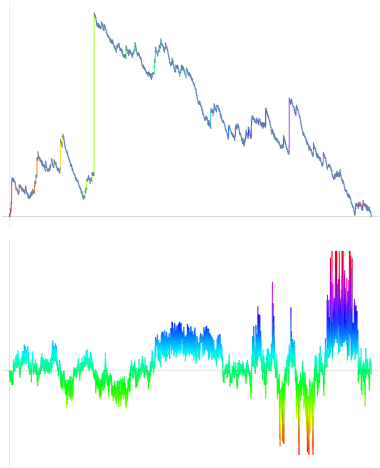


Label path



The scaling limits of the encoding processes

Scaling limits (Le Gall - Miermont)

Discret	**Continuous**
	

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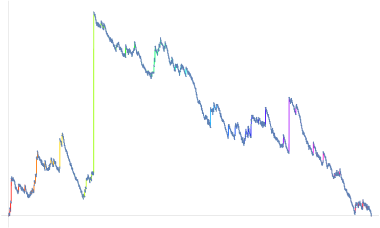

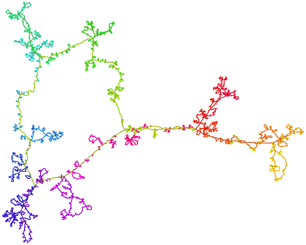

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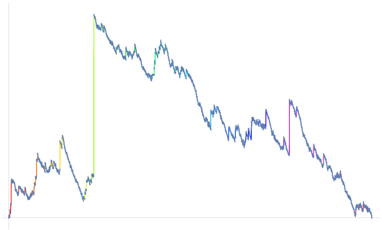

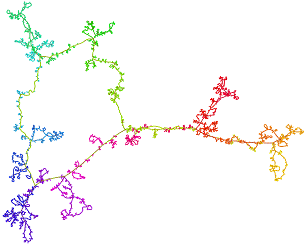

Roughly speaking, the jumps encode the loops. We write $s \sim_X t$ if s and t are the same points for the loop tree.

Second building block

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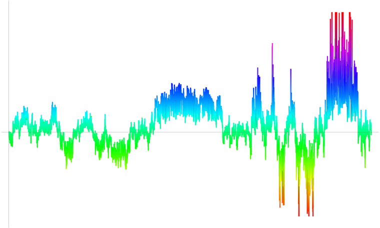

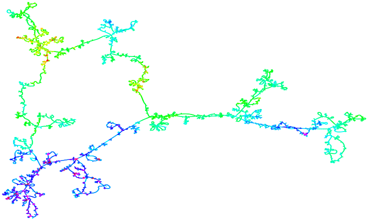

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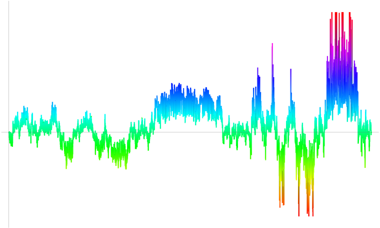

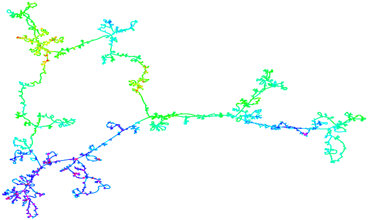

Some properties of Λ

Cyclically invariant

Distinct local miminum

Strong control of its module of continuity

Markovian with respect to the loop tree.

Construction of $\mathcal{S}_\alpha$

* For every $s, t \in [0, 1]$, we introduce the quantity :

$$\Delta_\alpha^\circ(s, t) := \Lambda_s + \Lambda_t - 2\max\Big(\min_{[s,t]} \Lambda, \min_{[t,s]} \Lambda\Big),$$

where we write $[a, b] = [0, b] \cup [a, 1]$ if $b \leqslant a$.

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* We then define Δ_α as the biggest pseudo-distance on $[0,1]$, such that :

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* We write $s \sim_{\Delta_\alpha} t$ if and only if $\Delta_\alpha(s, t) = 0$.

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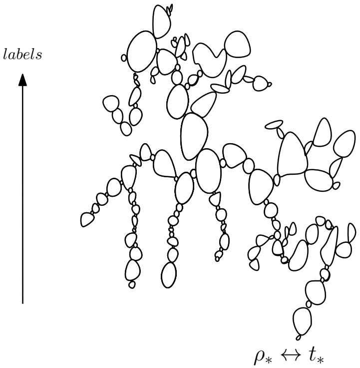

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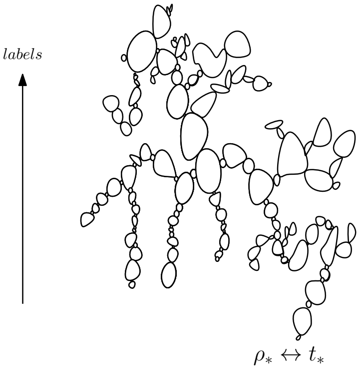

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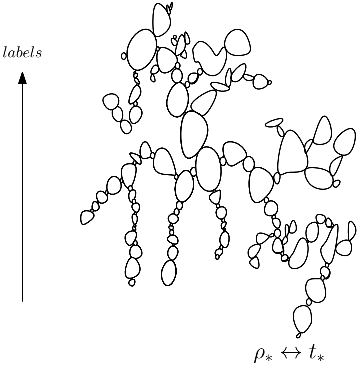

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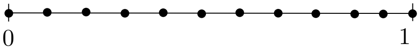
The Lebesgue measure induces a volume measure.

III. Some ideas of the proof : Topology

A useful representation

$\left(\mathfrak{M}_n, \mathrm{d}_{\mathrm{gr}}^{M_n}\right)$

$\{1, ..., n\}$ vertices in clockwise order



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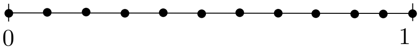

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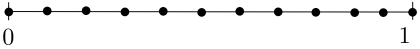

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Convergence of $n^{-\frac{1}{2\alpha}} \cdot \mathfrak{M}_n \iff$ Convergence of $n^{-\frac{1}{2\alpha}} \cdot d_n$

Topology of $\mathfrak{M}_n \iff \big\{(s, t) : \ d_n(s, t) = 0\big\}$.

Topology

$n^{-\frac{1}{2\alpha}} \cdot \mathrm{d_n} \to \mathrm{D}$ a.s. in $\mathcal{C}([0,1]^2, \mathbb{R}_+)$ (at least along a subsequence).

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The following equivalence hold a.s.

$$D(s,t) = 0 \iff \Delta_\alpha(s,t) = 0 \iff \begin{cases} s \sim_X t \\ \text{or} \\ \Delta_\alpha^\circ(s,t) = 0. \end{cases}$$

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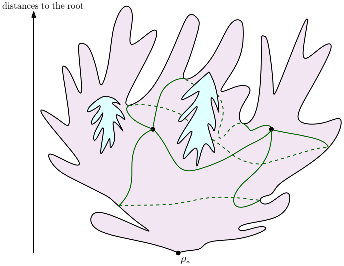
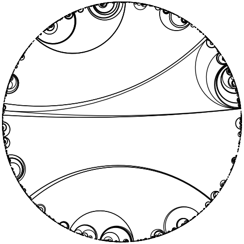


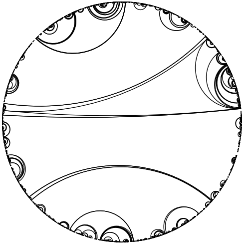
Points identification



Points identified by $\sim_X$

Points identified by Δ_α°

Points identification



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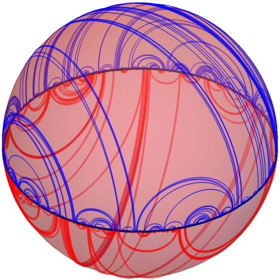
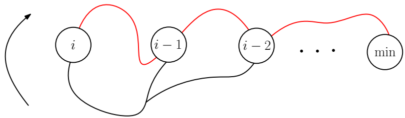


IV. Some ideas of the proof : Geodesics

Simple geodesics

A special type of geodesics in the discrete :



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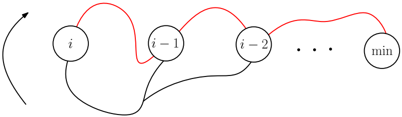

In the continuum : we construct geodesics to ρ_* following the running infimum.

Simple geodesics

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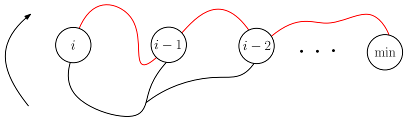

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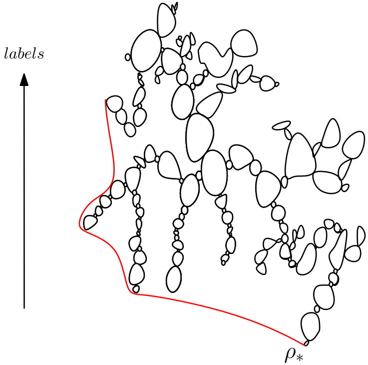

- All the geodesics to ρ_* are simple geodesics

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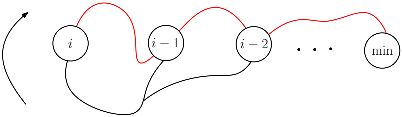

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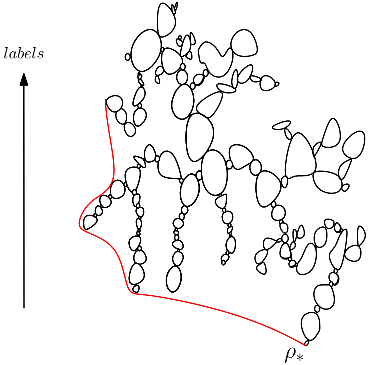

- All the geodesics to ρ_* are simple geodesics
- There is only one geodesic from a leave to the root

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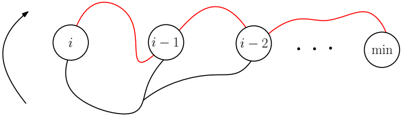

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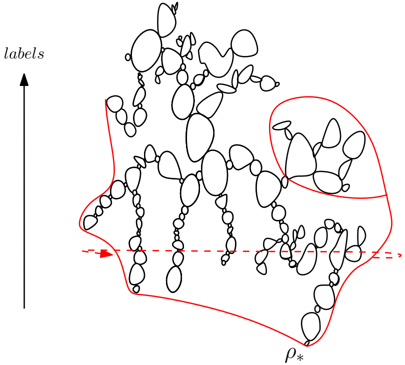

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- There are two geodesics from points with multiplicity 2

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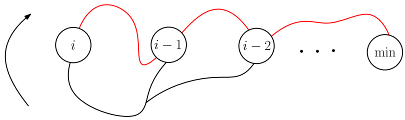

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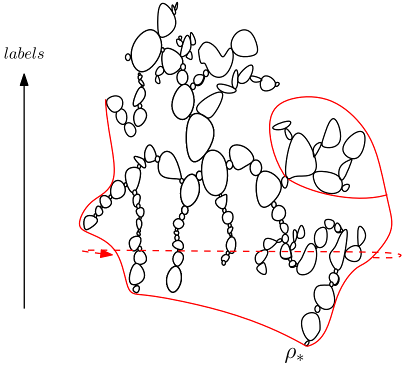

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- Simple geodesics hit the faces

Geodesic trap

Δ_α = Length of paths obtained by concatenation of simple geodesics.

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Our theorem is equivalent to show that all D-geodesic are determined by :

The topology and Simple geodesic.

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Typical local landscape of D-geodesic :

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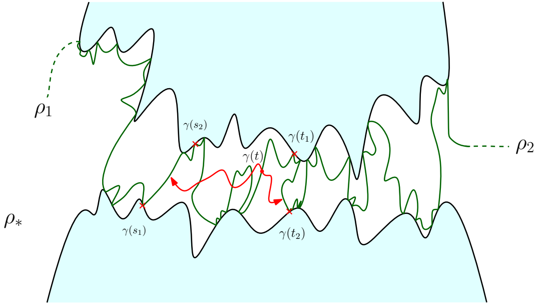

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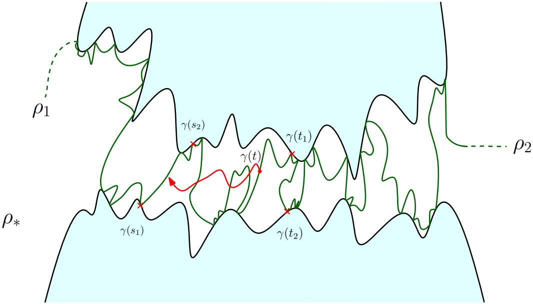

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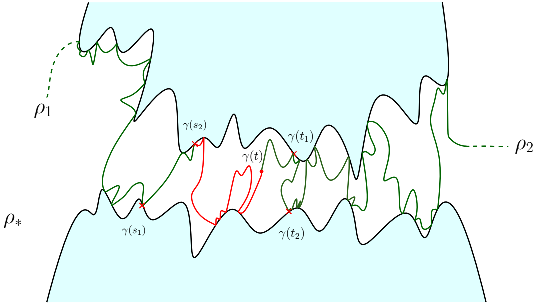

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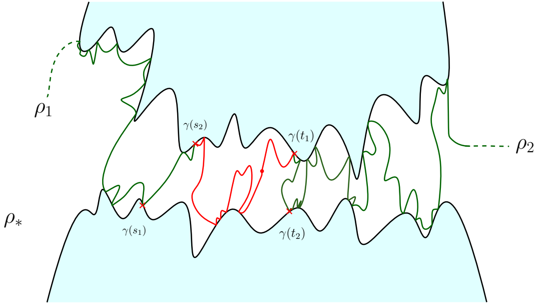

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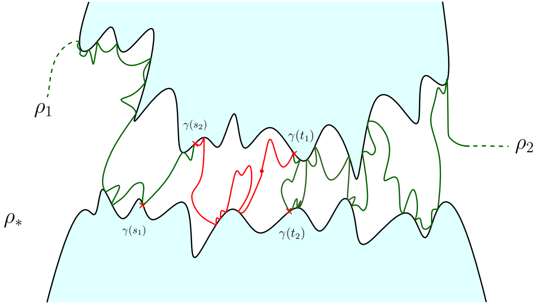

To conclude we need to show that the complement of these points has dimension smaller than 1.

Thank you for your attention