Maps of unfixed genus and blossoming trees

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joint work with Emmanuel Guitter

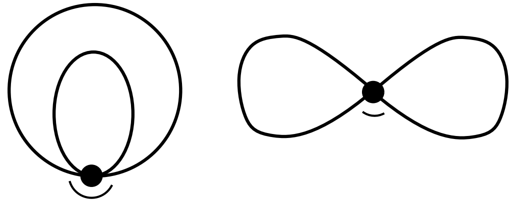
Random Geometry, Jan. 19 2022

Original observation/question

The generating function of rooted 4-regular planar maps is $R_1(t)$, with $R_1(t), R_2(t), \ldots$ solutions of the system (with $R_0 = 0$)

$$\boxed{R_i(t) = 1 + t\, R_i(t)(R_{i-1}(t) + R_i(t) + R_{i+1}(t)) \qquad i \geq 1} \qquad (\star)$$

$$R_1(t) = 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \cdots$$



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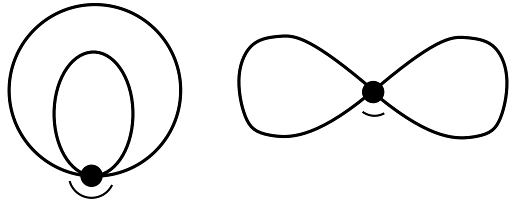

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tools: bijections

($R_i(t) =$ 2-point function of planar quadrangulations)

(Rk: simpler expression $R_1(t) = \sum_{n\geq 1} \frac{2}{n+2} 3^n \frac{(2n)!}{n!(n+1)!} t^n$)

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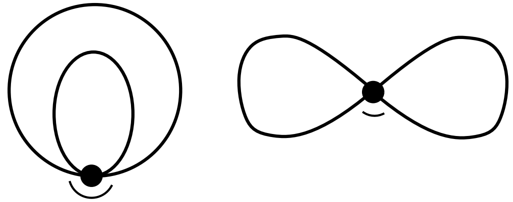

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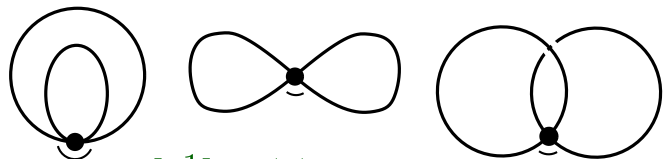

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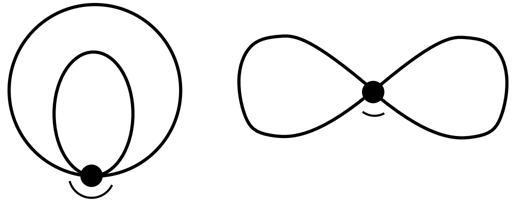

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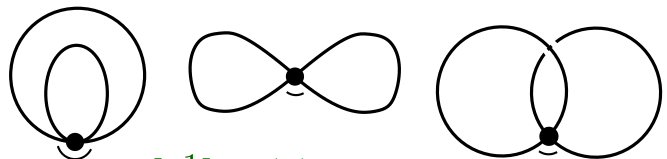

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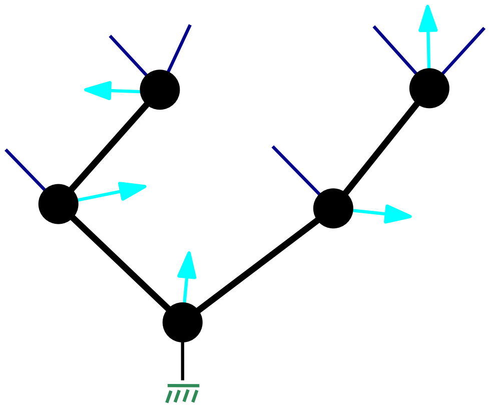
Question: bijective interpretation of $(\star\star)$? (unified with $(\star)$?)

Planar case

Blossoming trees

Blossoming tree $=$ rooted binary tree where each node carries an arrow, called a bud



$$\#\text{ blossoming trees on } n \text{ nodes} = 3^n \frac{(2n)!}{n!(n+1)!}$$

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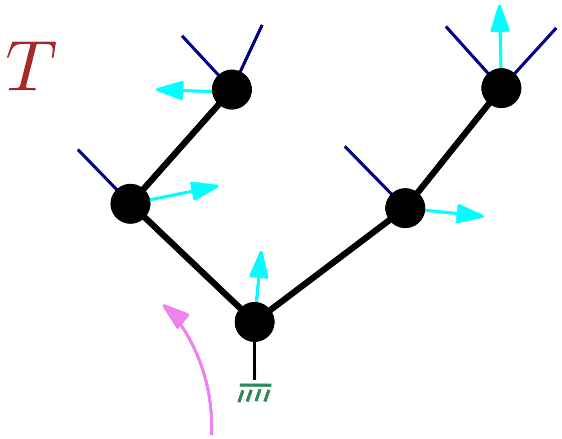

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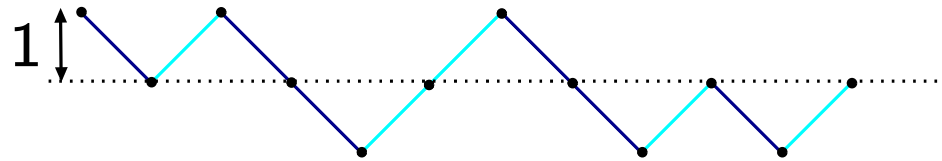
leaf-path $w(T) =$



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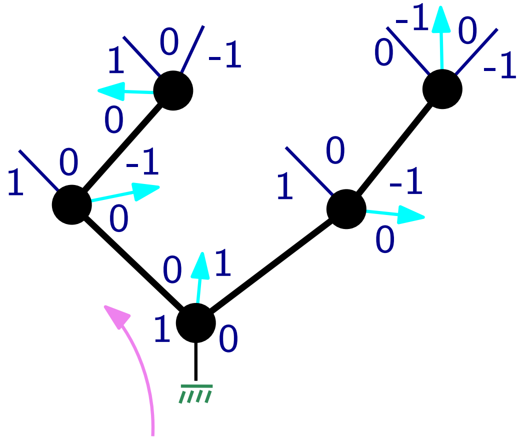
T



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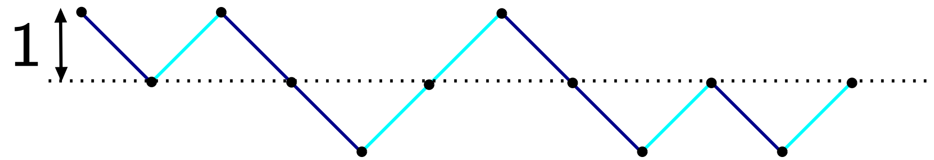

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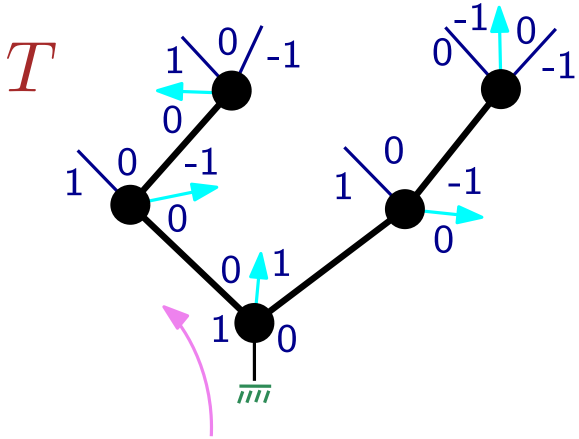

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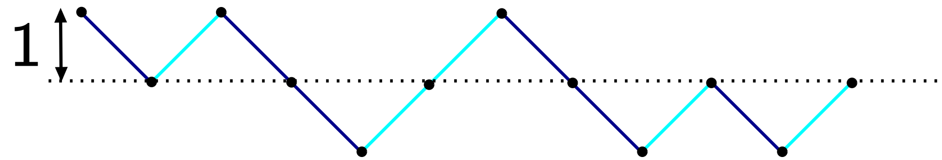
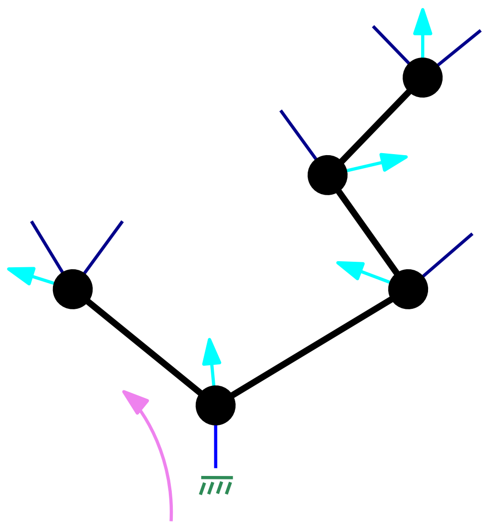
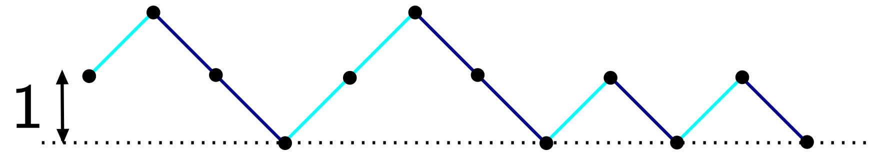


A blossoming tree T is balanced if $w(T)$ does not go below the x-axis





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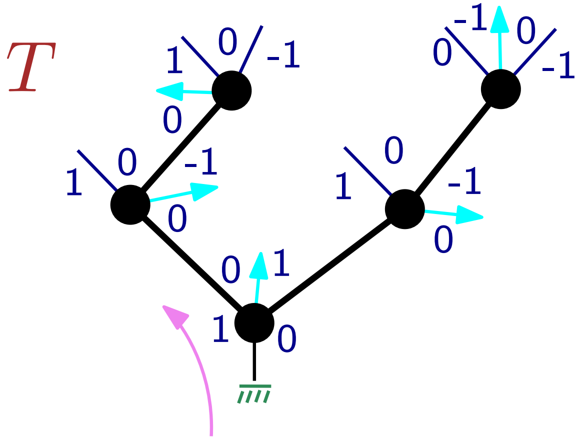

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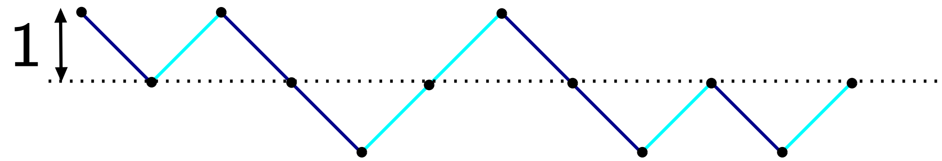

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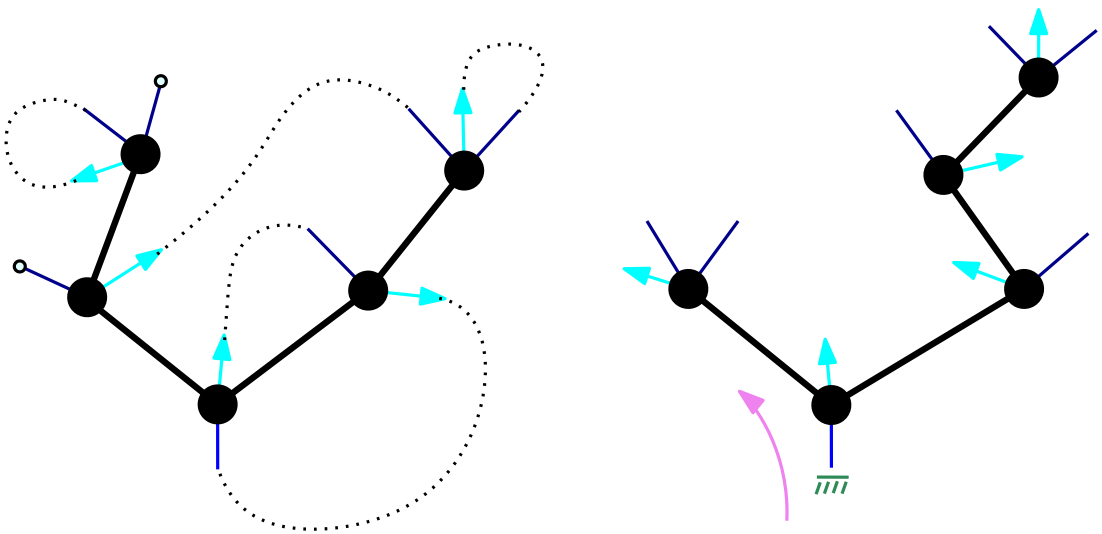

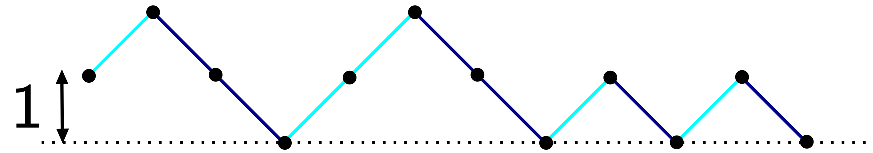

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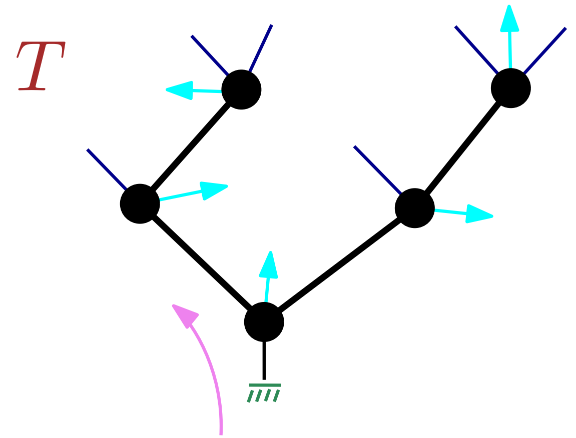

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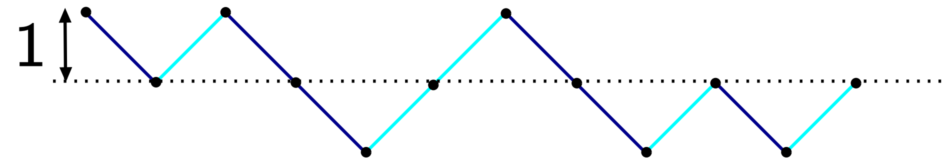

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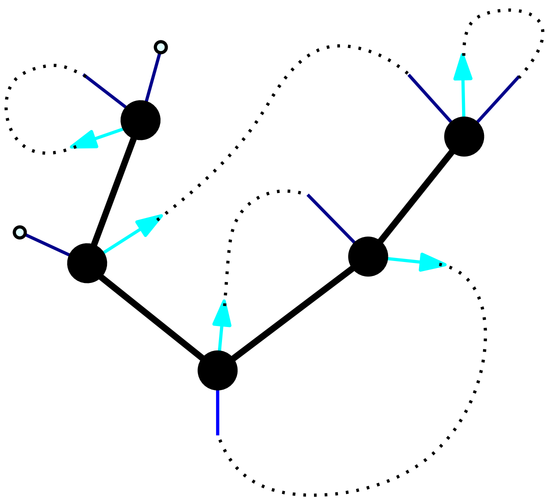
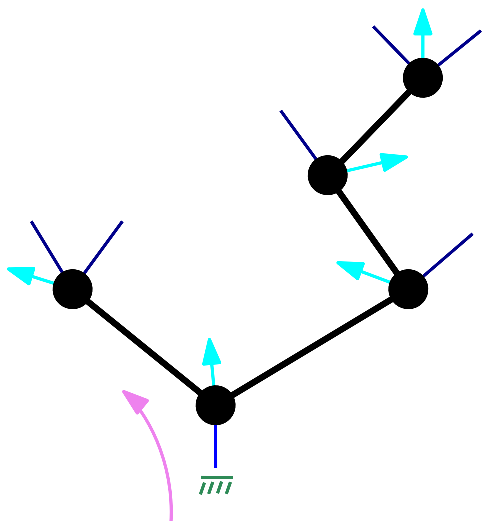
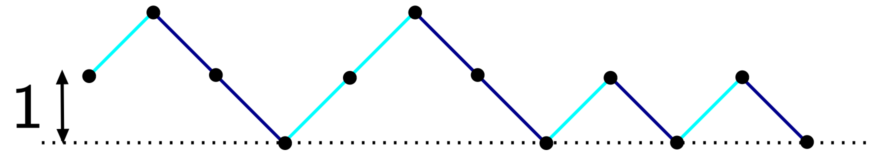


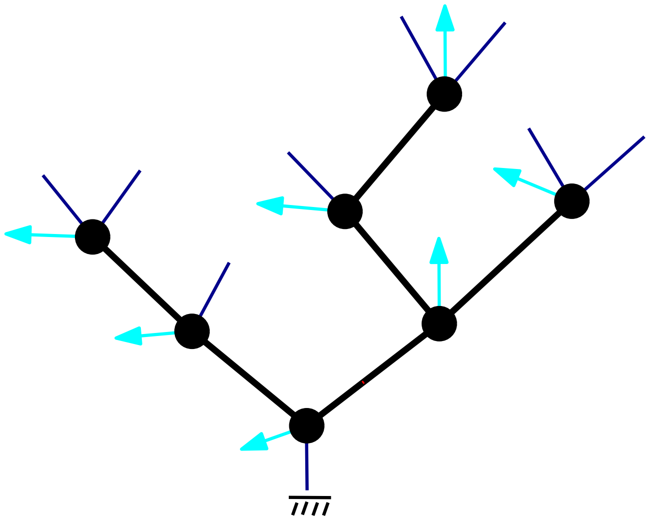


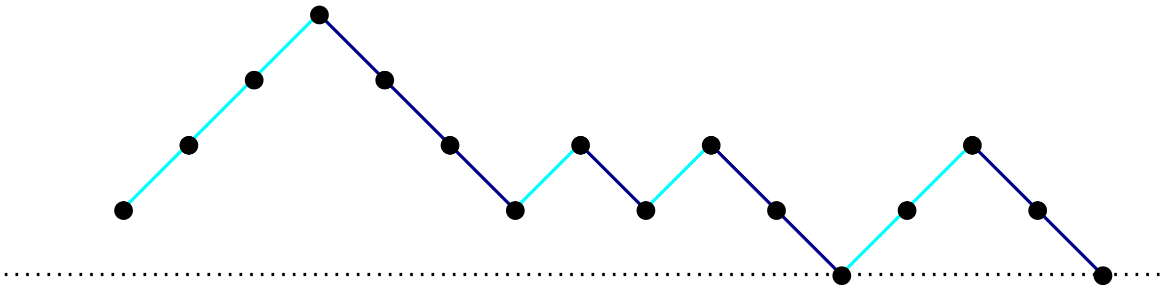


There are $\dfrac{2}{n+2} 3^n \dfrac{(2n)!}{n!(n+1)!}$ balanced blossoming trees on n nodes

Balanced blossoming tree $\rightarrow$ rooted 4-regular planar map

[Schaeffer'97]





leaf-path

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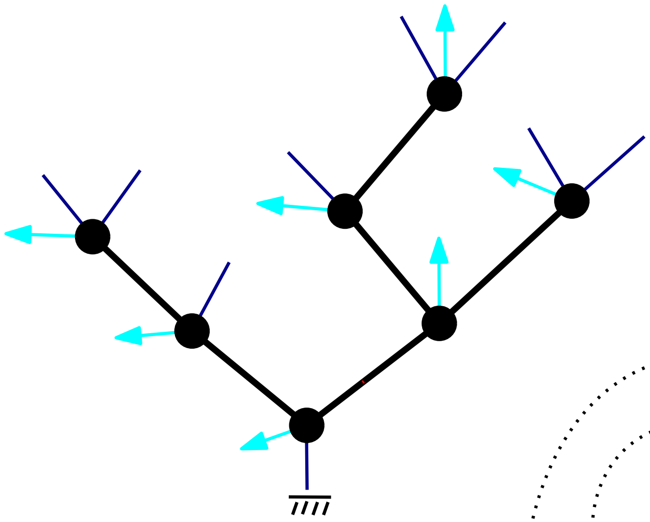

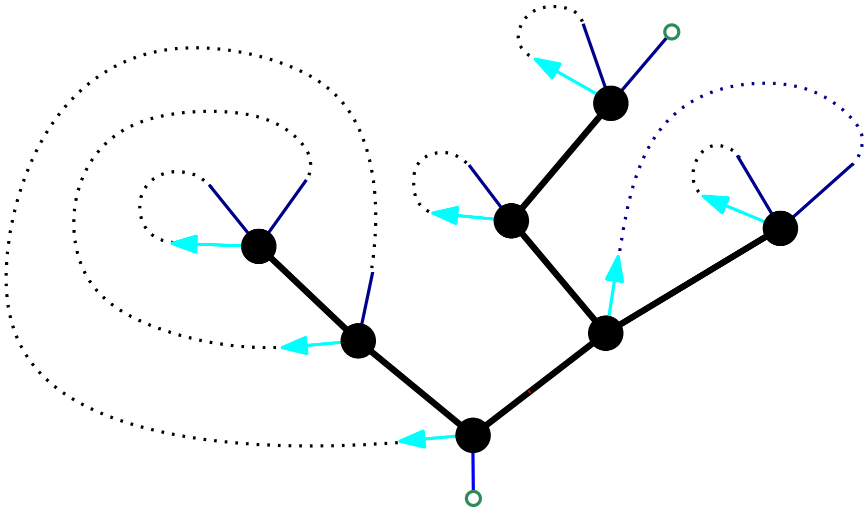

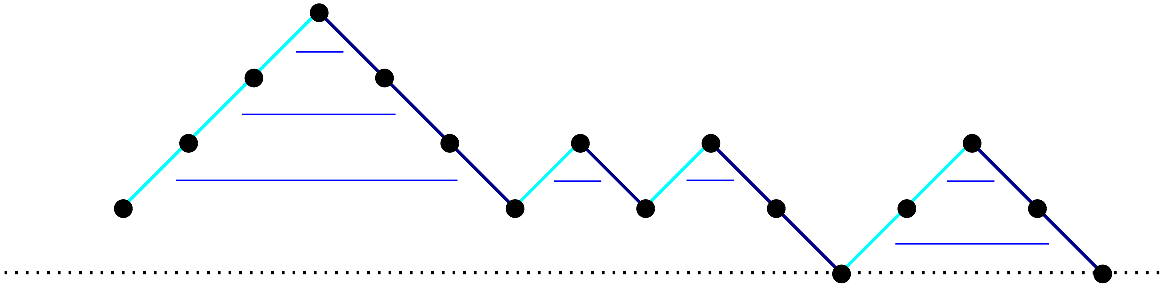

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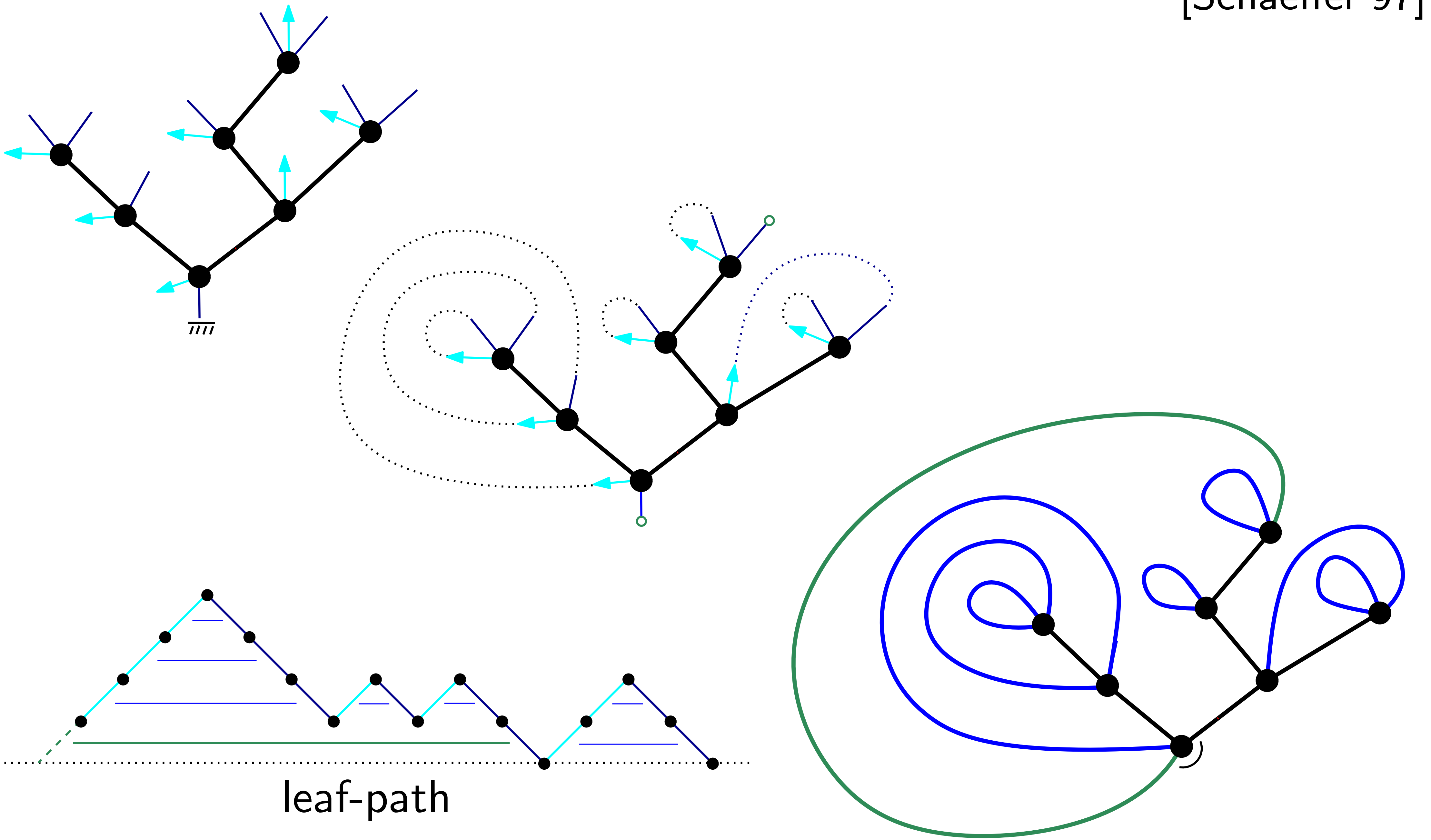

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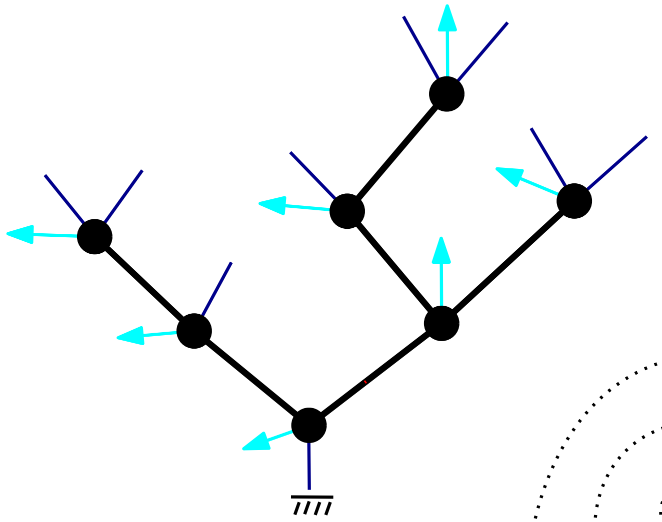
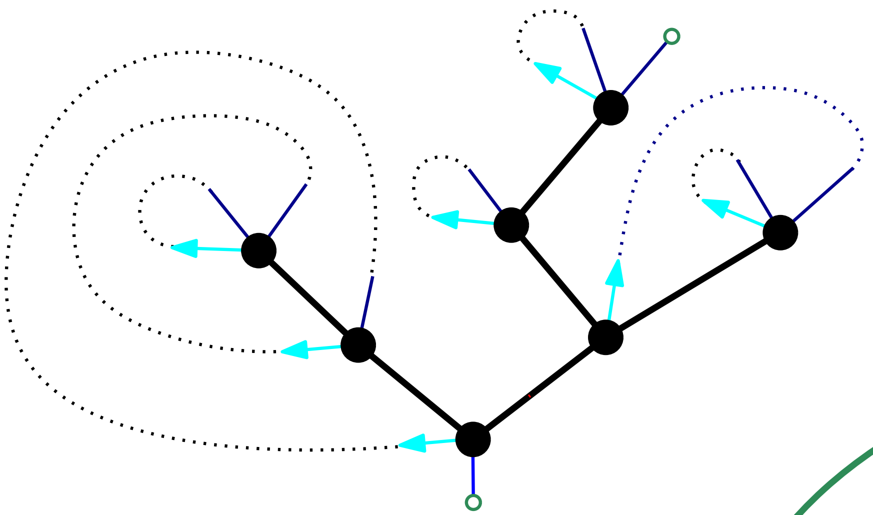
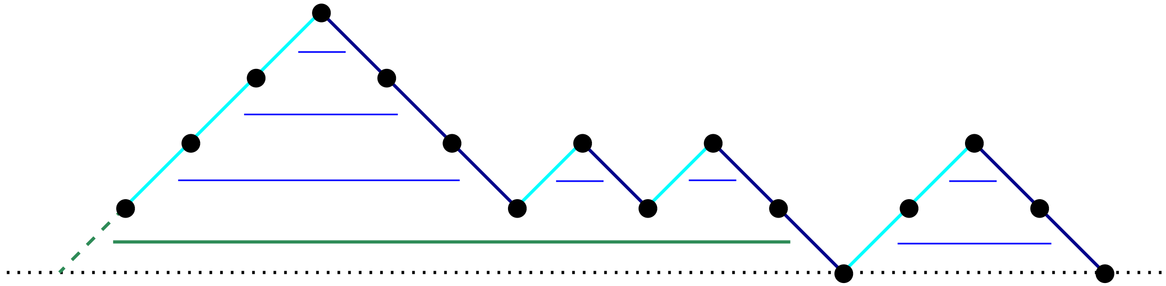
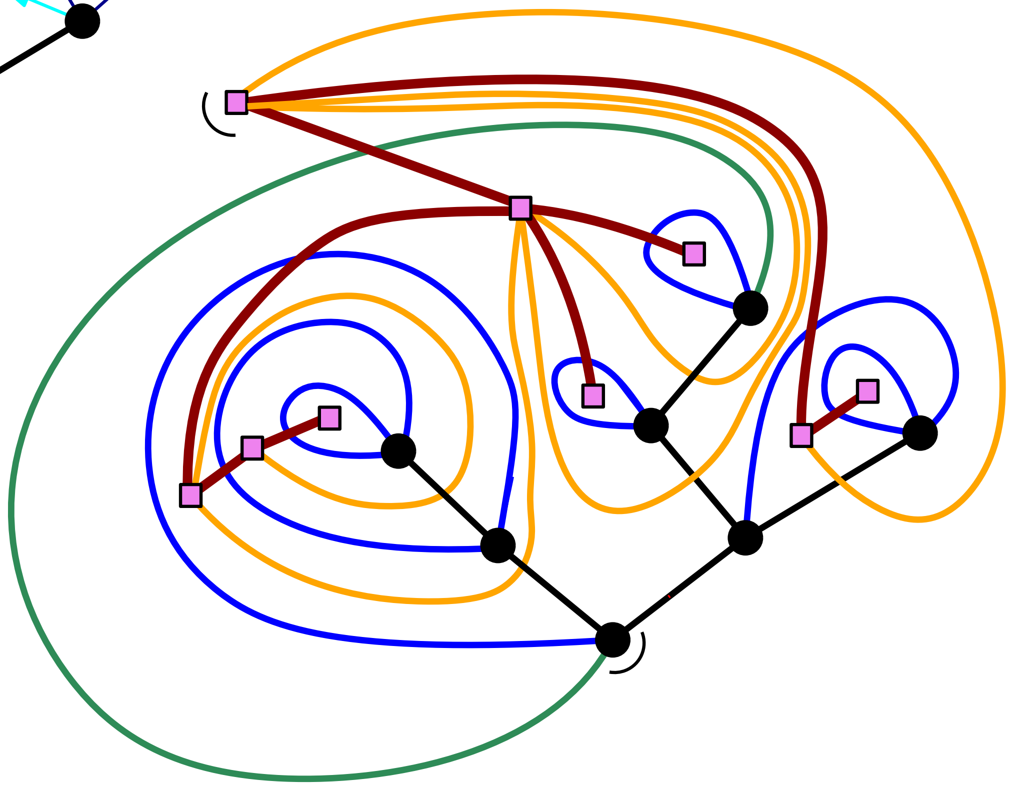






leaf-path

Inverse mapping
cut the edges dual to those in
the leftmost BST of the dual map



Balanced blossoming tree $\to$ rooted 4-regular planar map

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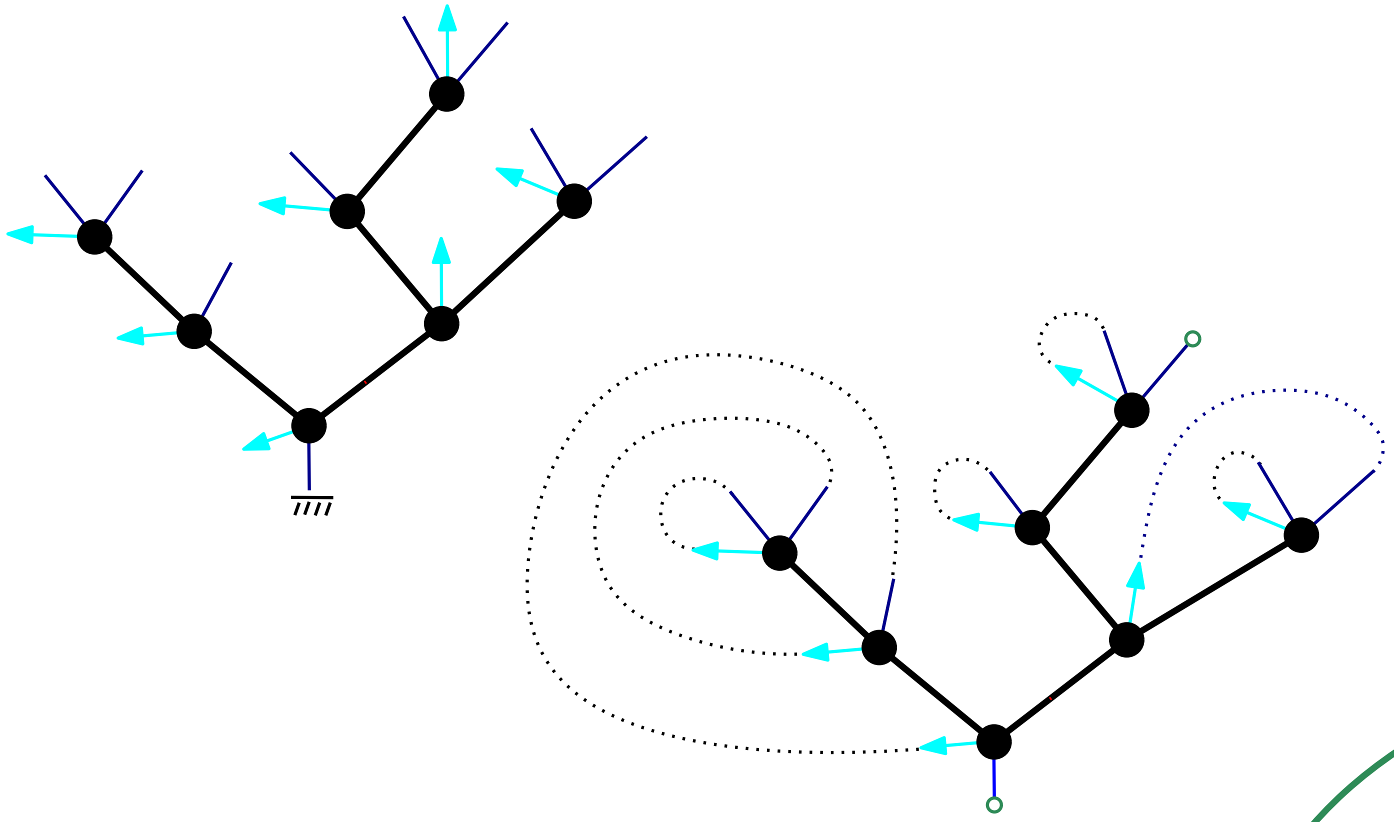

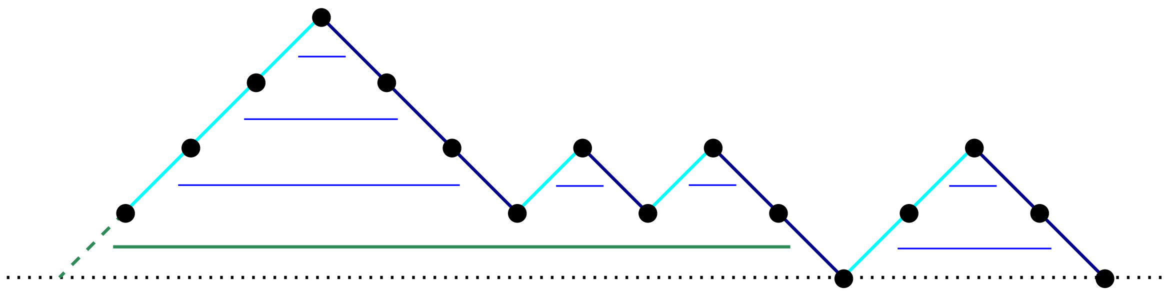

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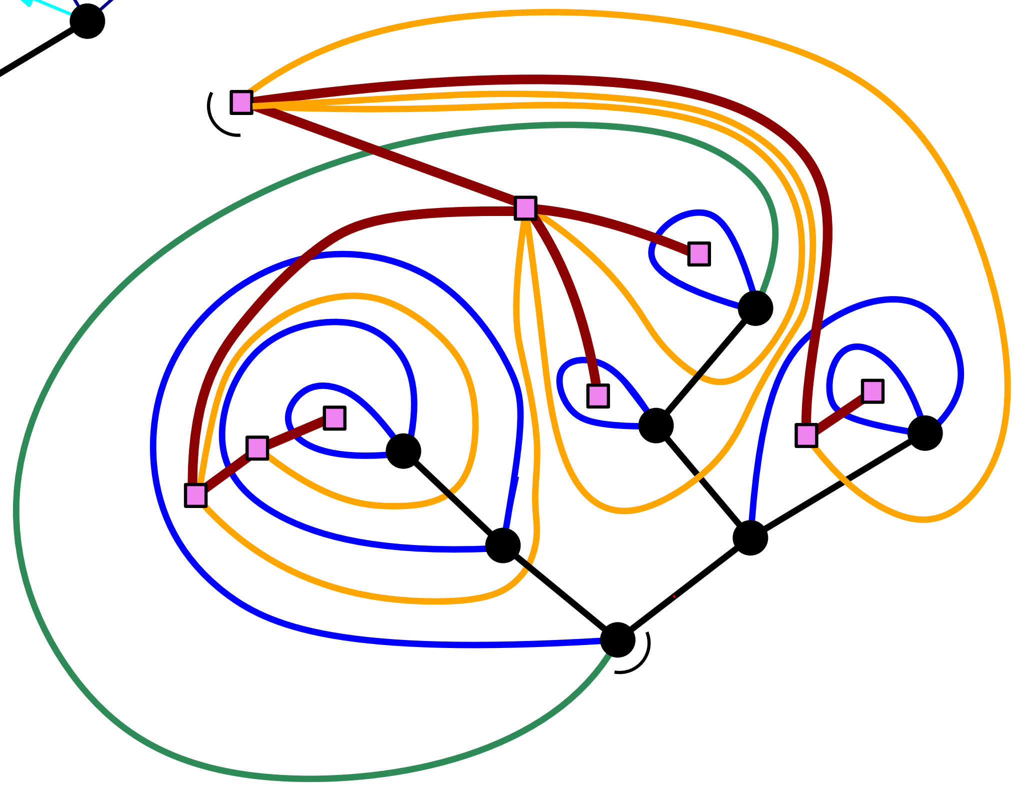
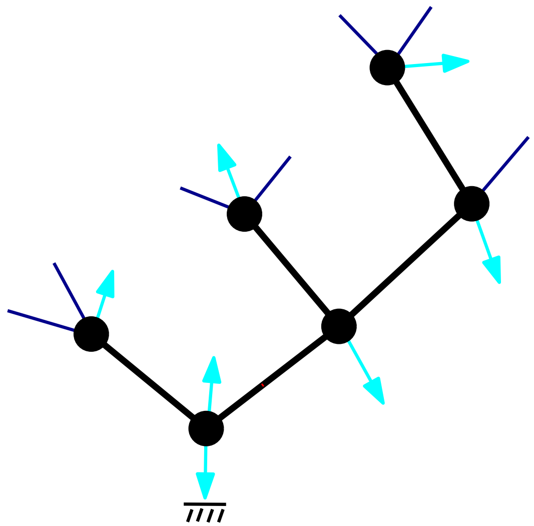


$\Rightarrow$ there are $\frac{2\cdot 3^n (2n)!}{n!(n+2)!}$ rooted 4-regular planar maps on n vertices

Blossoming tree → bi-pointed 4-regular planar map

[Bouttier-Di Francesco-Guitter'03]



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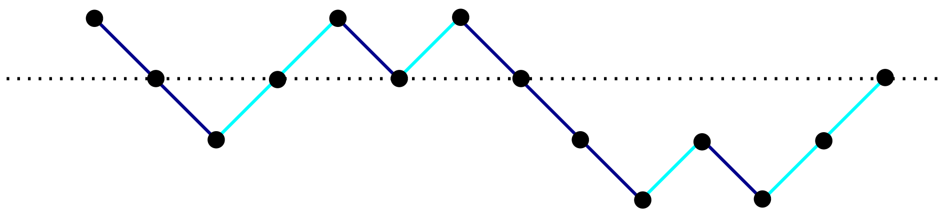

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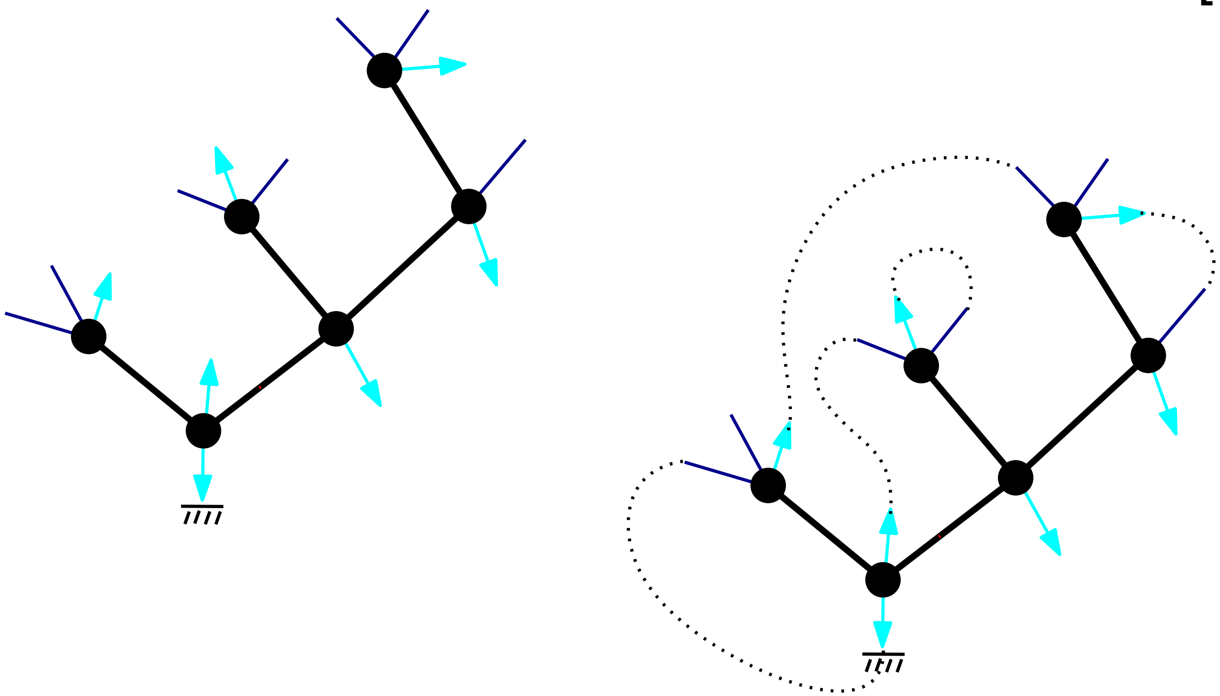

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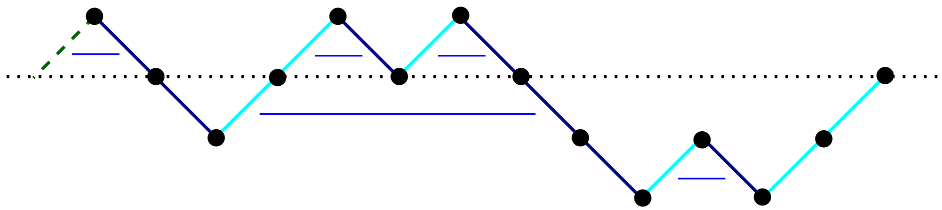

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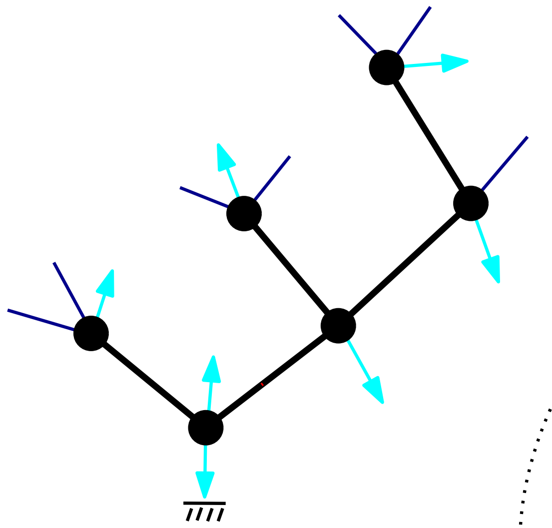

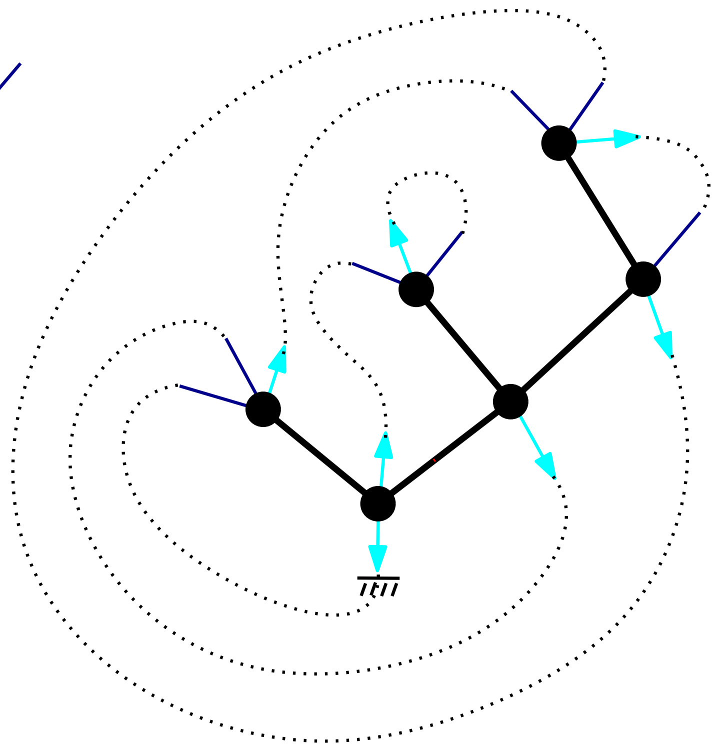

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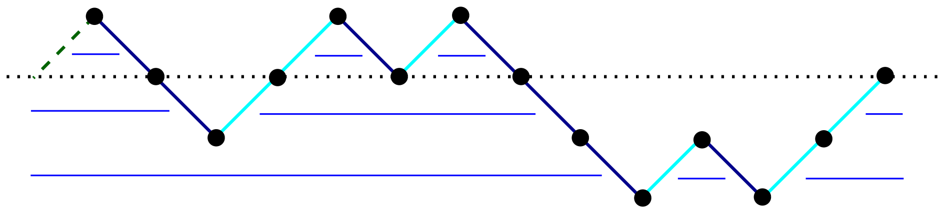

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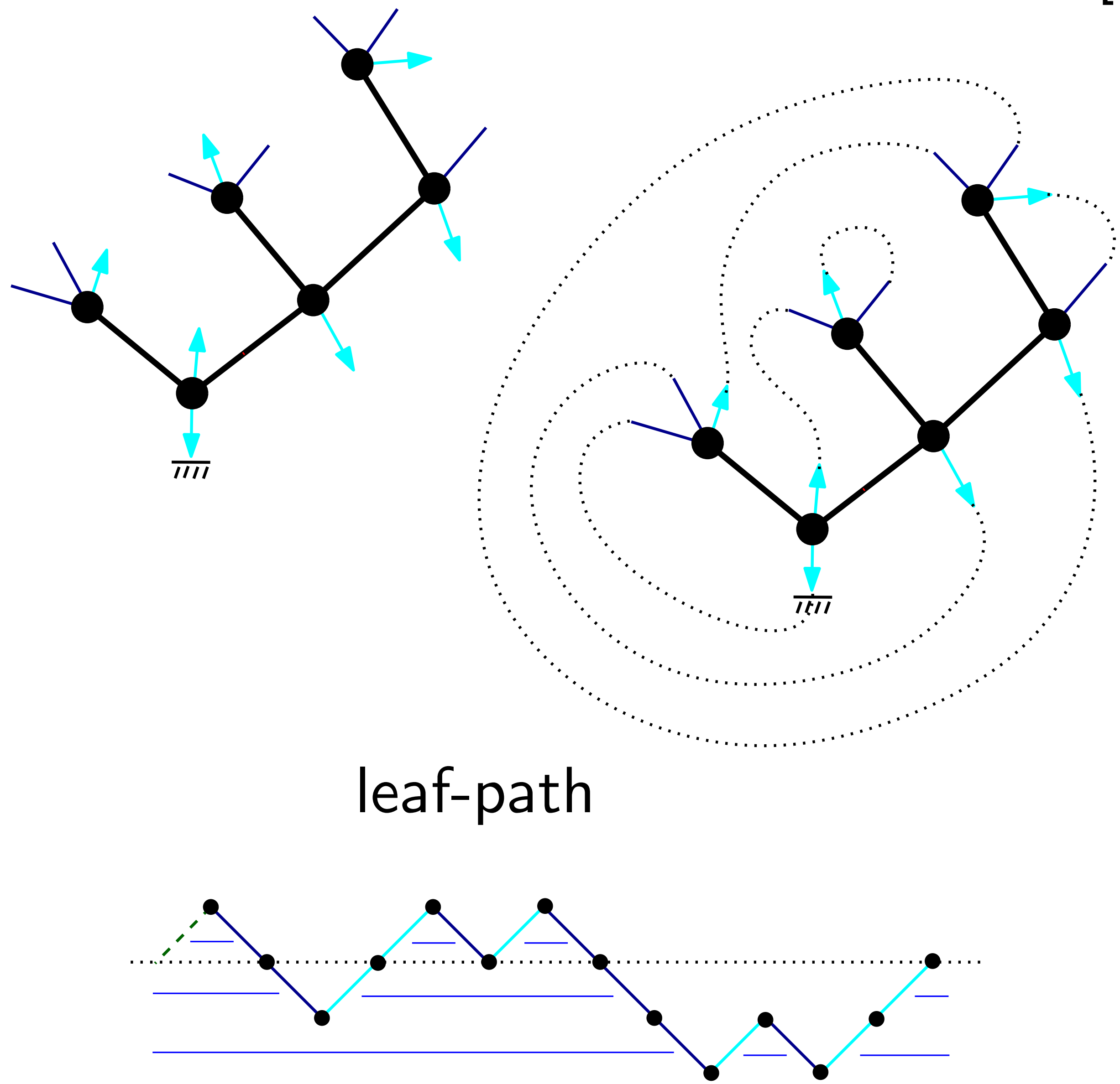

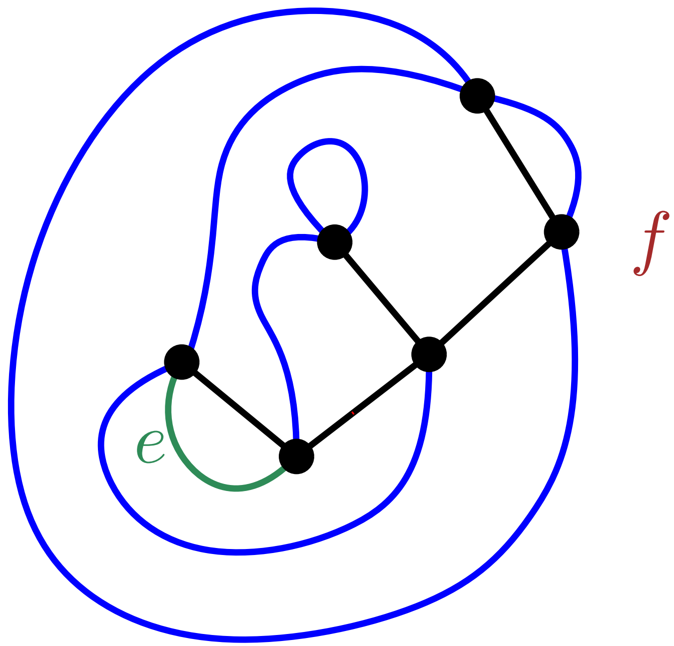

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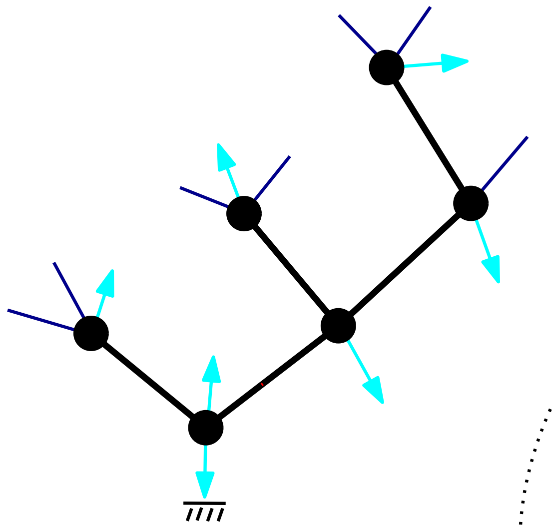
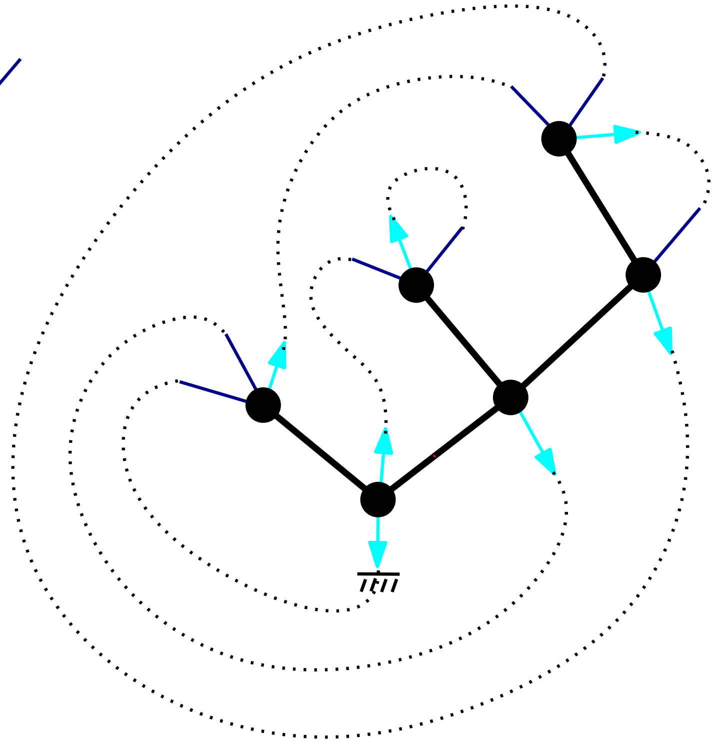
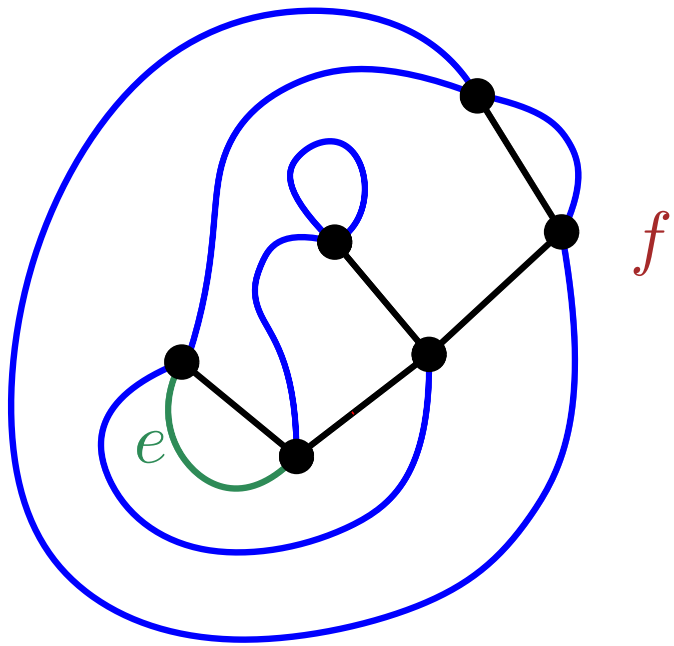




bi-pointed 4-regular map

(marked face f +marked edge e)



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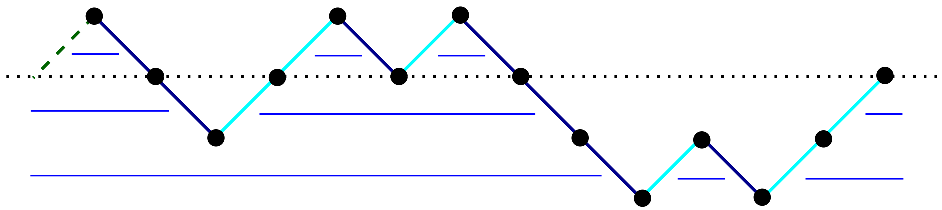

Bijection gives $\quad 3^n \mathrm{Cat}_n = \dfrac{n+2}{2} \cdot \#$ rooted 4-regular maps

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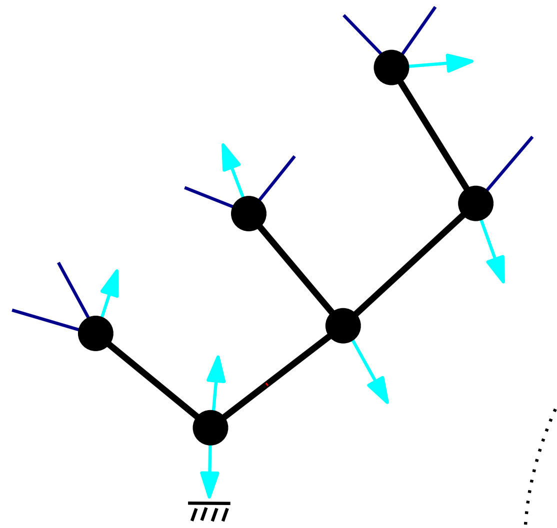

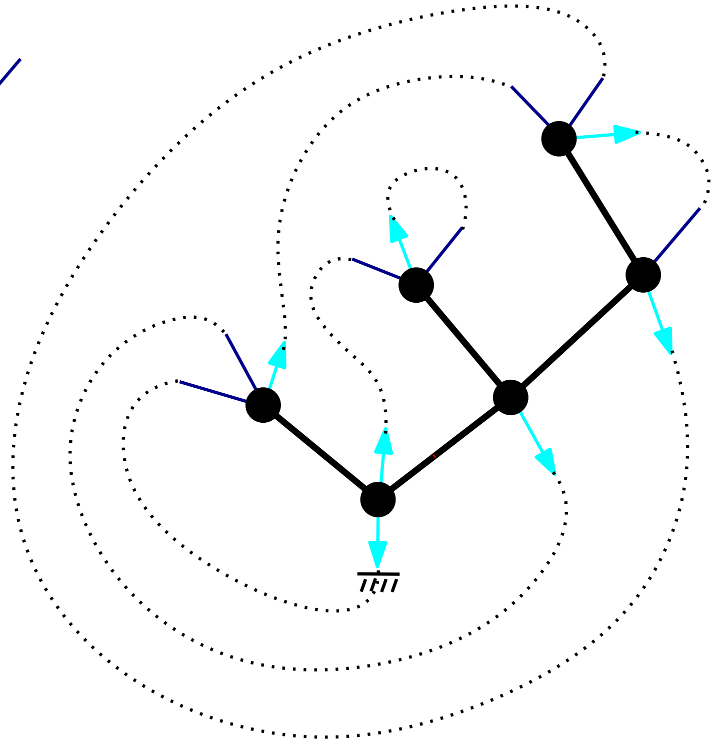

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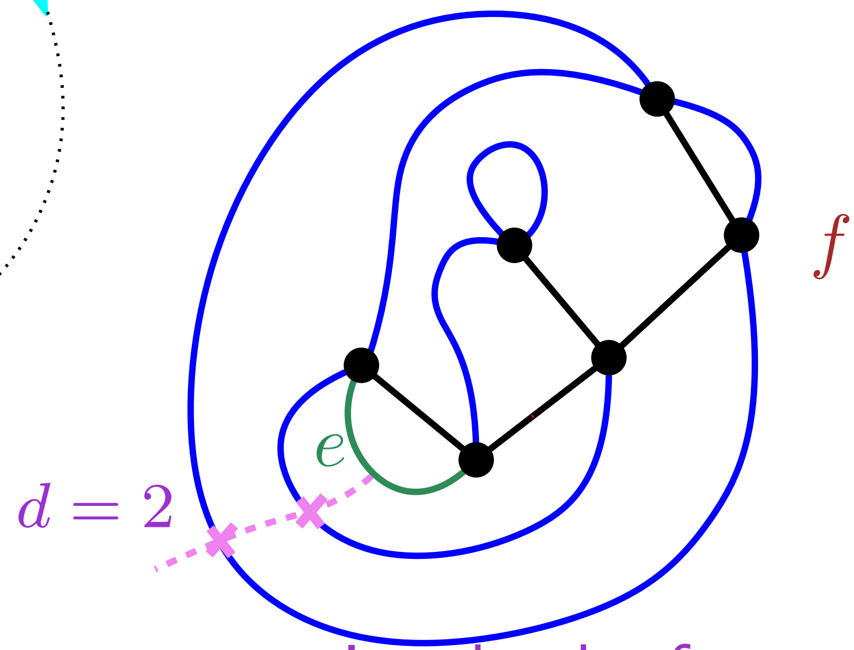

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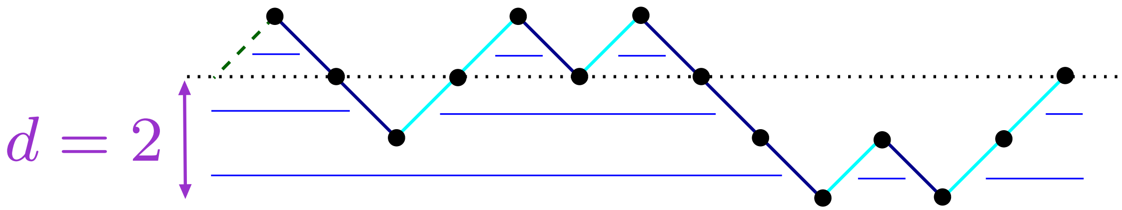

nesting depth of e
$=$ dual distance d from e to f

d is the depth ($-$ minimal level) of leaf-path

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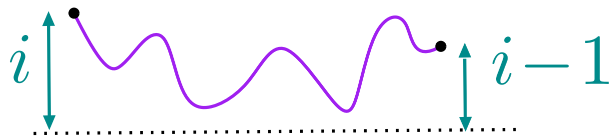
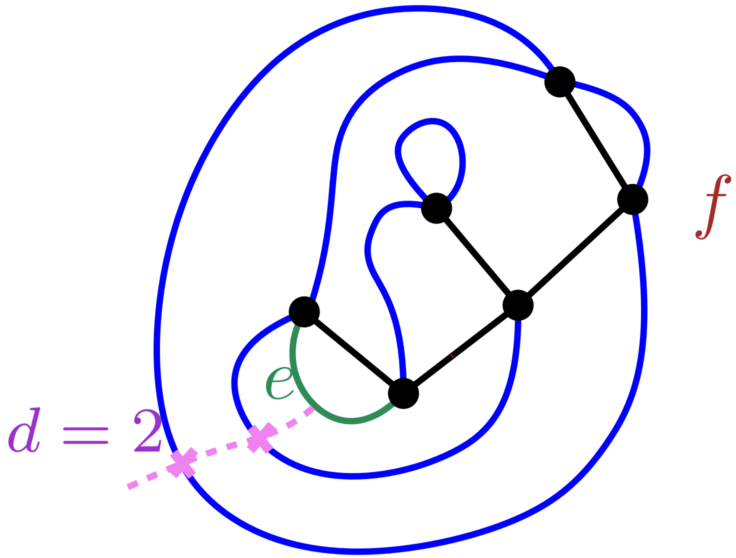
Two-point function $R_i(t)$

For $i \geq 1$ let $R_i(t)$ be the counting series of blossoming trees of depth $d < i$

$$\Updownarrow$$

leaf-path does not go below the x-axis when vertically shifted to start at height i





$R_i(t)$ counts those maps with dual distance $d < i$

Two-point function $R_i(t)$

For $i \geq 1$ let $R_i(t)$ be the counting series of blossoming trees of depth $d < i$

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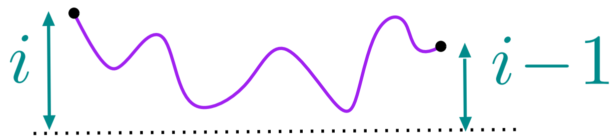

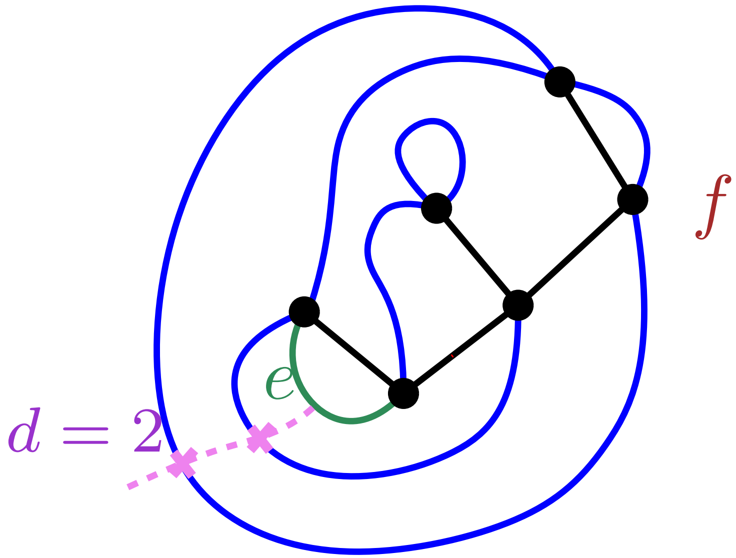

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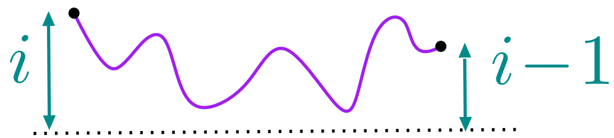

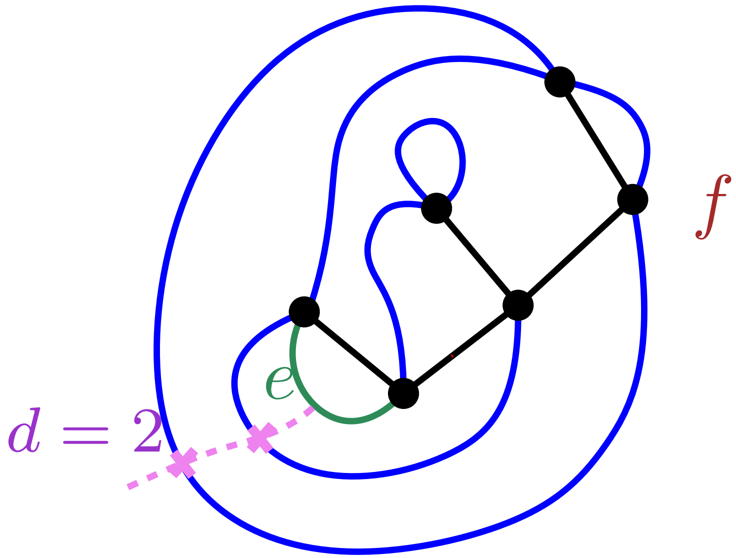

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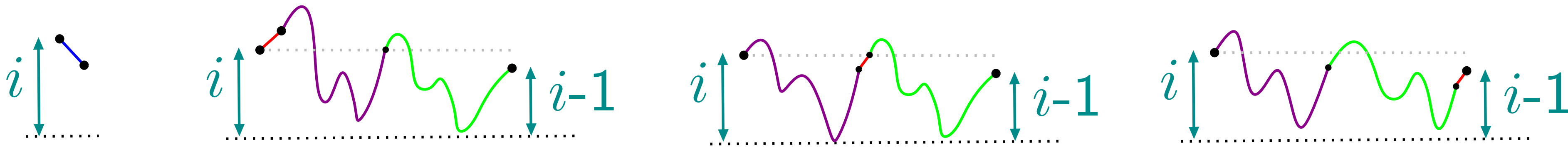
Equation:

$$R_i(t) \quad = \quad 1 \quad + \quad t\,R_{i+1}(t)R_i(t) \quad + \quad t\,R_i(t)R_i(t) \quad + \quad t\,R_i(t)R_{i-1}(t)$$

leaf-path



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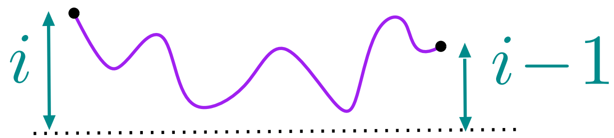

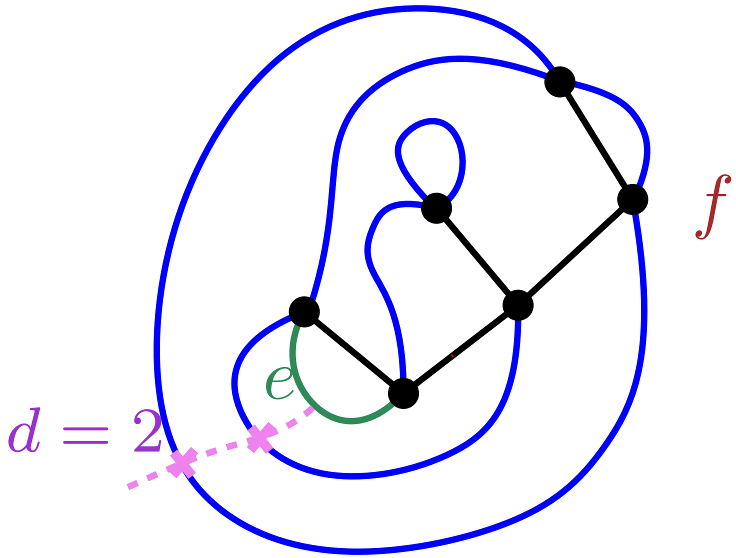

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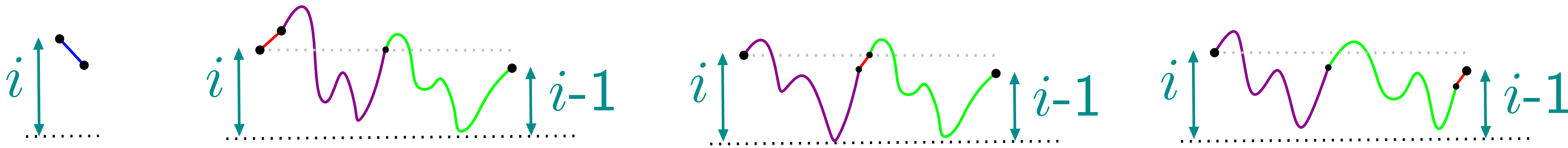
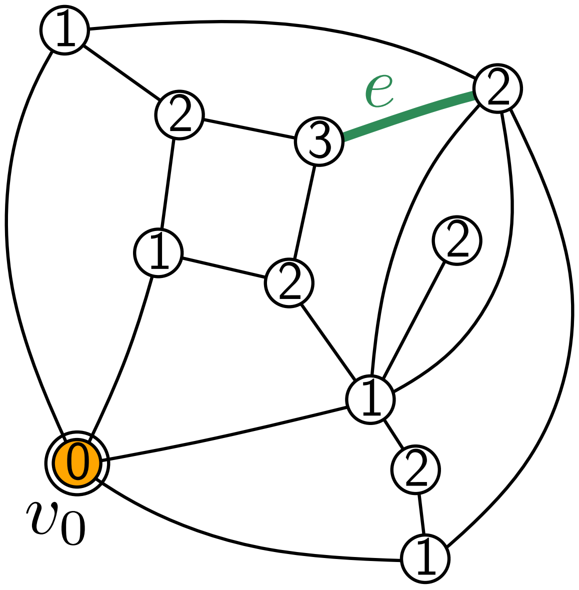
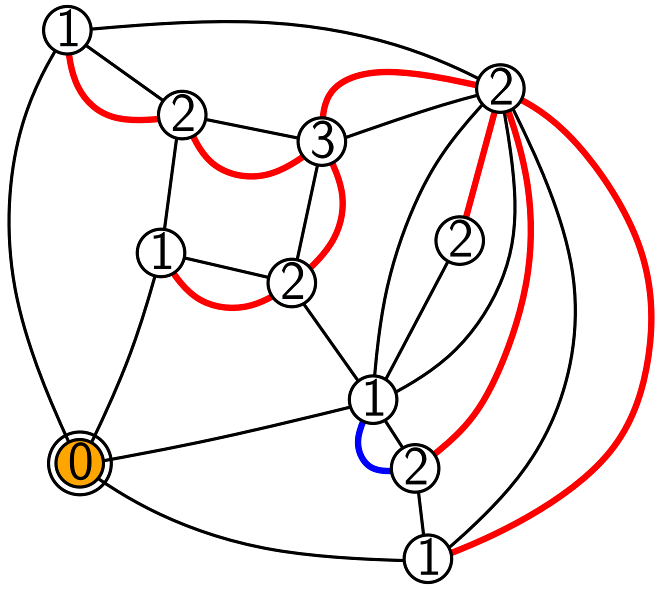


$R_0 = 0 \qquad \boxed{R_i(t) = 1 + tR_i(t)\cdot(R_{i-1}(t) + R_i(t) + R_{i+1}(t))} \qquad$ for $i \geq 1$

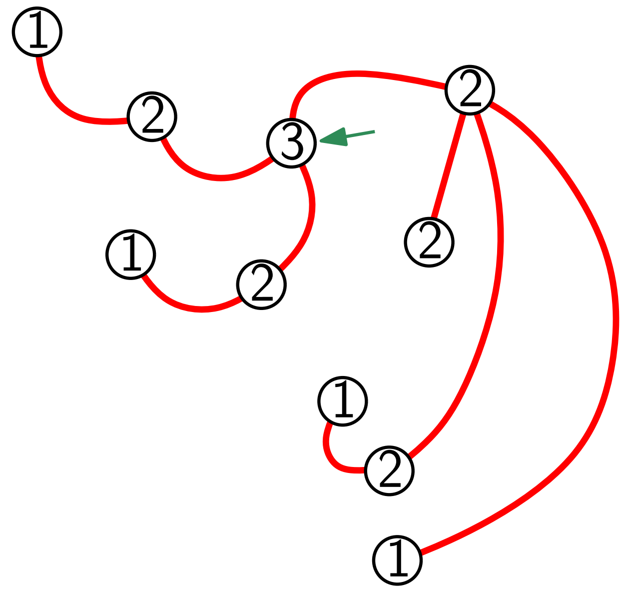
Cori-Vauquelin-Schaeffer bijection

n faces



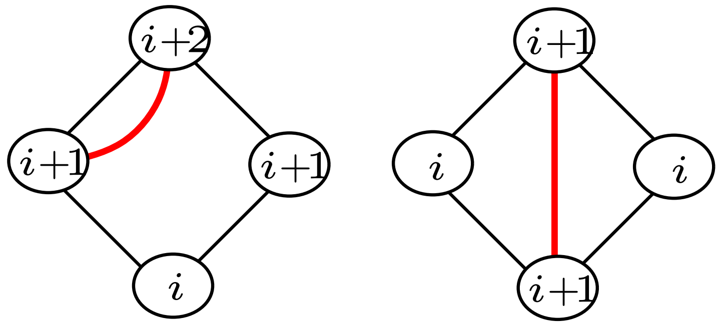


n edges



pointed quadrangulation
$+$ distance-labeling

Local rule in each face:



conditions:

- min-label $= 1$
- all edges satisfy

$|i - j| \leq 1$

Cori-Vauquelin-Schaeffer bijection

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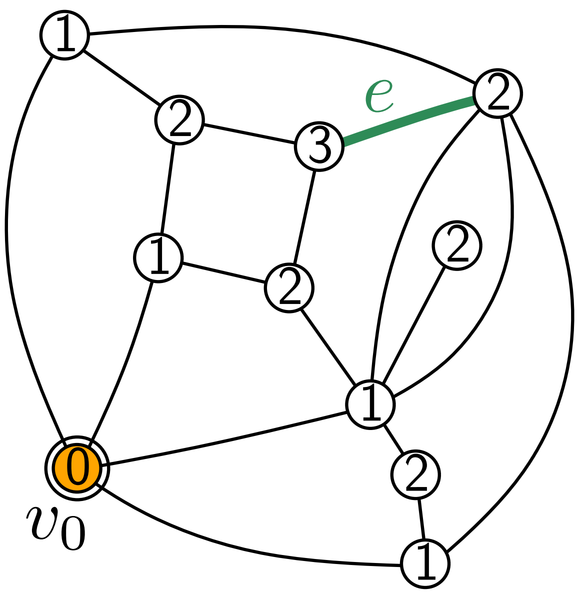

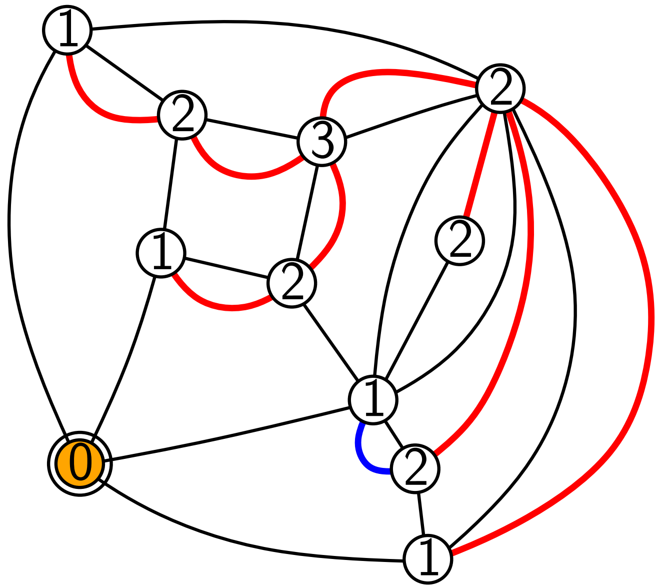

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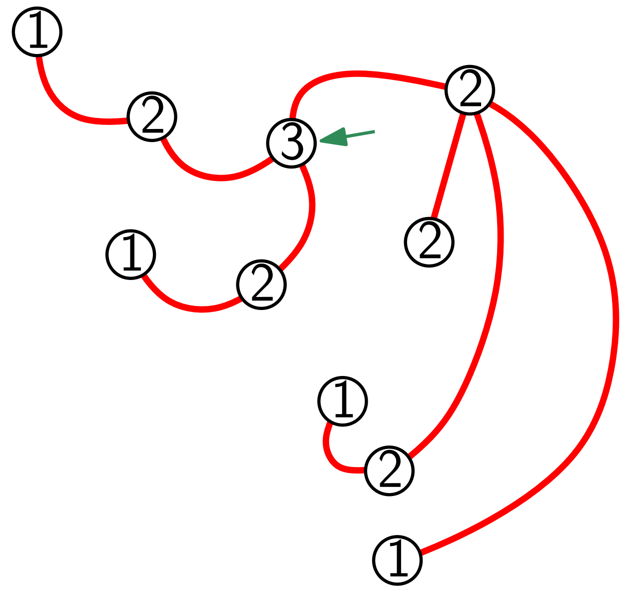

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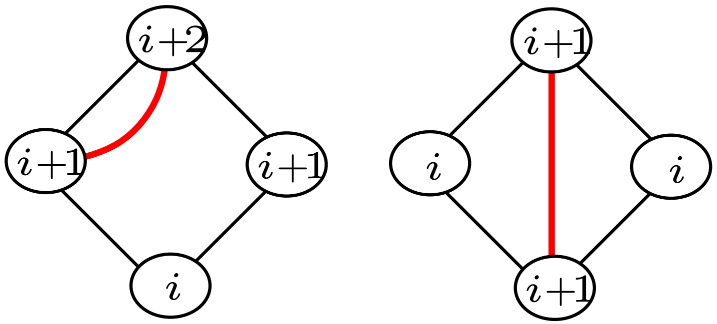

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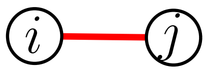
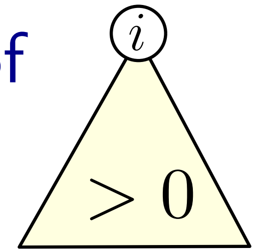
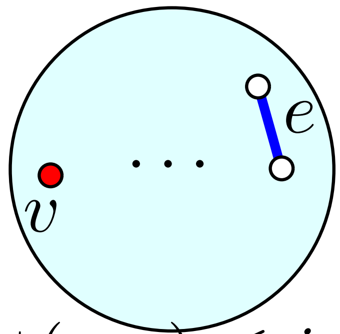
 $|i-j| \leq 1$

$R_i(t) =$ GF of



bijection yields $R_i(t) =$ GF of



$\mathrm{dist}(v, e) \leq i - 1$

Exact expression for $R_i(t)$ [Bouttier-Di Francesco-Guitter'03]

Let $R \equiv R_\infty$ given by $R = 1 + 3tR^2$

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Then $$\boxed{R_i = R \cdot \frac{(1 - x^i)(1 - x^{i+3})}{(1 - x^{i+1})(1 - x^{i+2})}}$$ for all $i \geq 1$

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Let X_n = distance(root-edge, random vertex) in Q_n

$$\boxed{\mathbb{P}(X_n \leq i) = \frac{[g^n]R_i(g)}{[g^n]R(g)}}$$

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$x(g) \sim \tau - c(1 - 12g)^{1/4} \quad \Rightarrow \quad X_n/n^{1/4}$ converges to explicit law

Unfixed genus

Classical counting approaches

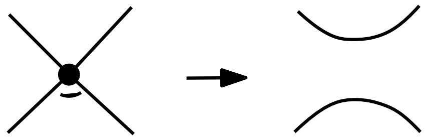
- **1st approach: deletion of root-vertex** v_0

cf [Arquès-Béraud'00, Vidal-Petitot'10, Courtiel-Yeats-Zeilberger'17]

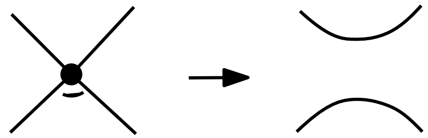
$$M(t) = 3t \; + \; 6t\, M(t) \; + \; \left(4t^2 \frac{\mathrm{d}}{\mathrm{d}t} M(t) - 2tM(t)\right) \; + \; t\, M(t)^2$$

2 loops at v_0

1 loop at v_0



still connected



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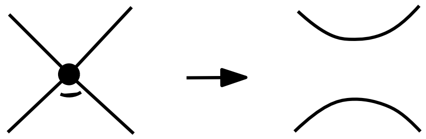
2 loops at v_0 (for $3t$)

1 loop at v_0 (for $6t\,M(t)$)



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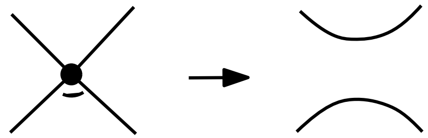
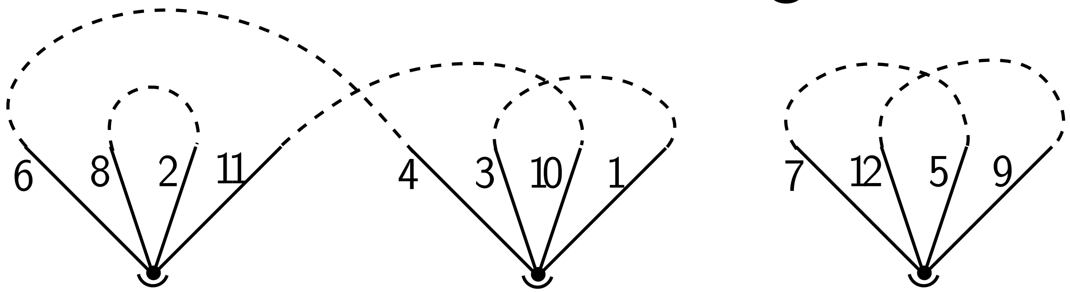


disconnected

• 2nd approach: configuration model

Let $\mathcal{U}_n :=$ family of 4-regular maps on n vertices that are unrooted & half-edge-labeled & not necessarily connected



$$|\mathcal{U}_n| = \frac{1}{4^n n!}(4n)!(4n-1)!!$$

($|\mathcal{U}_0| = 1$ with convention $(-1)!! = 1$)

$\Rightarrow$ EGF of $\mathcal{U} = \cup_n \mathcal{U}_n$ is $U(t) = \displaystyle\sum_{n\geq 0} \frac{|\mathcal{U}_n|}{(4n)!} t^n = \sum_{n\geq 0} \frac{(4n-1)!!}{4^n n!} t^n$

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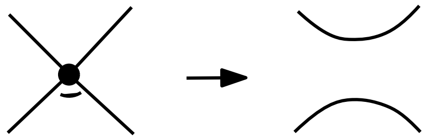

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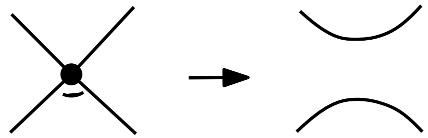

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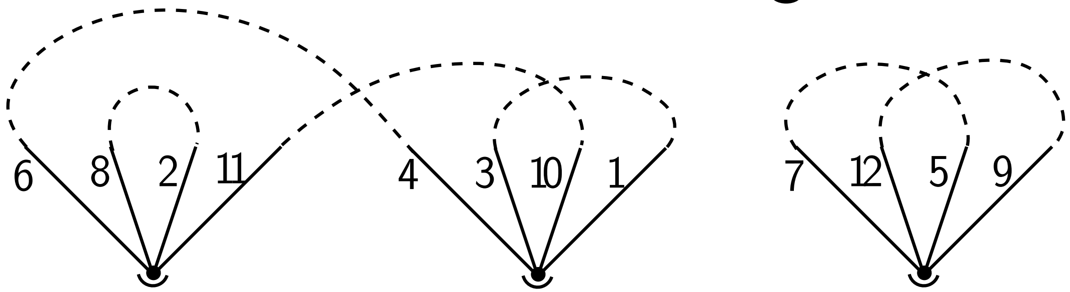

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$\Rightarrow$ GF of (rooted) 4-regular maps is $M(t) = 4t\frac{\mathrm{d}}{\mathrm{d}t}\log(U(t))$

Another expression (via Gaussian integrals)

[Bessis-Itzykson-Zuber'80]

Configuration model yields $\boxed{M(t) = \frac{\int x^2 e^{tx^4/4 - x^2/2} \mathrm{d}x}{\int e^{tx^4/4 - x^2/2} \mathrm{d}x} - 1}$

(uses $(2n-1)!! = \frac{1}{\sqrt{2\pi}} \int x^{2n} e^{-x^2/2} \mathrm{d}x$)

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(t-deformation of Hermite polynomials)

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and moreover $r_1(t), r_2(t), \ldots$ satisfy the recurrence

$r_0 = 0$ $\boxed{r_i(t) = i + tr_i(t)\cdot(r_{i-1}(t) + r_i(t) + r_{i+1}(t))}$ for $i \geq 1$

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$$\Rightarrow r_i(t) = i + 3i^2t^2 + (18i^3 + 6i)t^4 + (135i^4 + 162i^2)t^6 + \cdots$$

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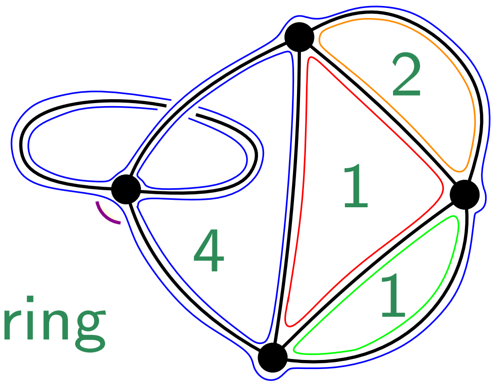
in particular $M(t) = r_1(t) = 1 + 3t + 24t^2 + 297t^3 + \cdots$

Extension to face-colored maps (via matrix integrals)



4-face-coloring

N-face-colored map = map where each face receives a color in $[1..N]$

For $N \geq 1$, let $\overline{M}(t,N) :=$ GF of N-face-colored rooted 4-regular maps i.e., each map with f faces is counted with weight N^f

(Rk: $M(t) = \overline{M}(t,1)$)

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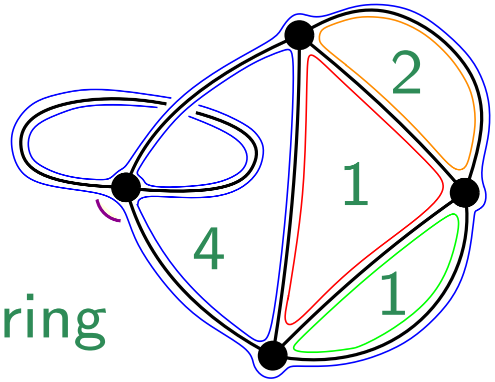

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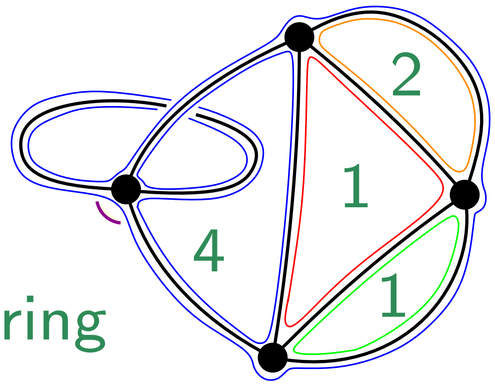
(Rk: $M(t) = \overline{M}(t, 1)$**)**

Using matrix integrals (over $N \times N$ Gaussian hermitian matrices) and orthogonal polynomials, one obtains

$$\overline{M}(t, N) = -N^2 + \sum_{i=1}^{N-1} \left(r_i(t) + r_{i+1}(t)\right)$$

Extension to face-colored maps (via matrix integrals)



4-face-coloring

N-face-colored map = map where each face receives a color in $[1..N]$

For $N \geq 1$, let $\overline{M}(t,N) :=$ GF of N-face-colored rooted 4-regular maps
i.e., each map with f faces is counted with weight N^f

(**Rk:** $M(t) = \overline{M}(t,1)$)

Using matrix integrals (over $N \times N$ Gaussian hermitian matrices) and orthogonal polynomials, one obtains

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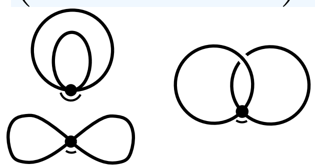
$$= (2N^3 + N)t^2 + (9N^4 + 15N^2)t^4 + (54N^5 + 198N^3 + 45N)t^6 + \cdots$$



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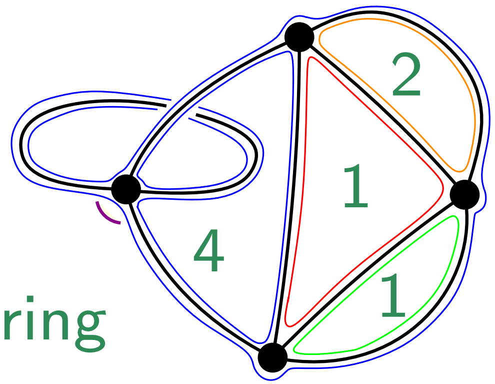

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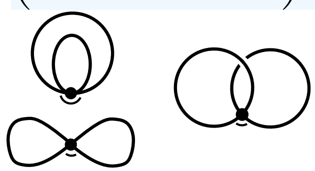

Planar case ($N \to \infty$)

$t\, M_{\text{planar}}(t) = 2 \int R(t)\mathrm{d}t$

with $R(t) = 1 + 3tR(t)^2$

Compared properties R_i vs r_i

$R_0 = 0$	$r_0 = 0$
$R_i = 1 + t\,R_i(R_{i-1} + R_i + R_{i+1})$	$r_i = i + t\,r_i(r_{i-1} + r_i + r_{i+1})$

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Exact expression of $R_i(t)$ $R_i = R \cdot \frac{(1-x^i)(1-x^{i+3})}{(1-x^{i+1})(1-x^{i+2})}$ where $x = t\,R^2(1 + x + x^2)$	

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$r_1(t)$ is differentially algebraic

iteratively, $r_i \in \mathrm{Rat}(t, r_1)$ for $i \geq 2$ is also differentially algebraic

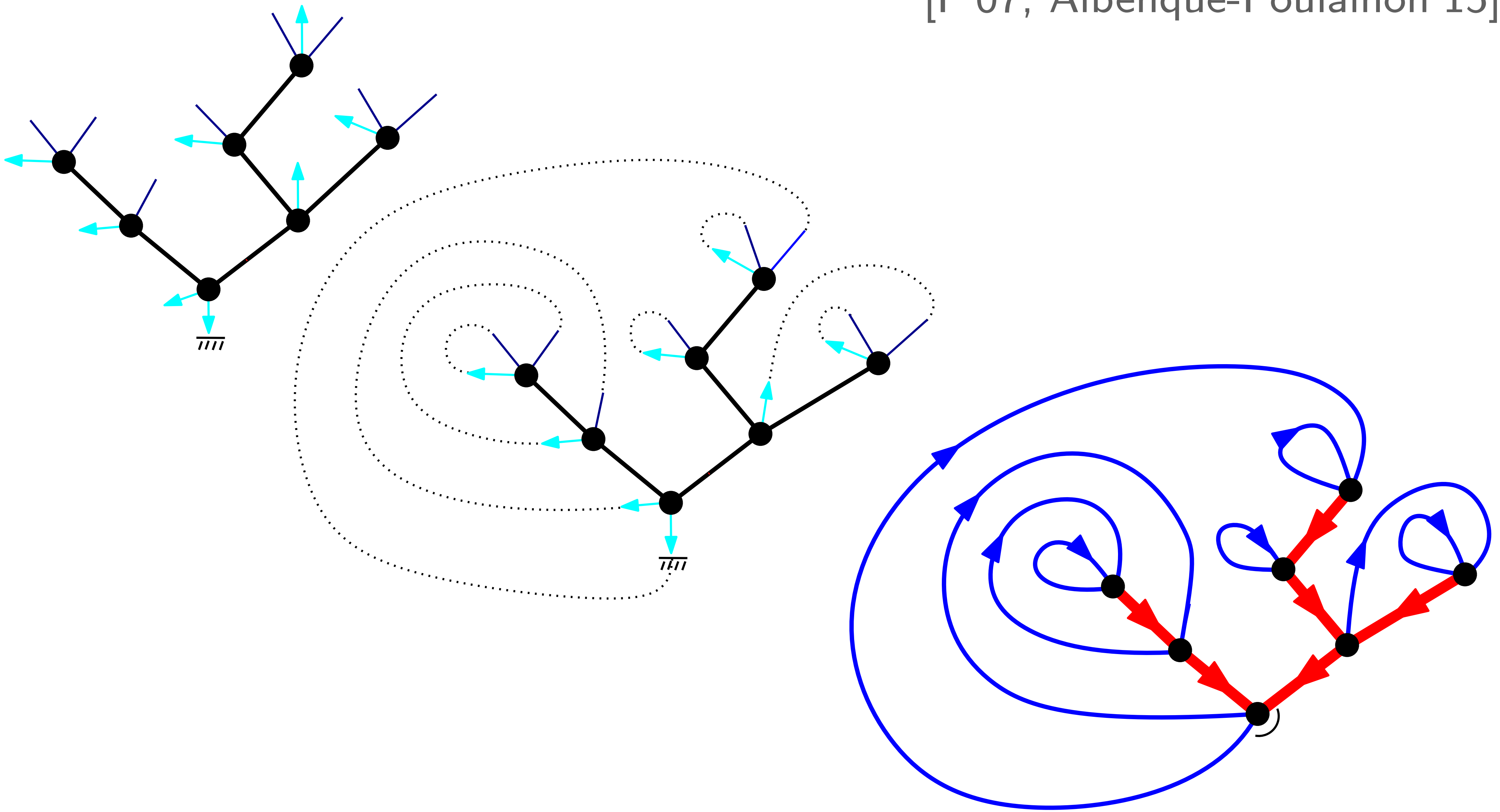
Polynomiality in i of coefficients

$$r_i(t) = i + 3i^2t + (18i^3 + 6i)t^2 + (135i^4 + 162i^2)t^3 + \cdots$$

Bijective proof that $M(t) = r_1(t) - 1$

The planar bijection via Eulerian orientations

[F'07, Albenque-Poulalhon'15]



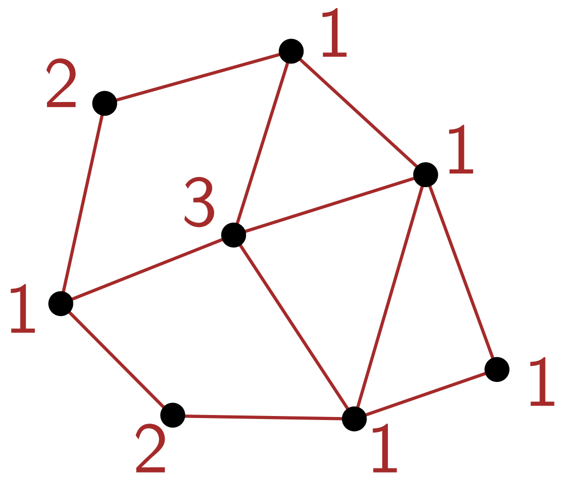
Rk: via the bijection, the 4-regular map is endowed with a spanning tree T and an orientation O such that edges of T are oriented toward the root and edges not in T turn clockwise around the tree

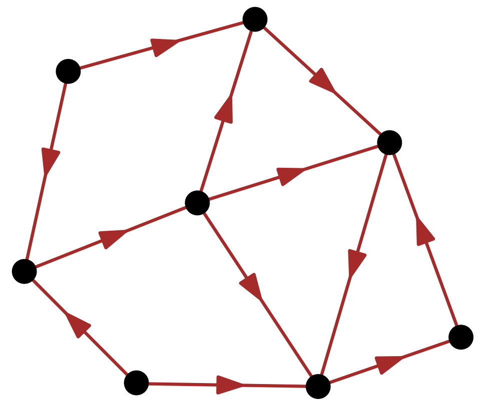
α-orientations

[Propp'02], [Felsner'03]

For $G = (V, E)$ a graph and $\alpha : V \to \mathbb{N}$

α-orientation of G = orientation where every vertex v has outdegree $\alpha(v)$





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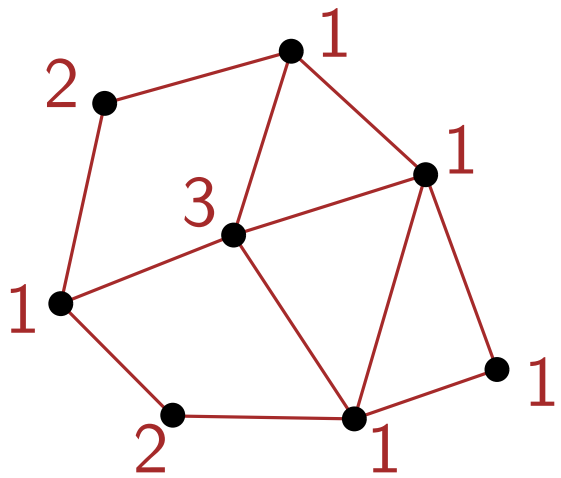

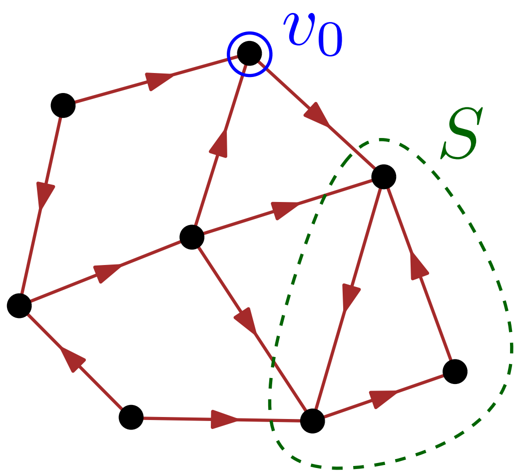

Let $v_0 \in V$ be a marked vertex

an orientation is **v_0-accessible** if $\forall v \in V$ there is a path from v to v_0

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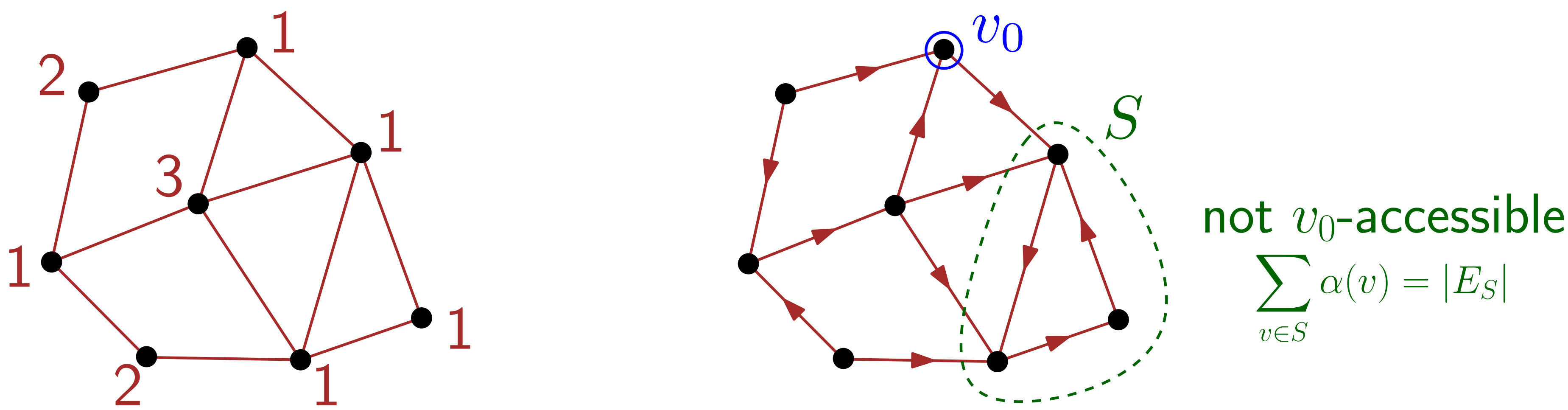

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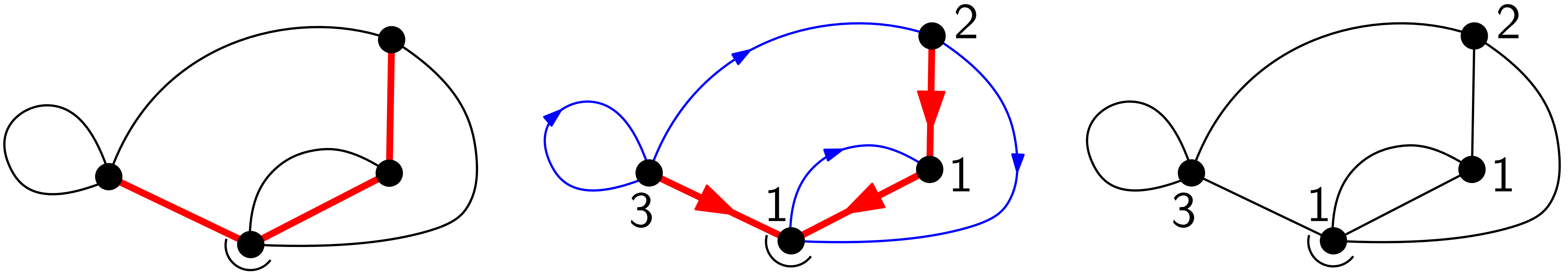
Property: either all α-orientations are v_0-accessible or none

In the first case (and non-emptiness), α is called **root-accessible**

Bernardi's bijection (planar case) [Bernardi'07]

Let M be a rooted planar map, with vertex-set V

spanning trees of M $\longleftrightarrow$ root-accessible $\alpha : V \to \mathbb{N}$



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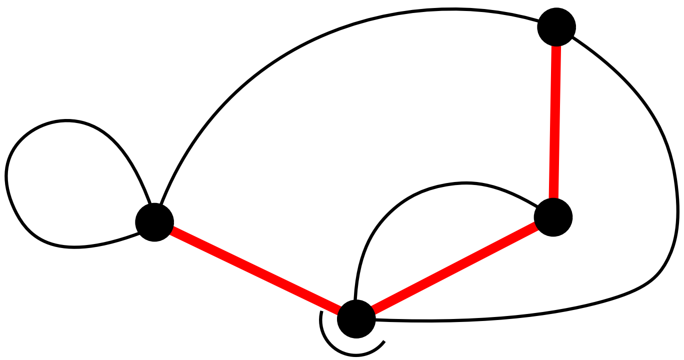

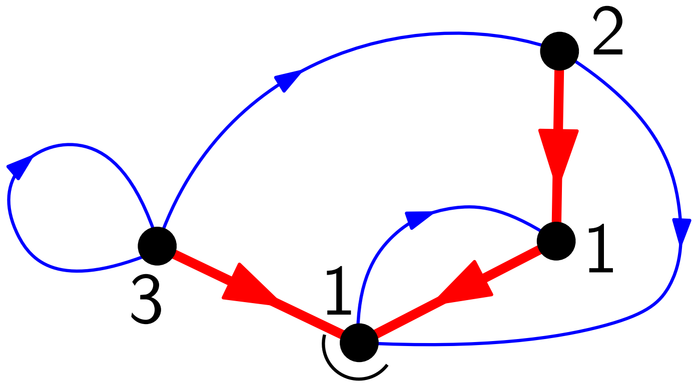

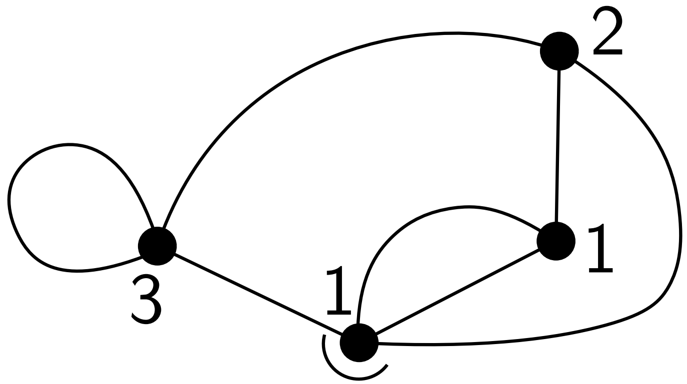

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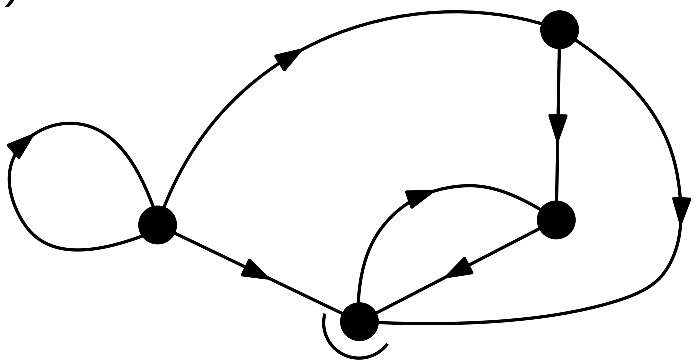
(unique without ccw cycle)

From the minimal α-orientation, the spanning tree is computed by a certain traversal ([Poulalhon-Schaeffer'06] for 3-orientations)



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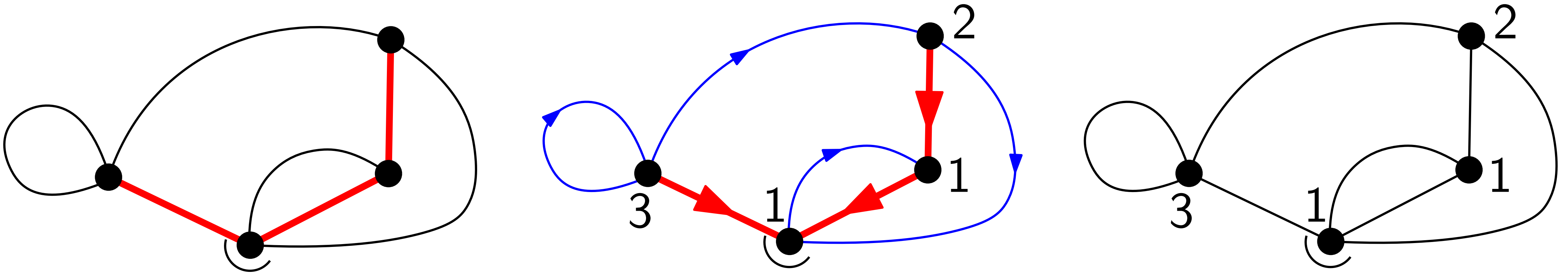

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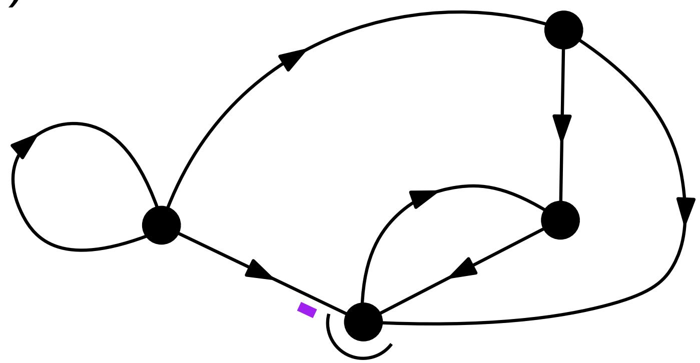

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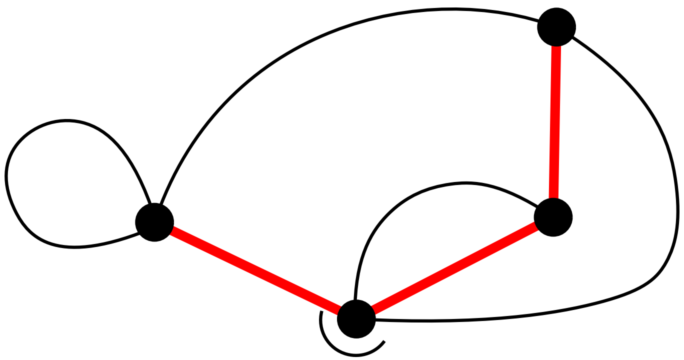

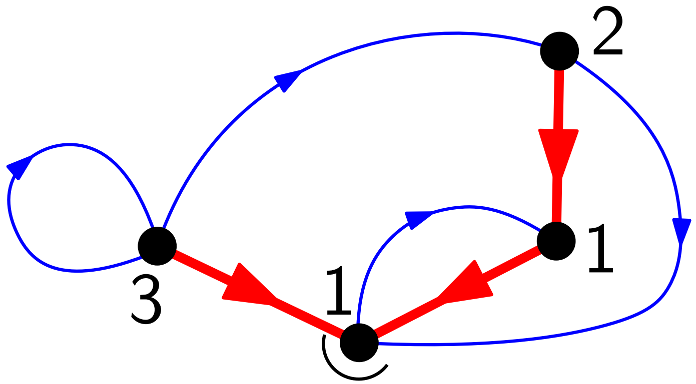

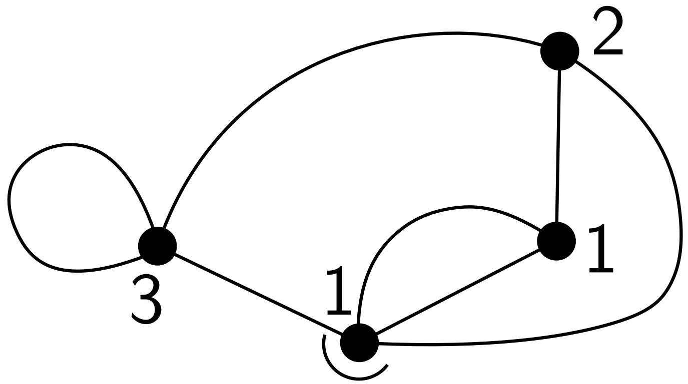

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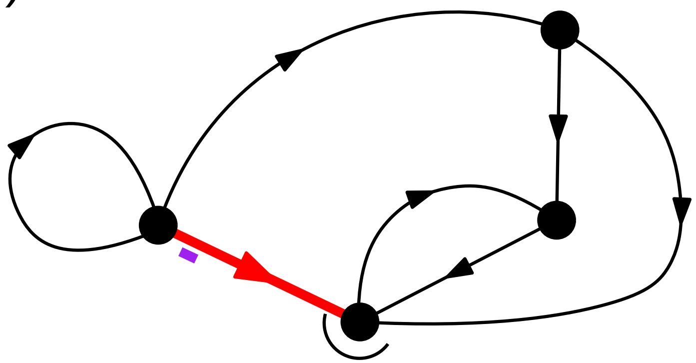

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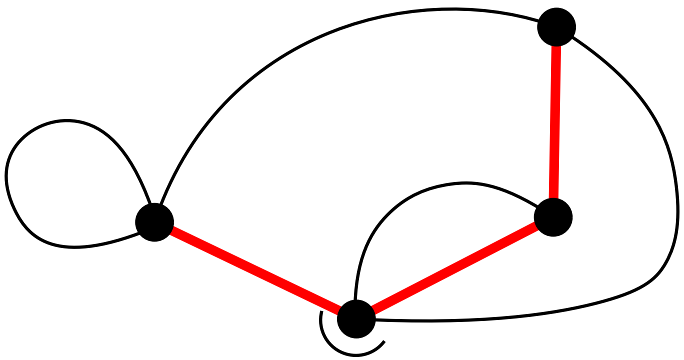

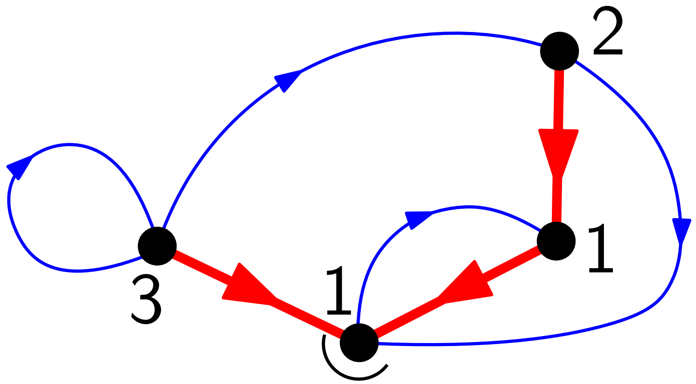

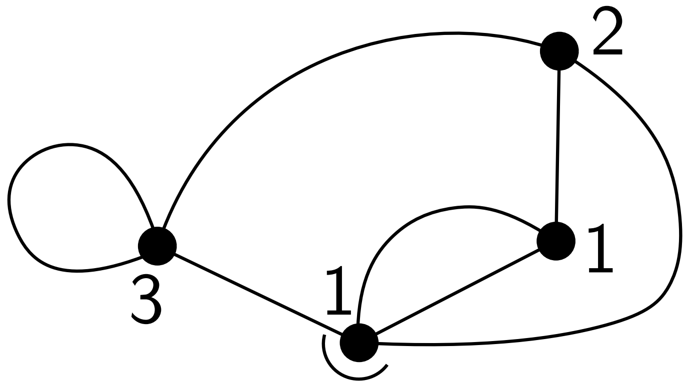

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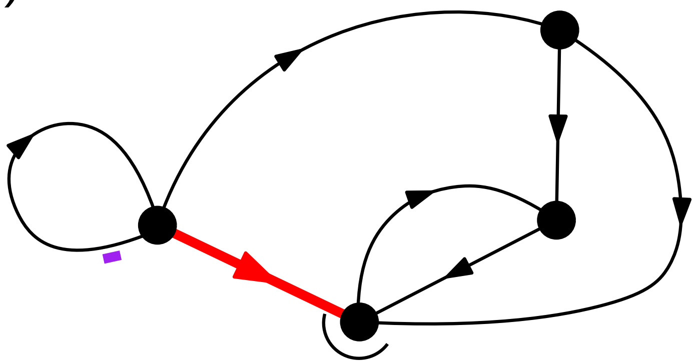

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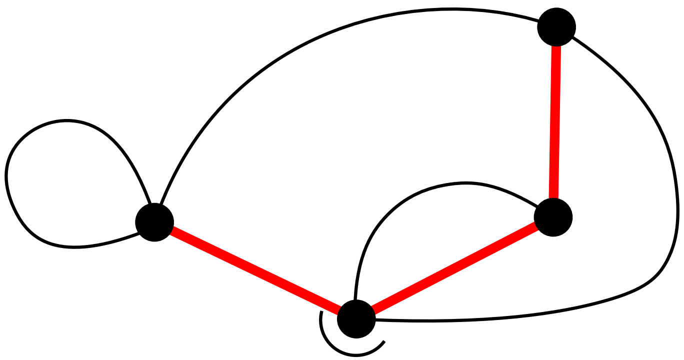

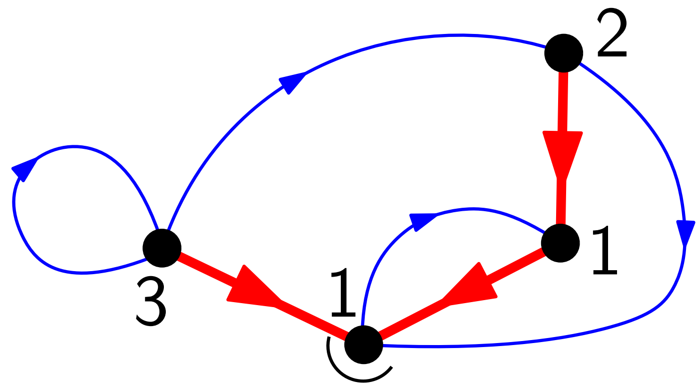

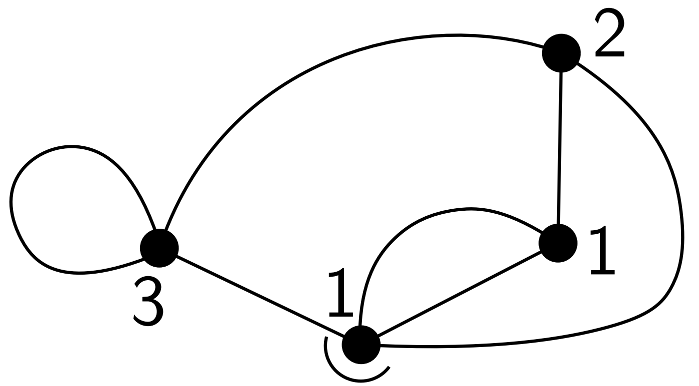

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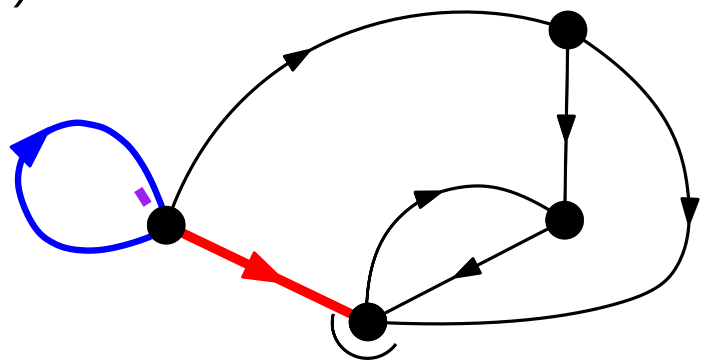

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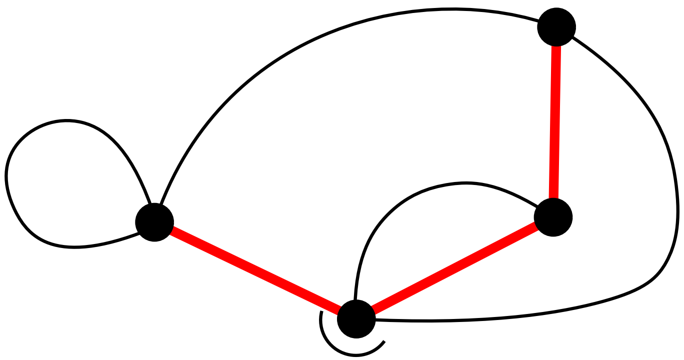

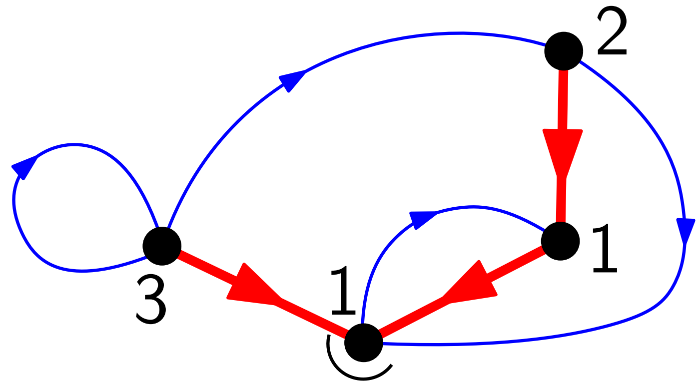

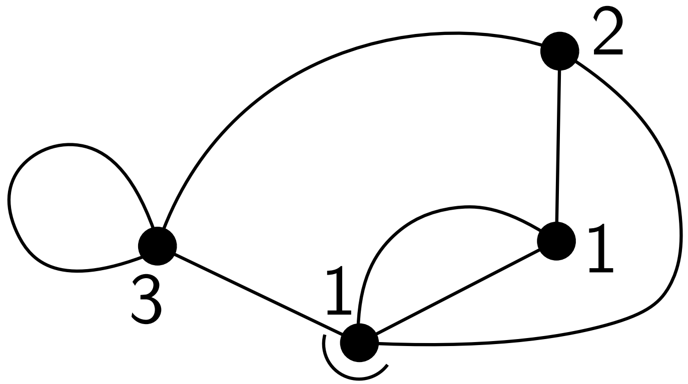

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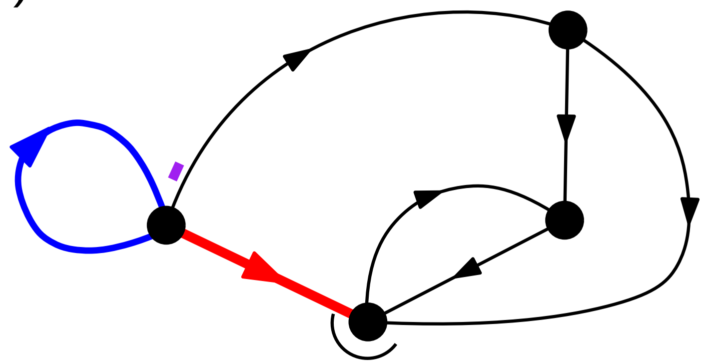

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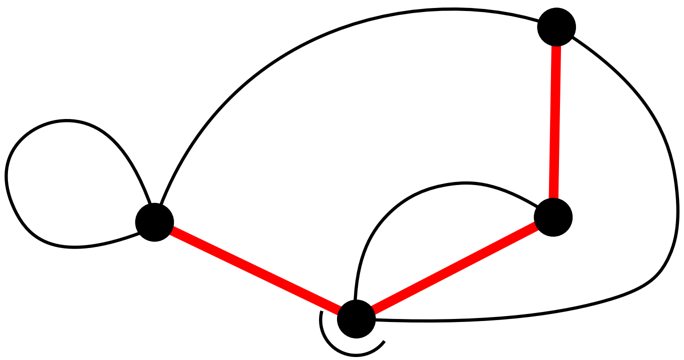

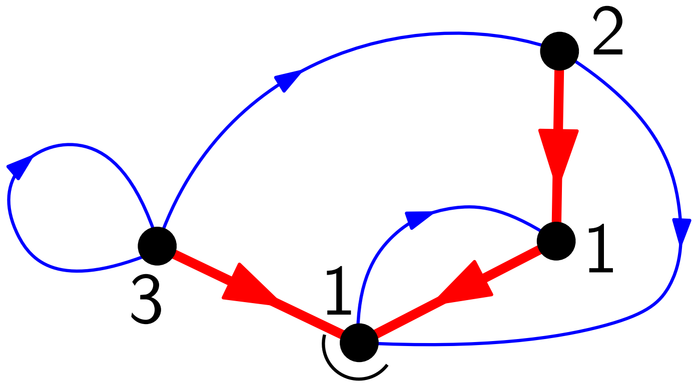

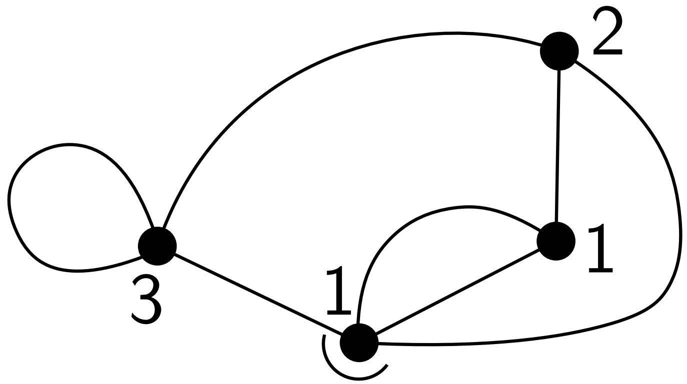

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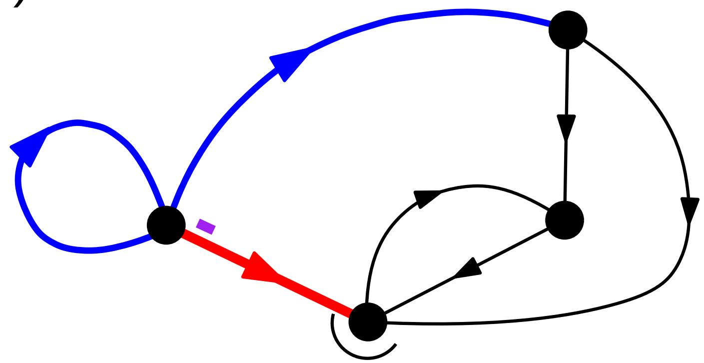

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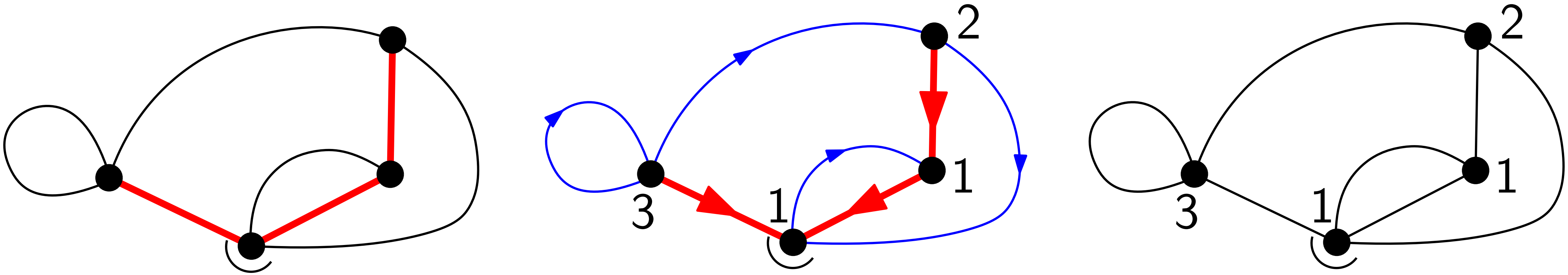

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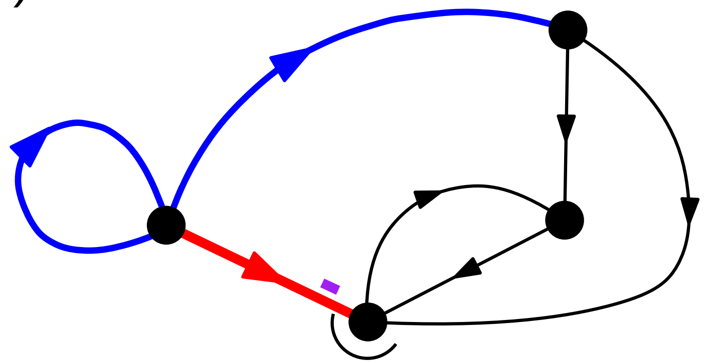

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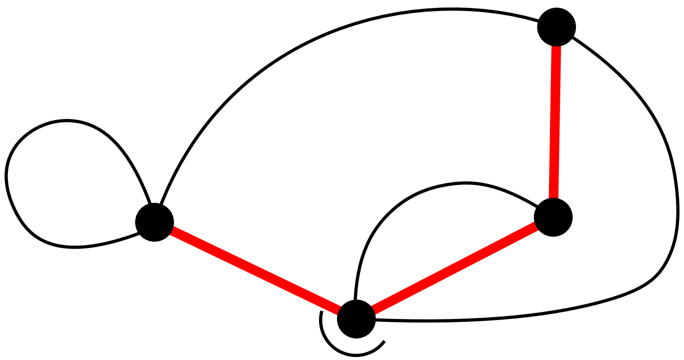

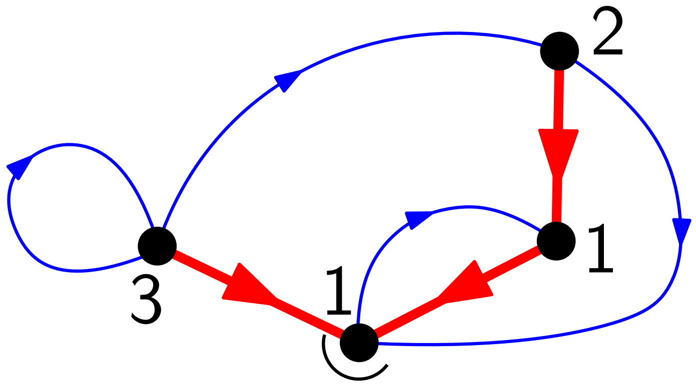

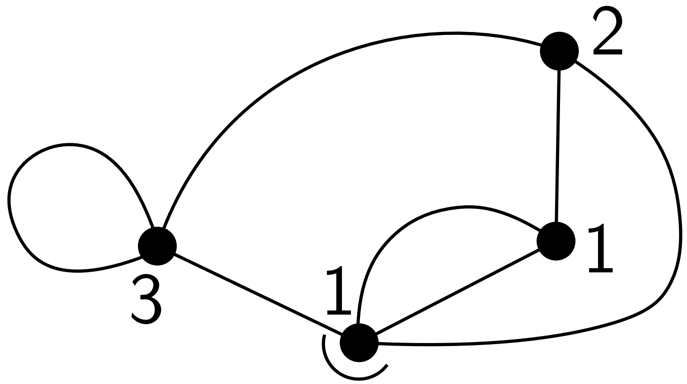

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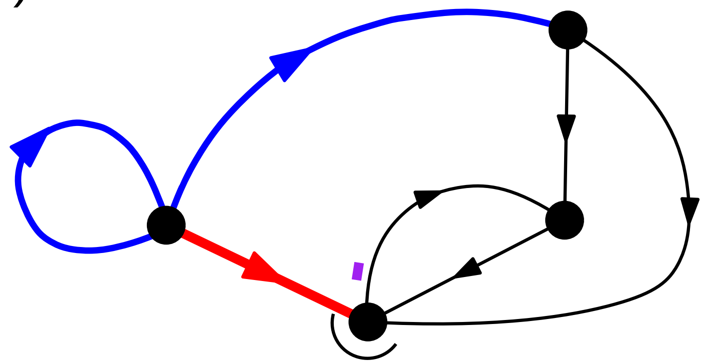

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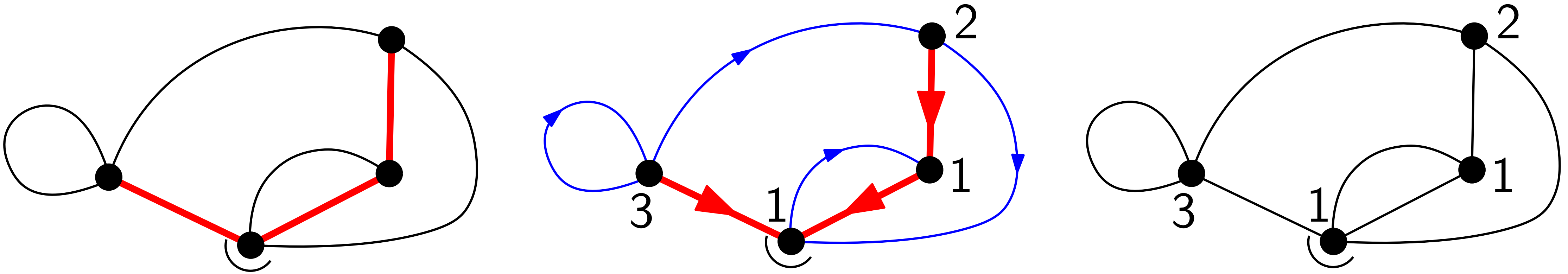

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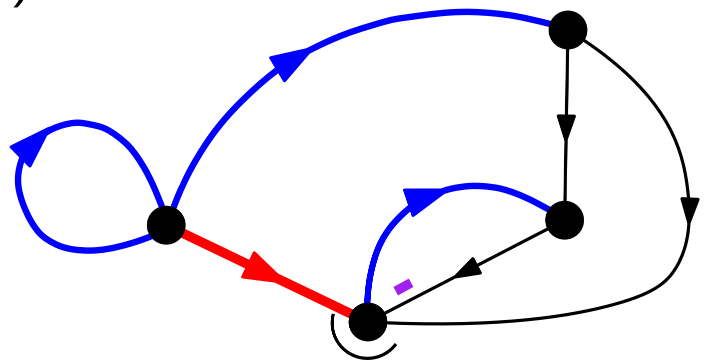

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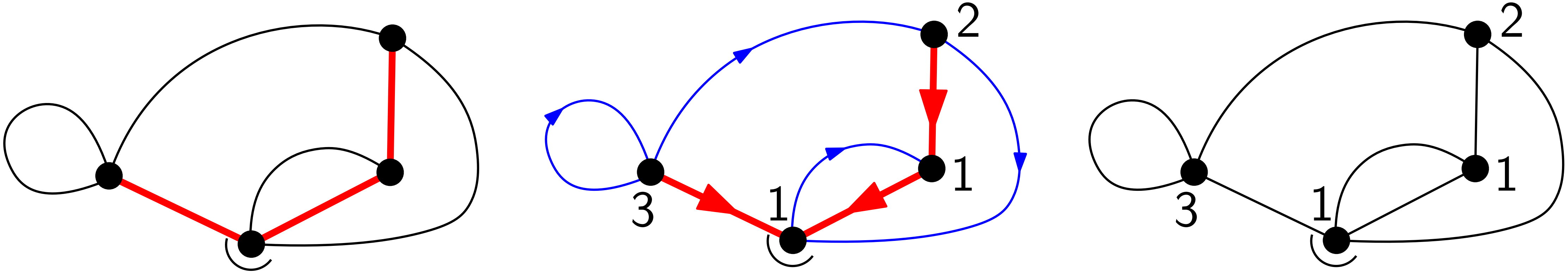

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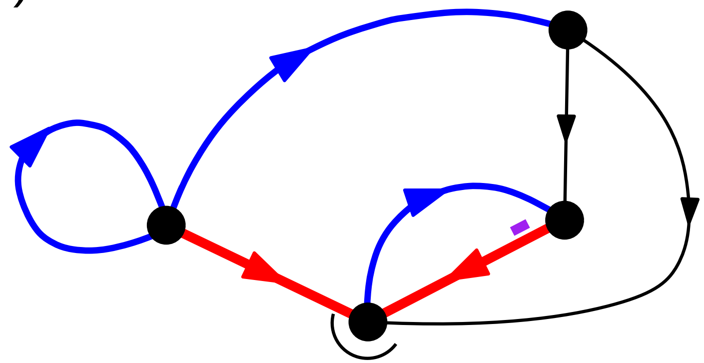

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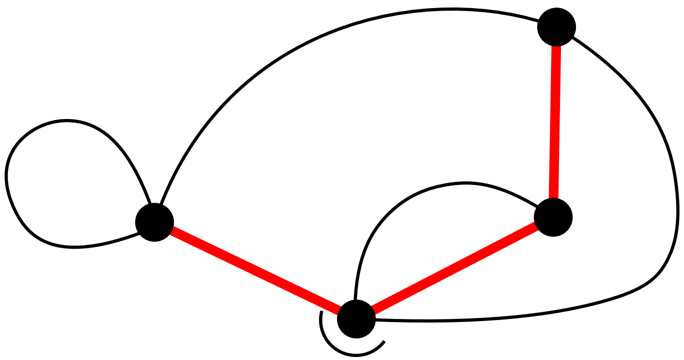

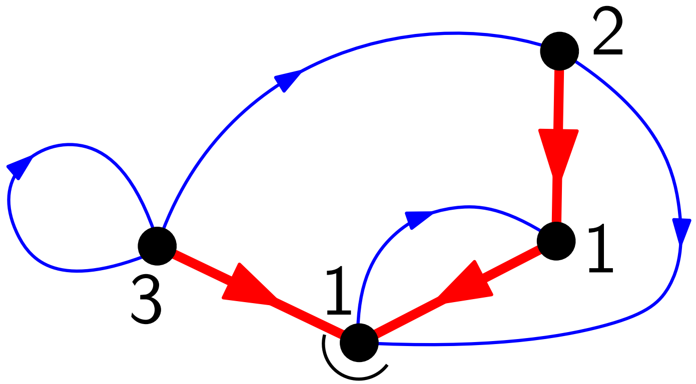

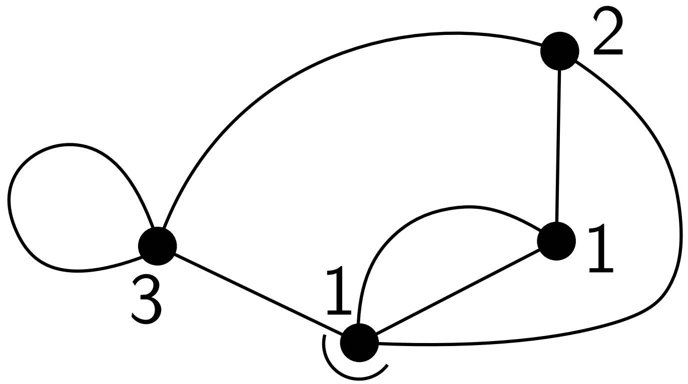

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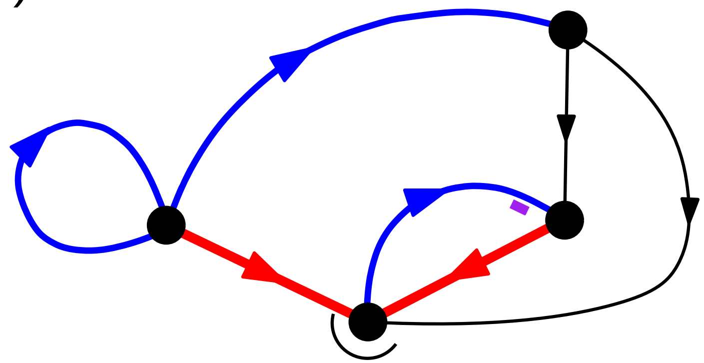

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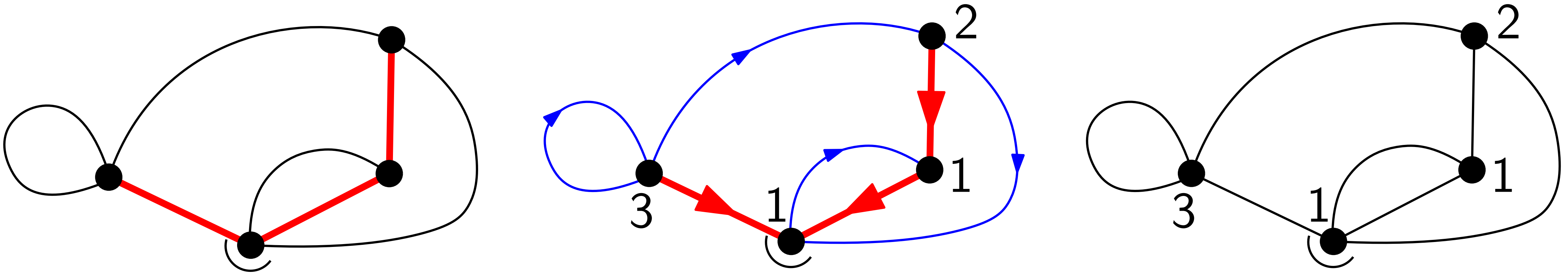

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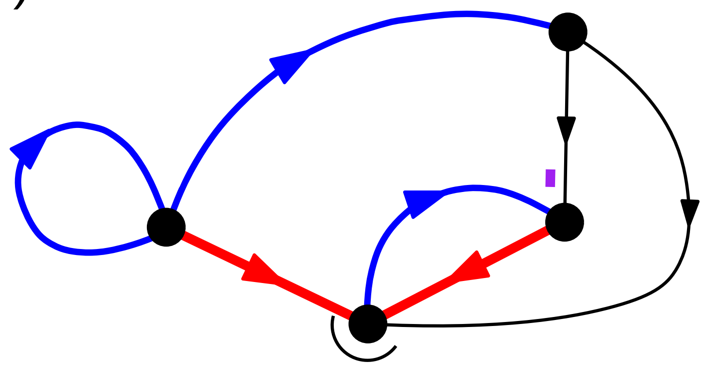

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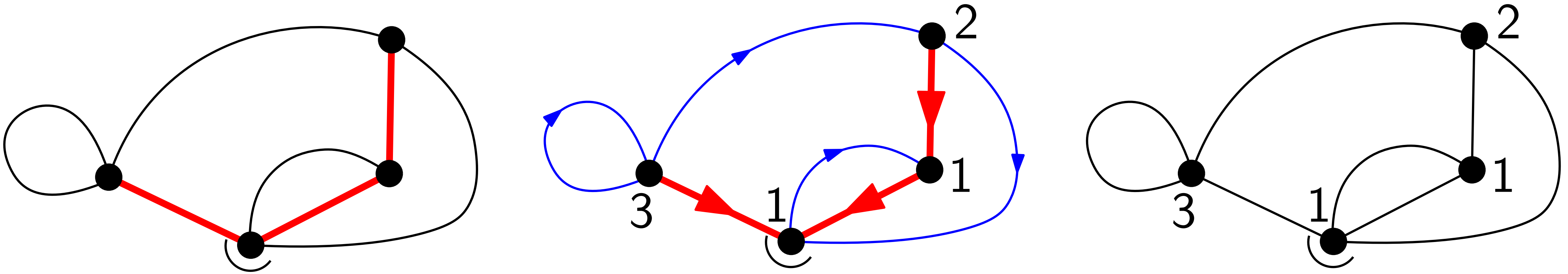

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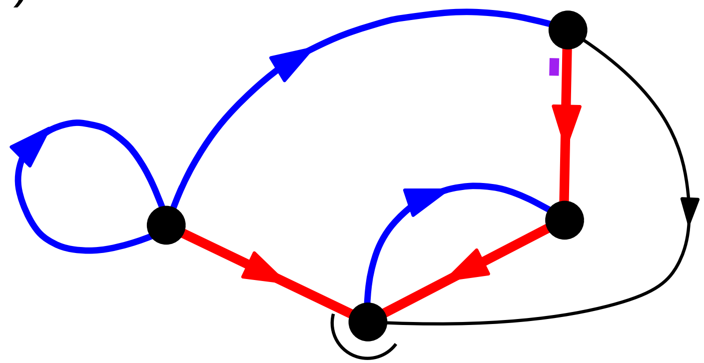

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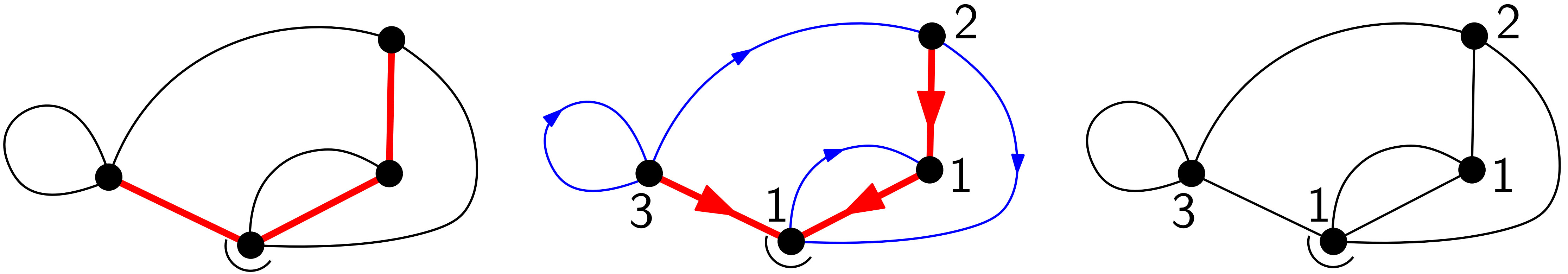

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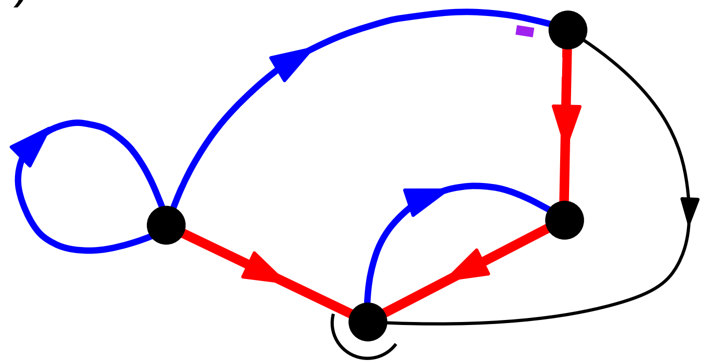

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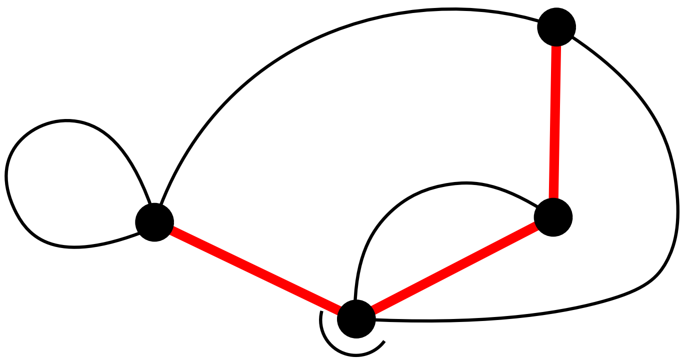

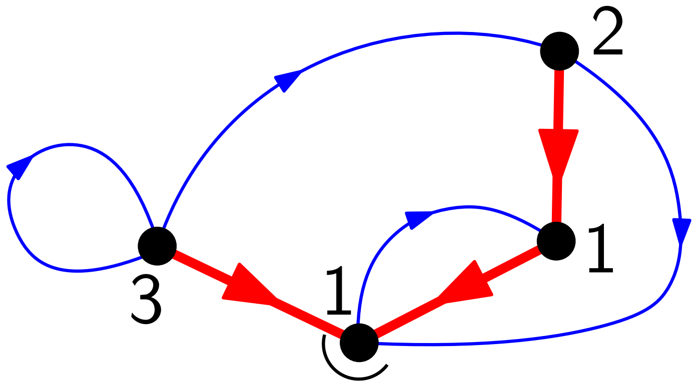

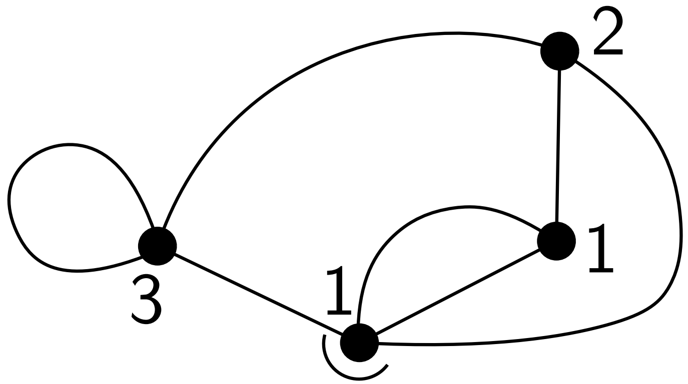

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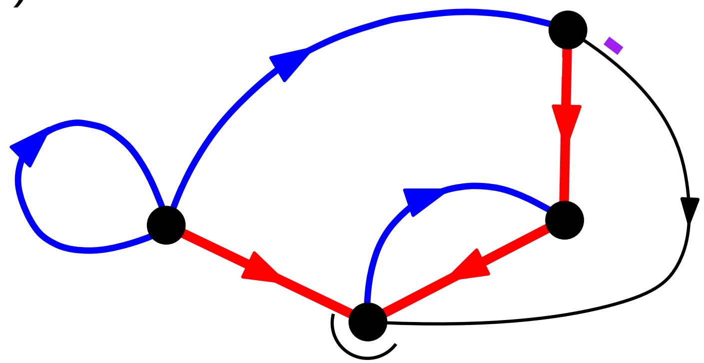

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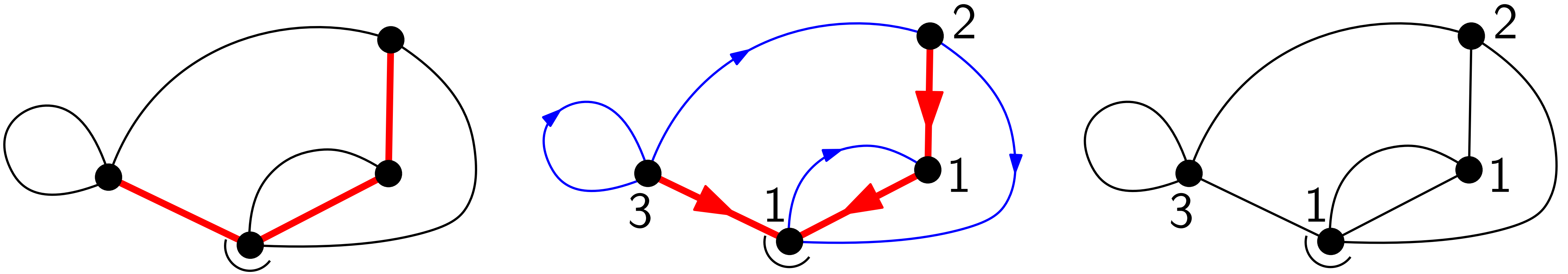

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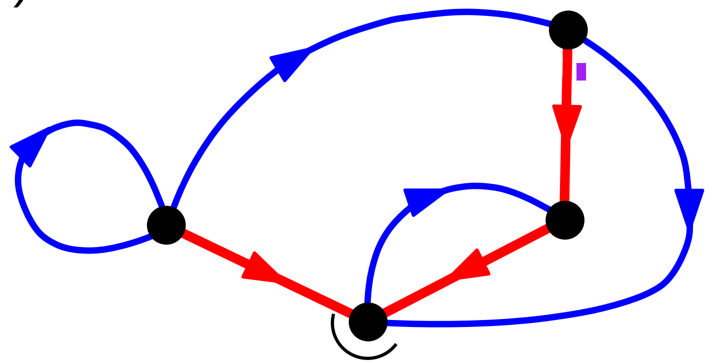

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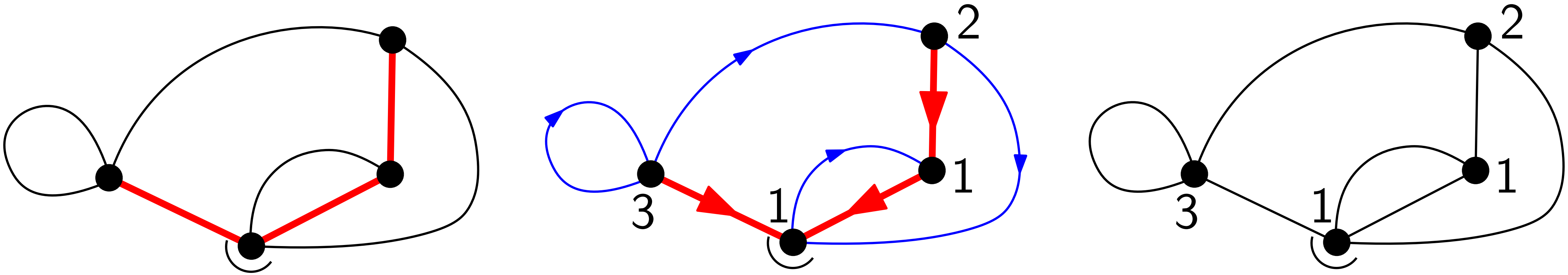

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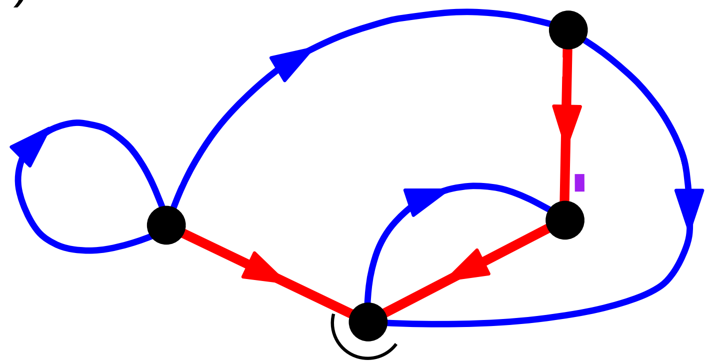

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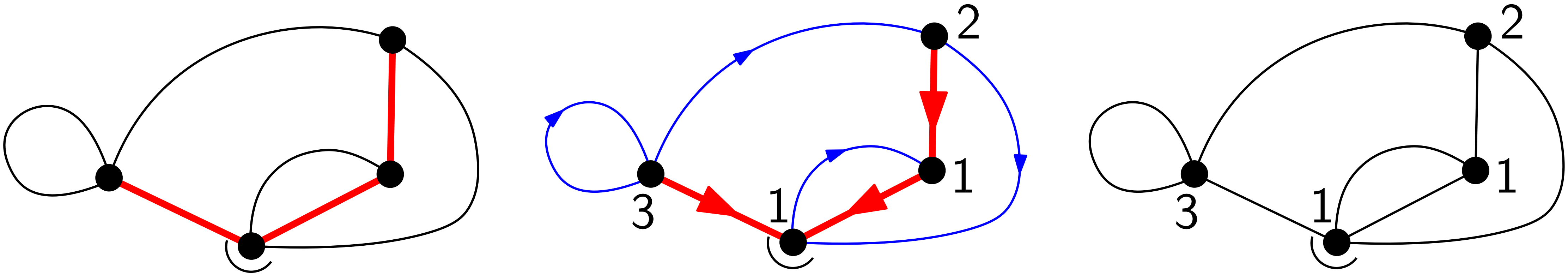

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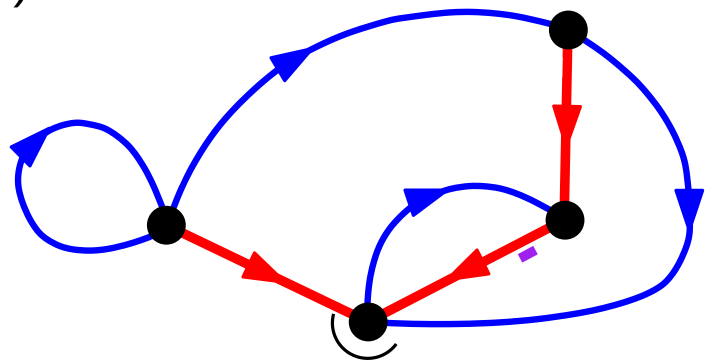

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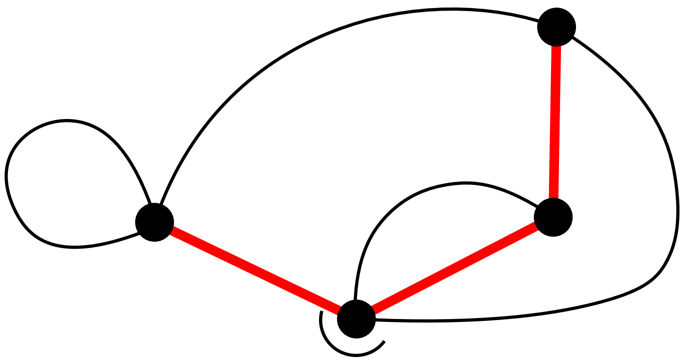

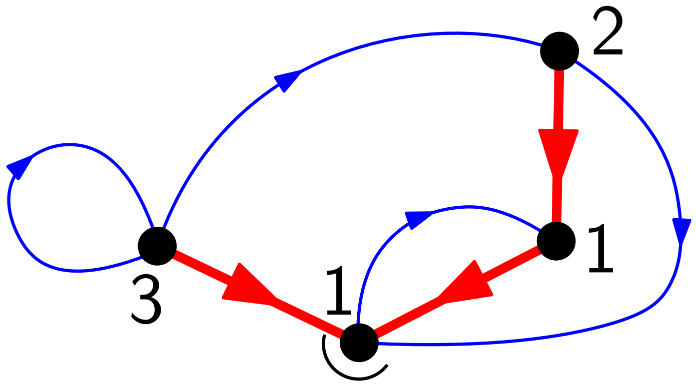

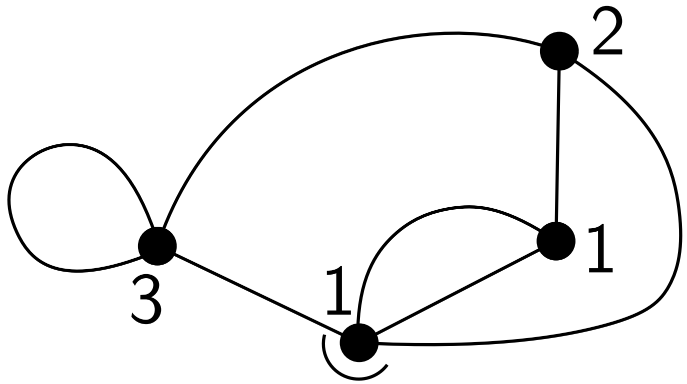

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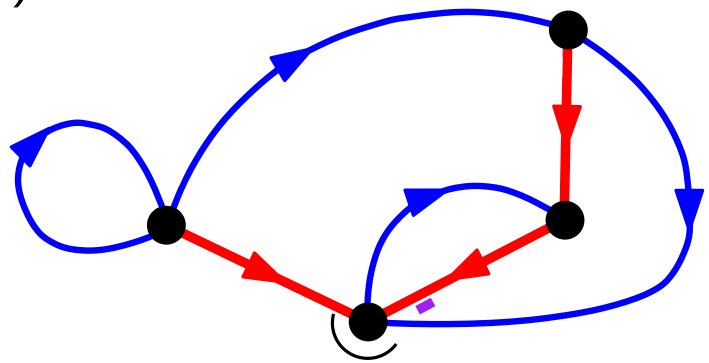

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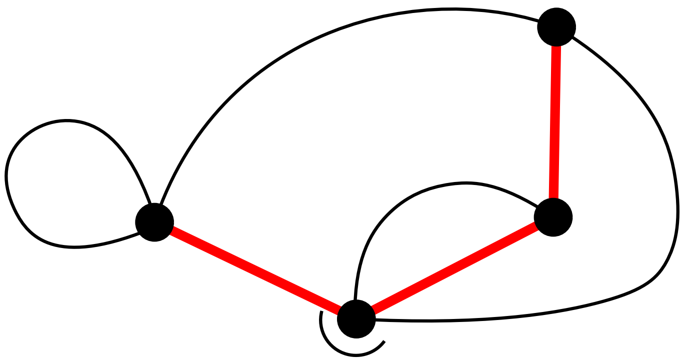

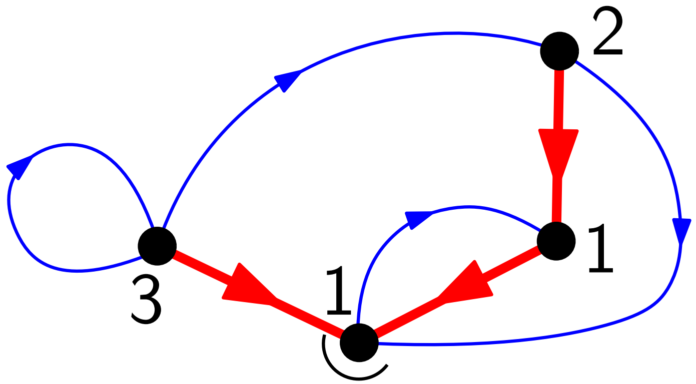

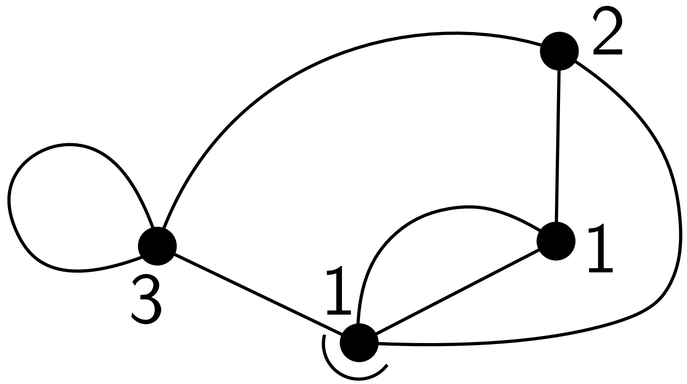

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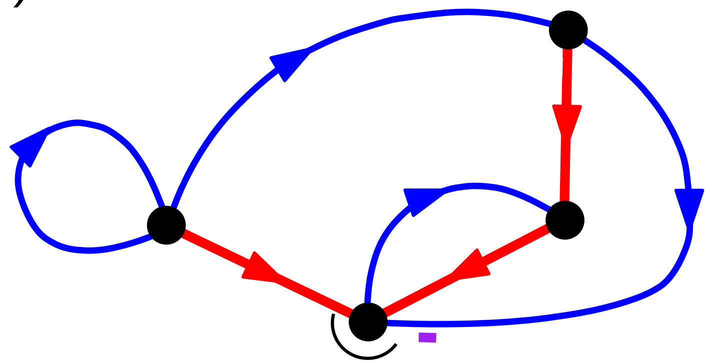

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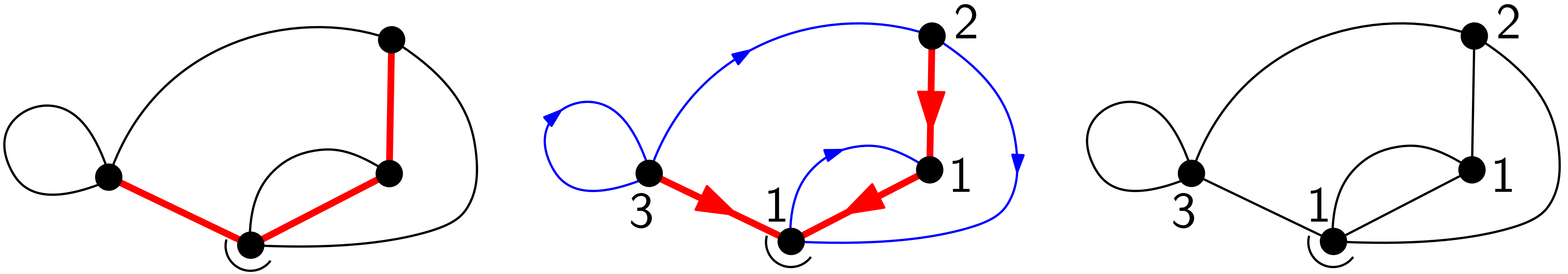

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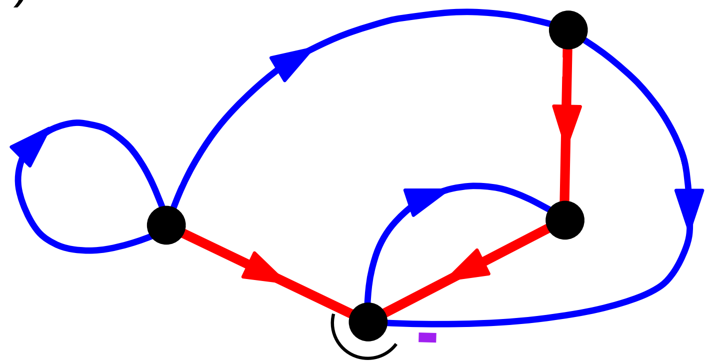

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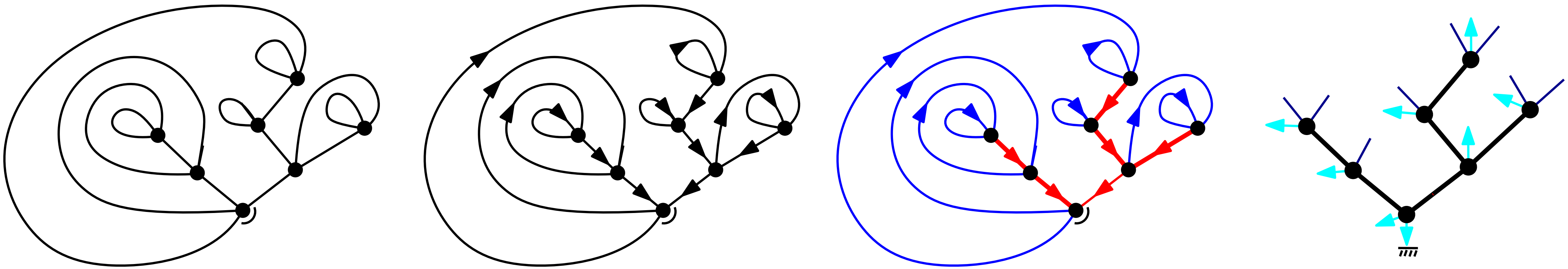

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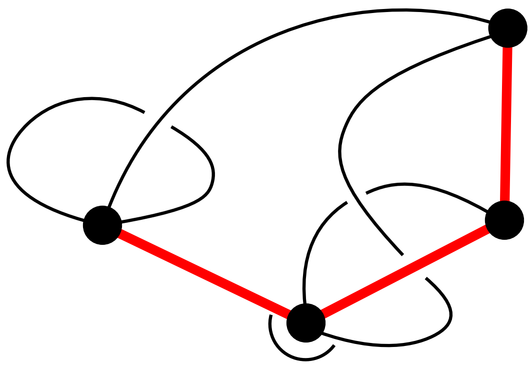

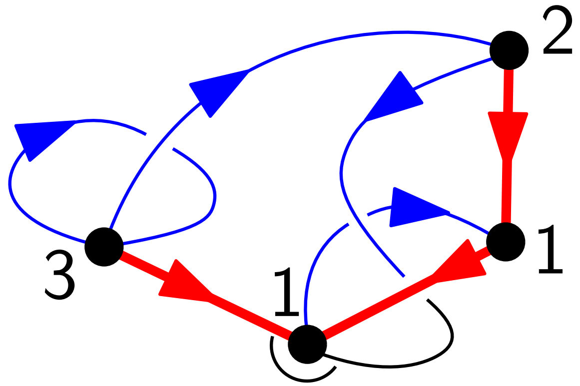

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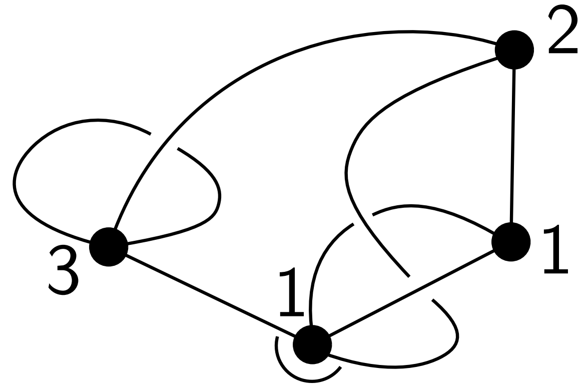

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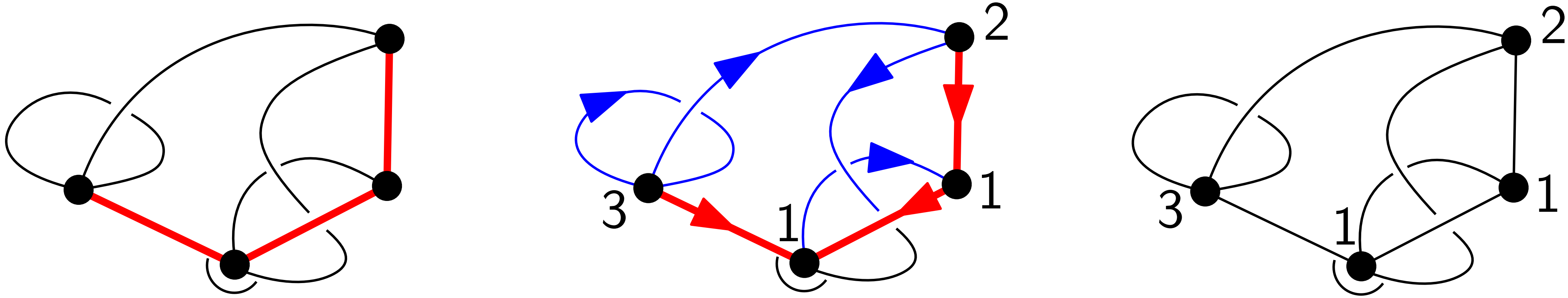

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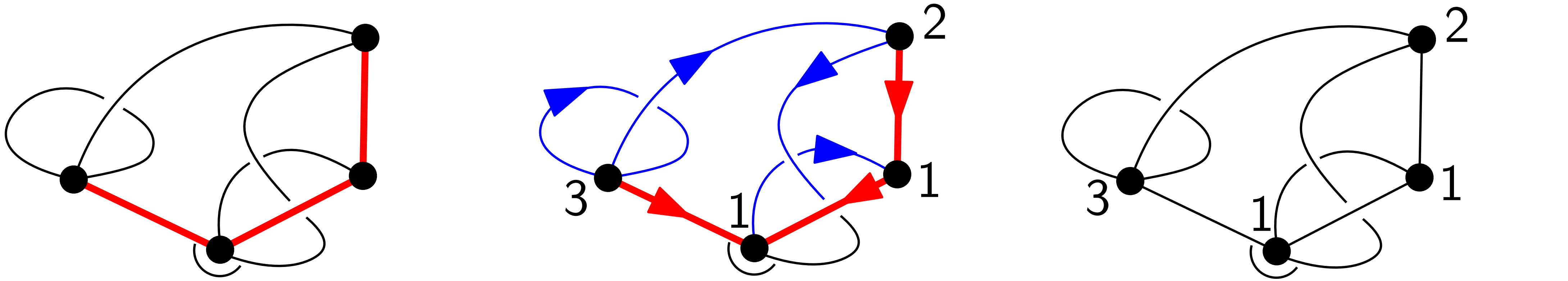

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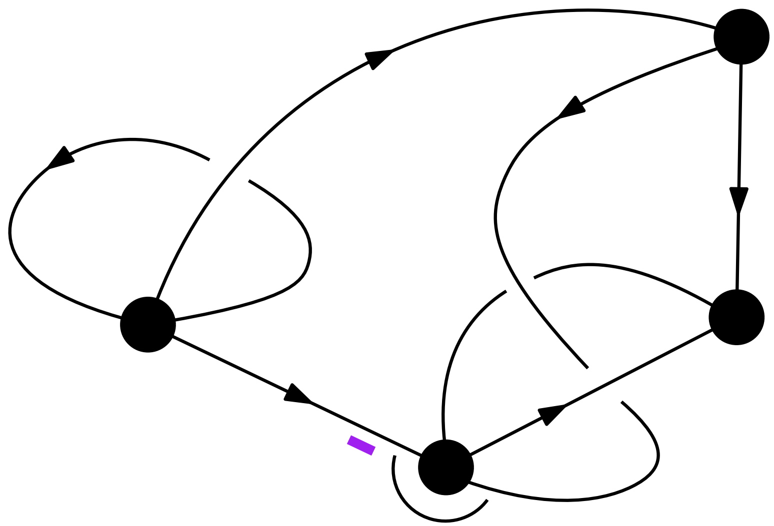

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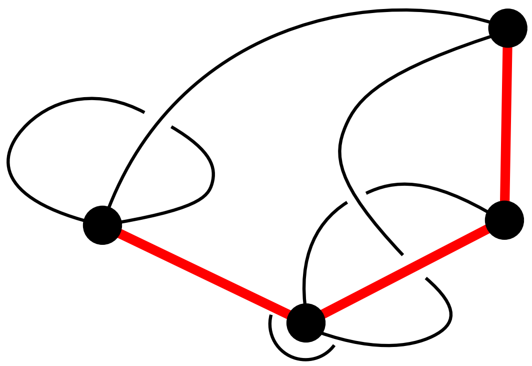

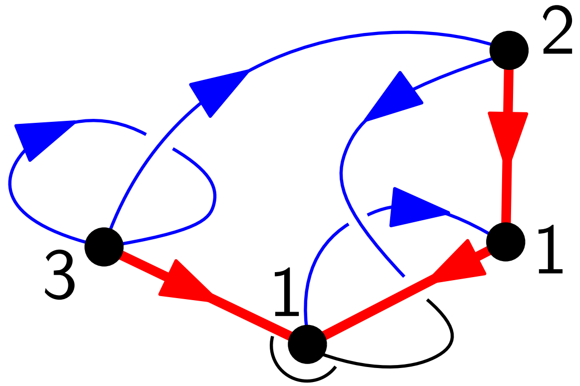

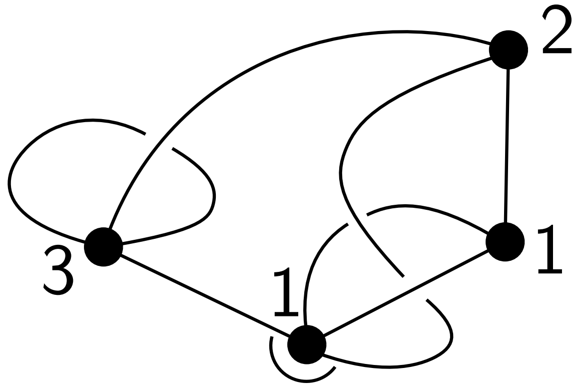

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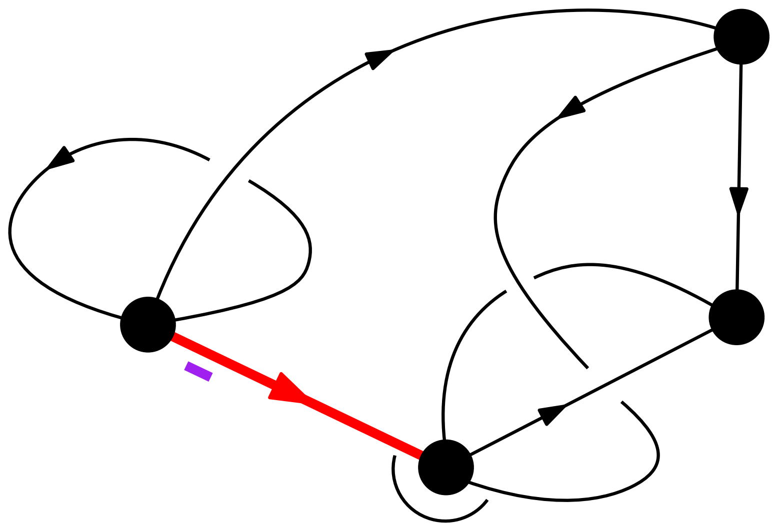

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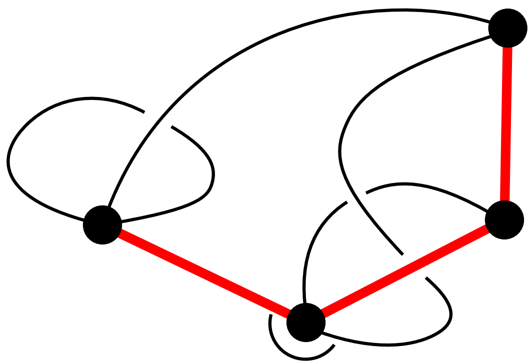

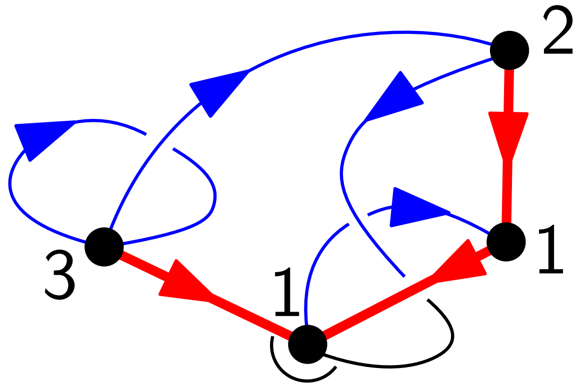

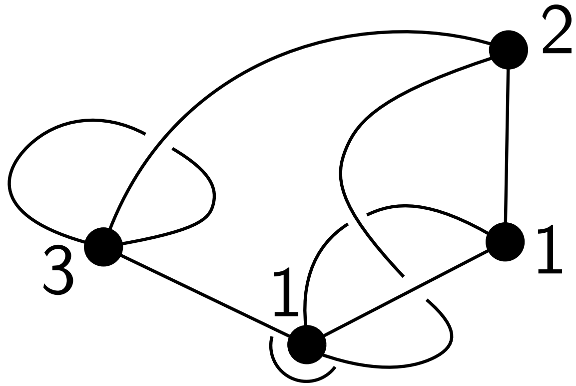

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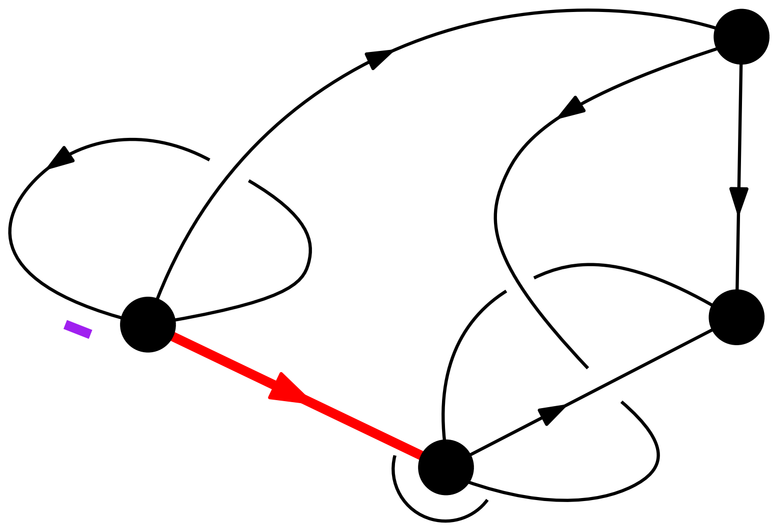

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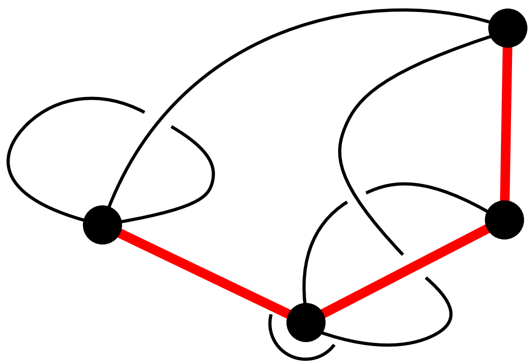

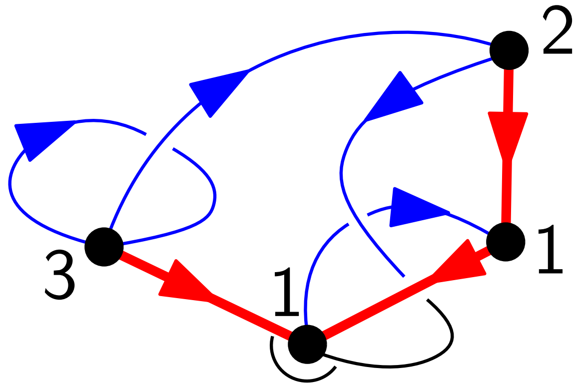

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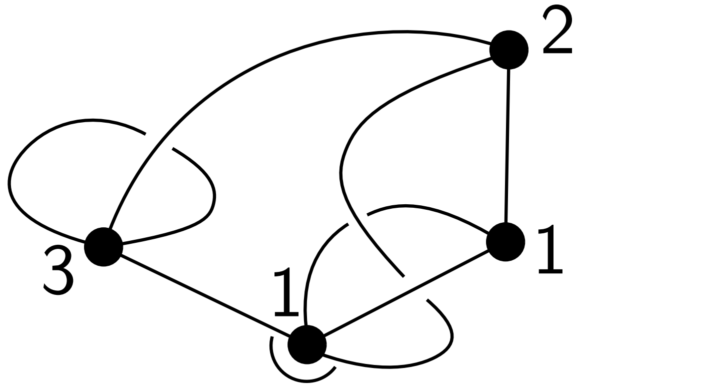

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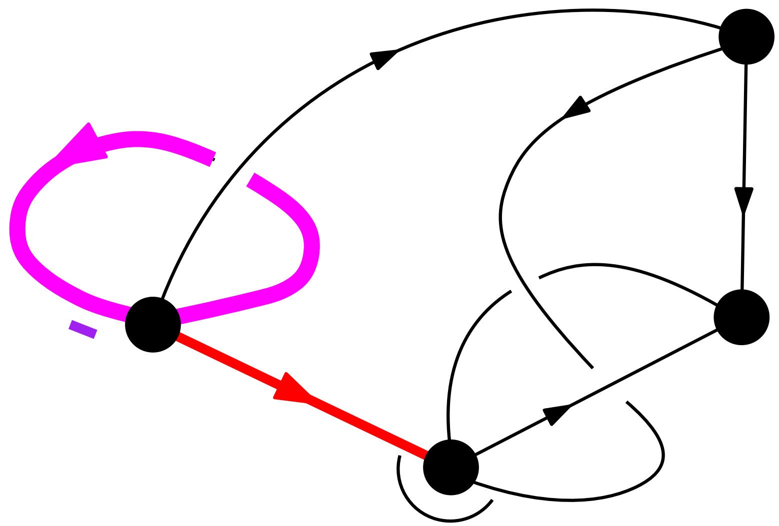

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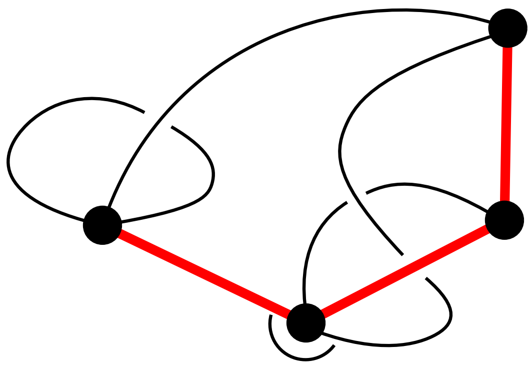

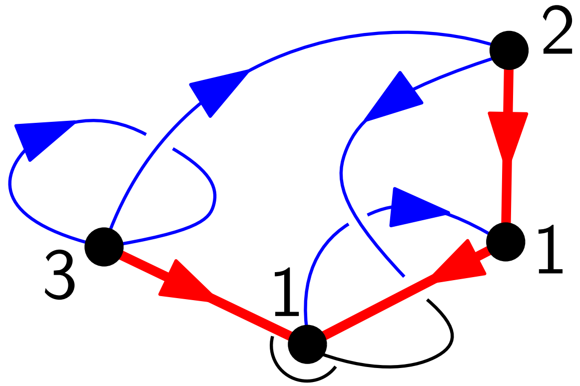

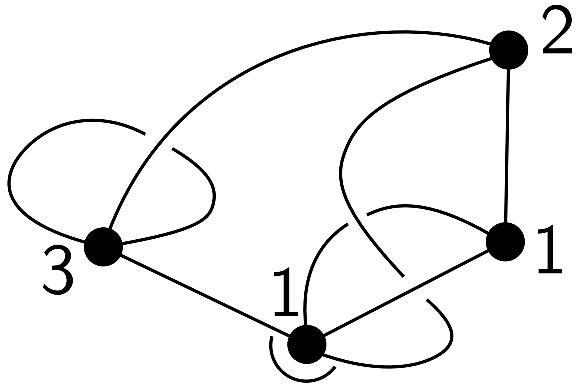

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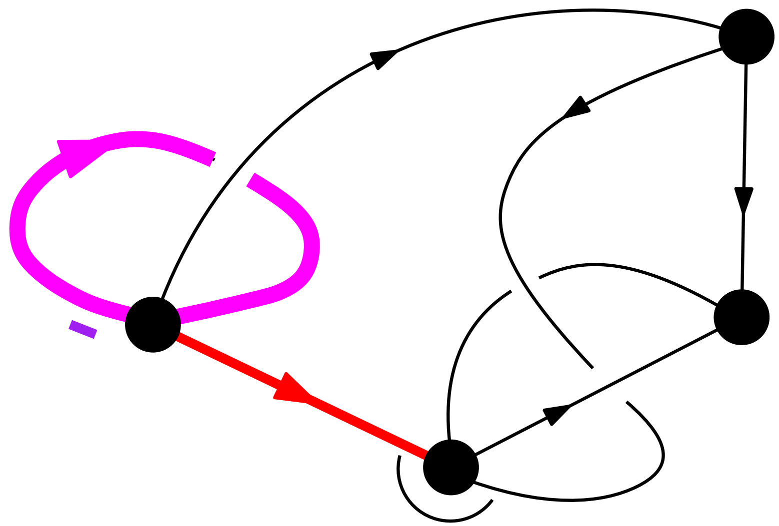

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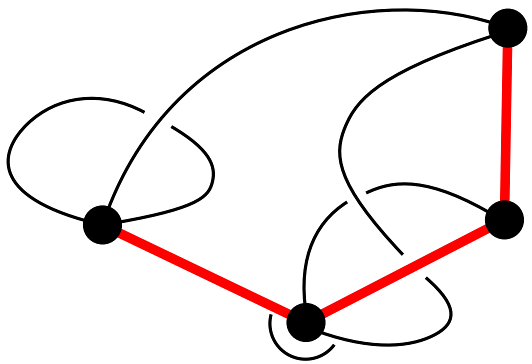

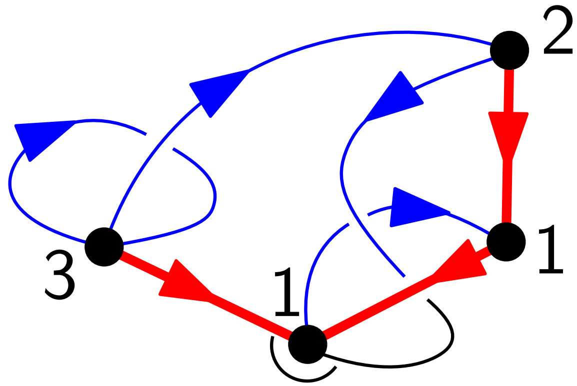

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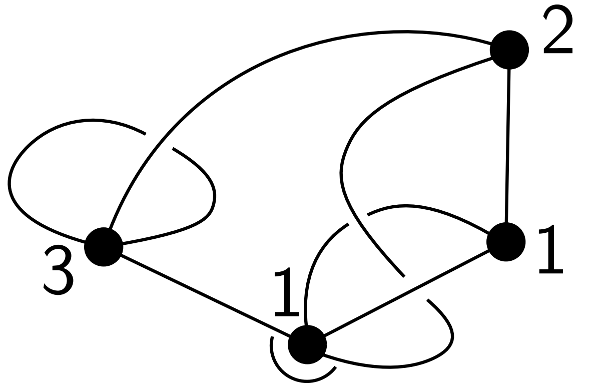

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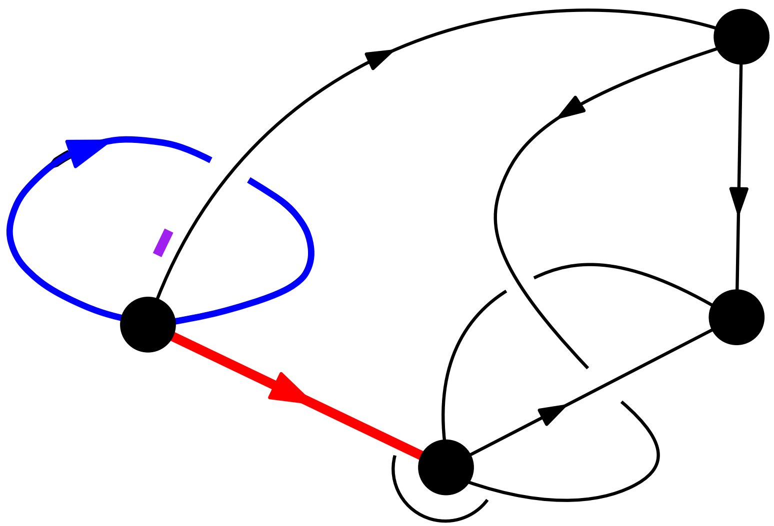

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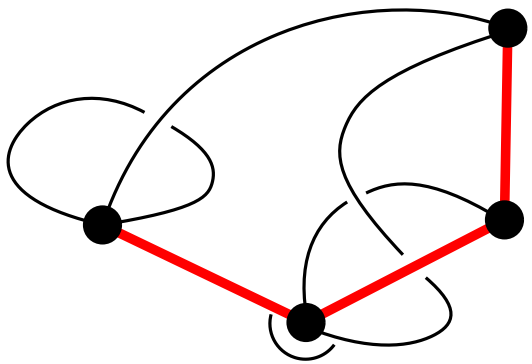

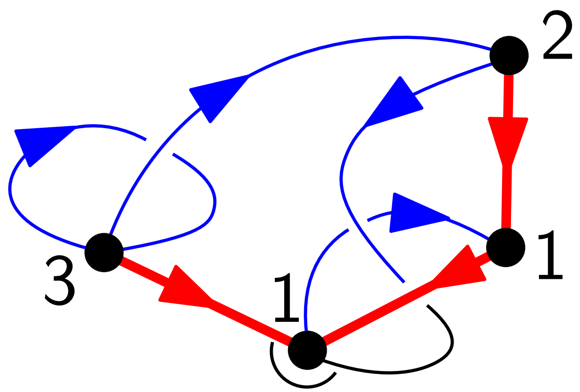

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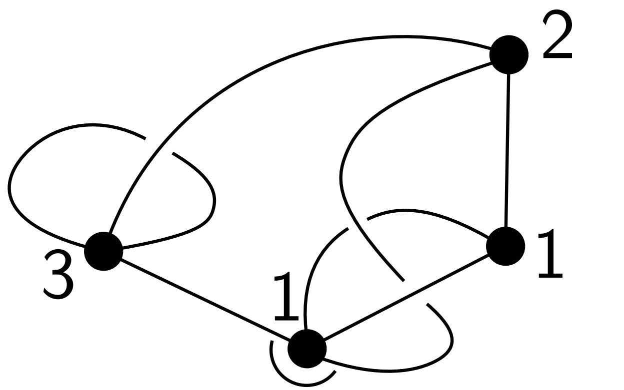

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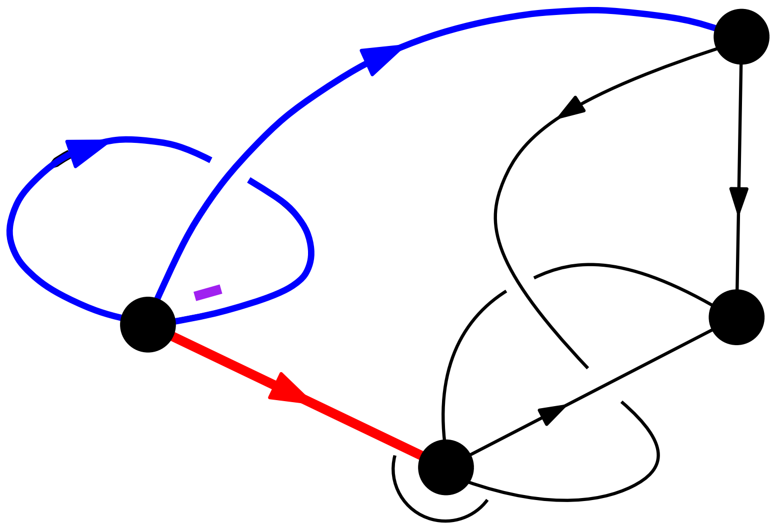

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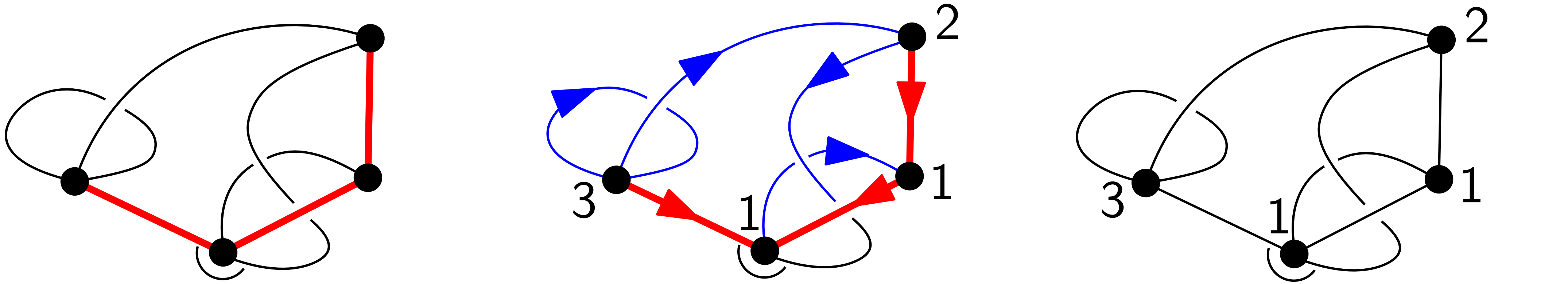

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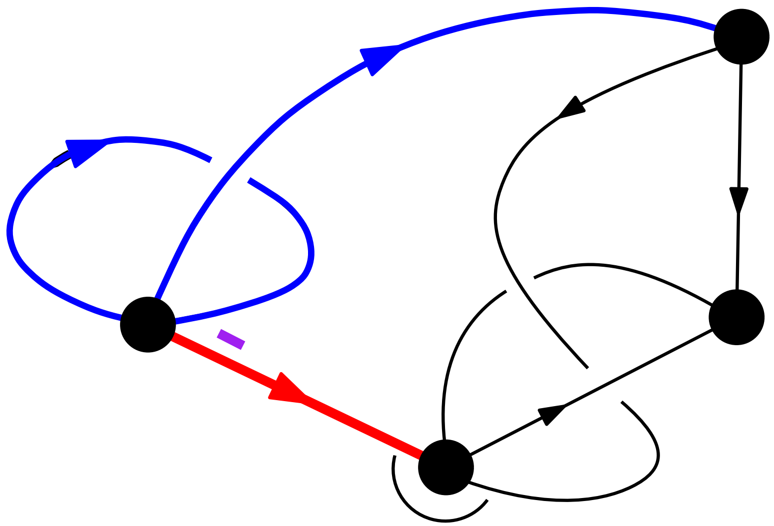

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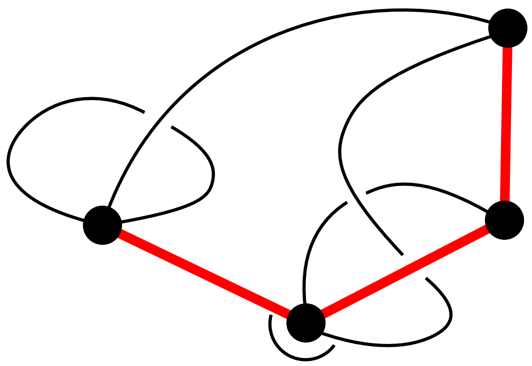

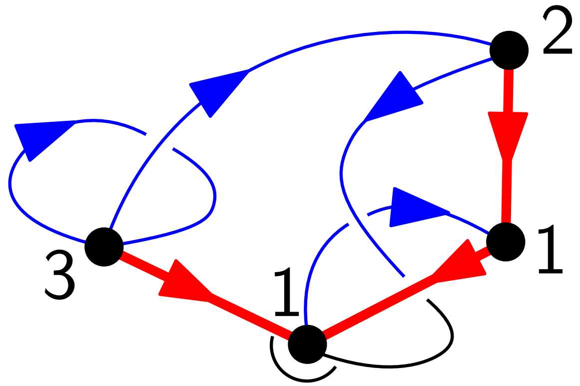

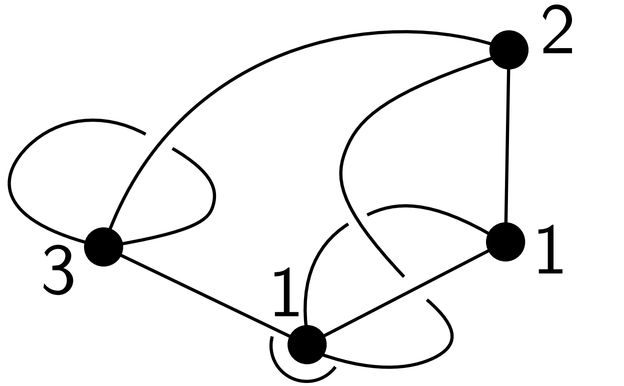

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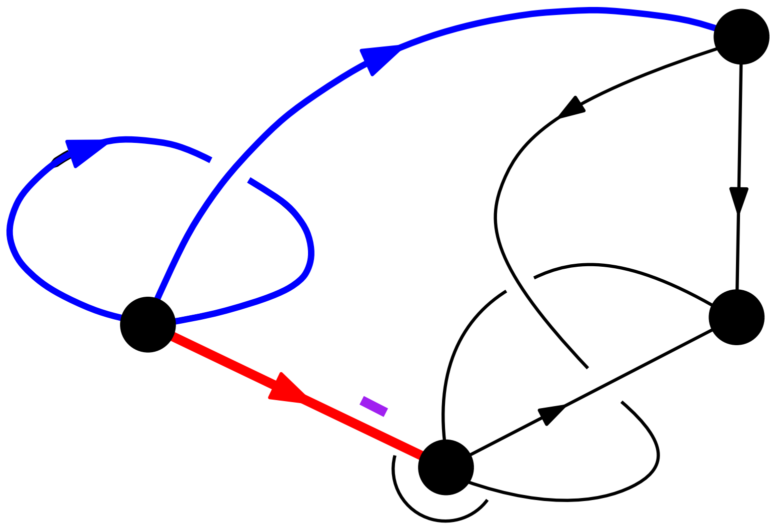

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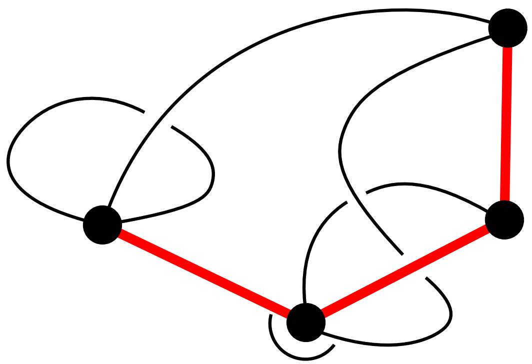

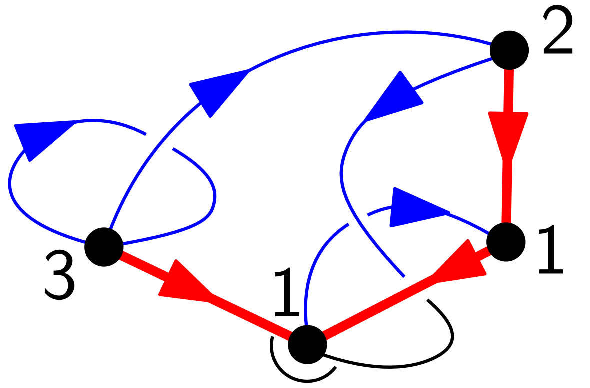

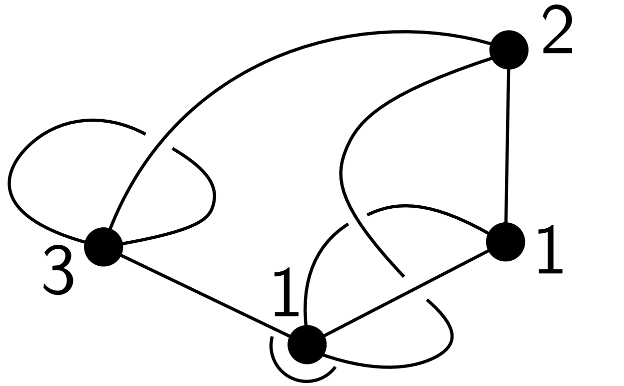

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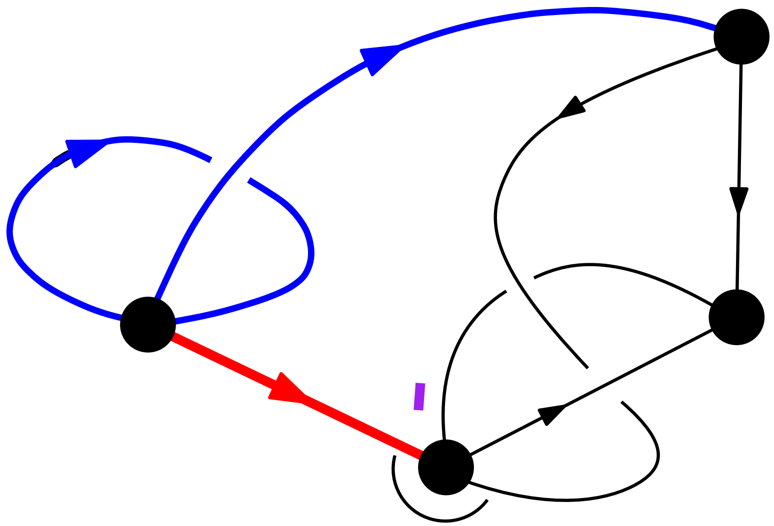

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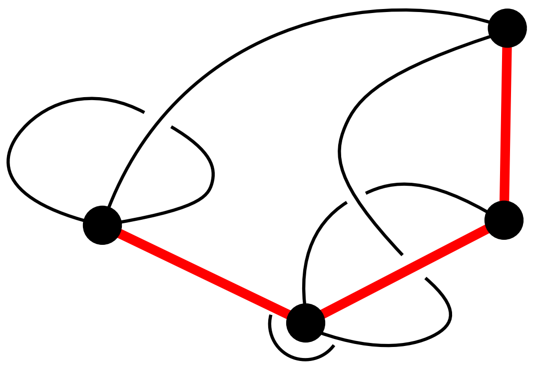

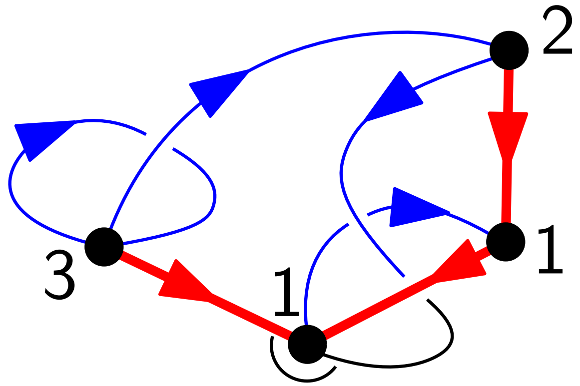

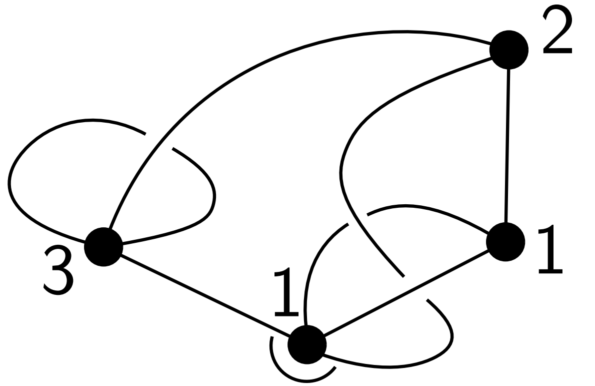

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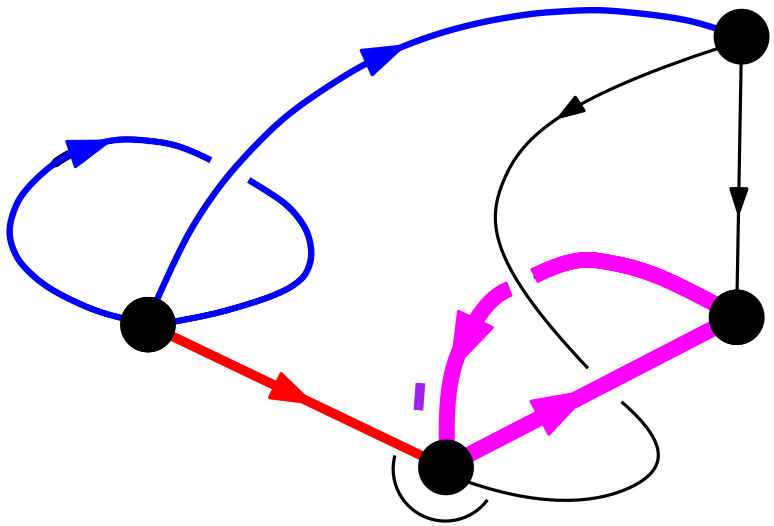

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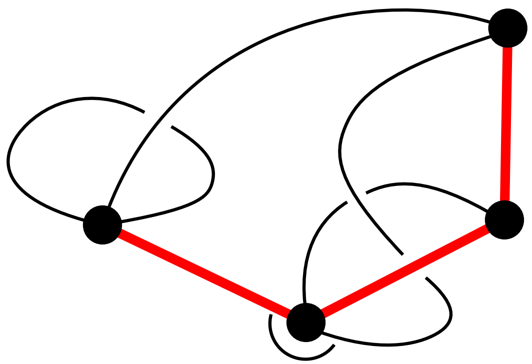

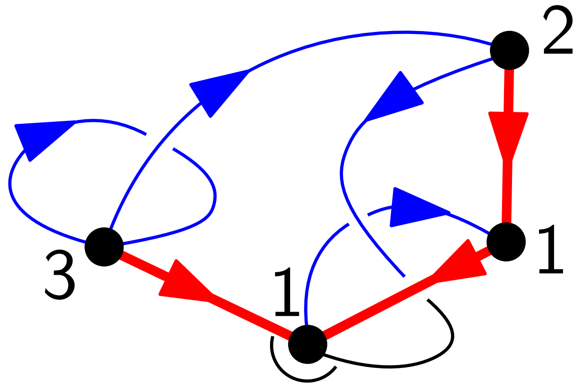

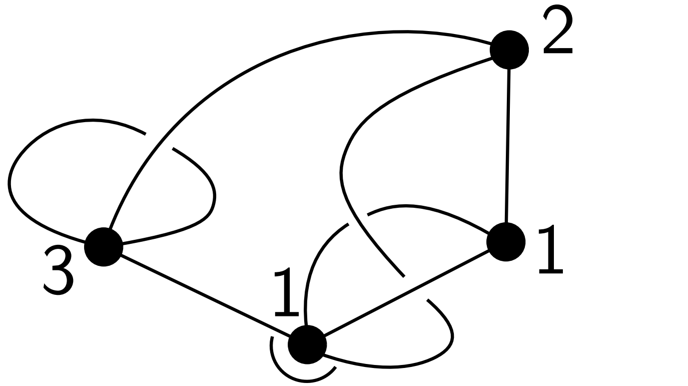

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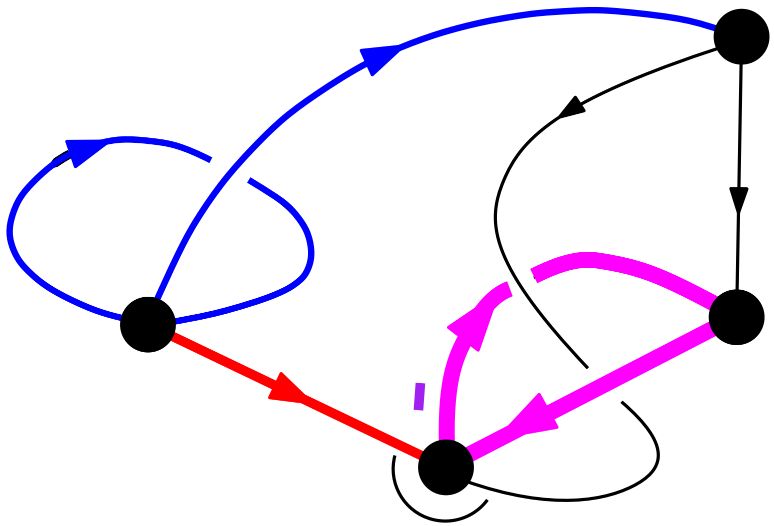

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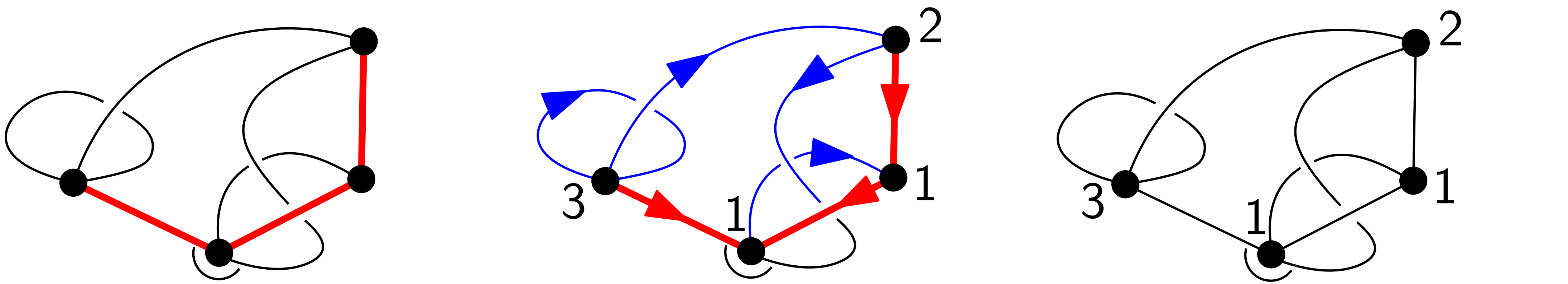

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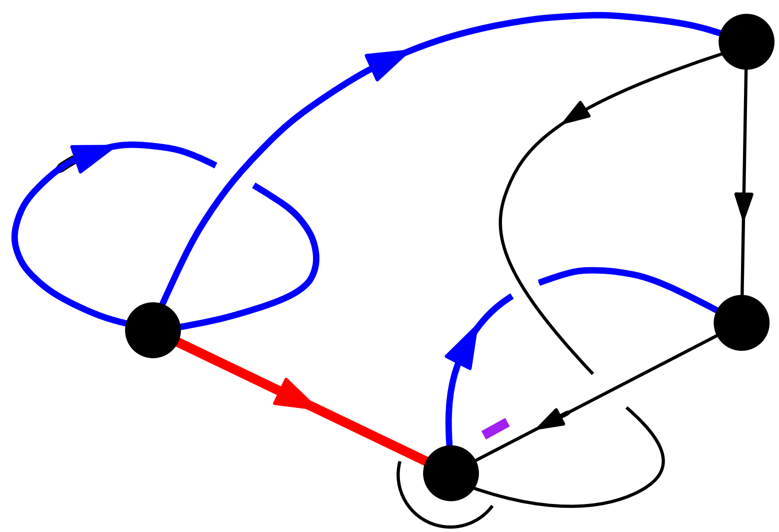

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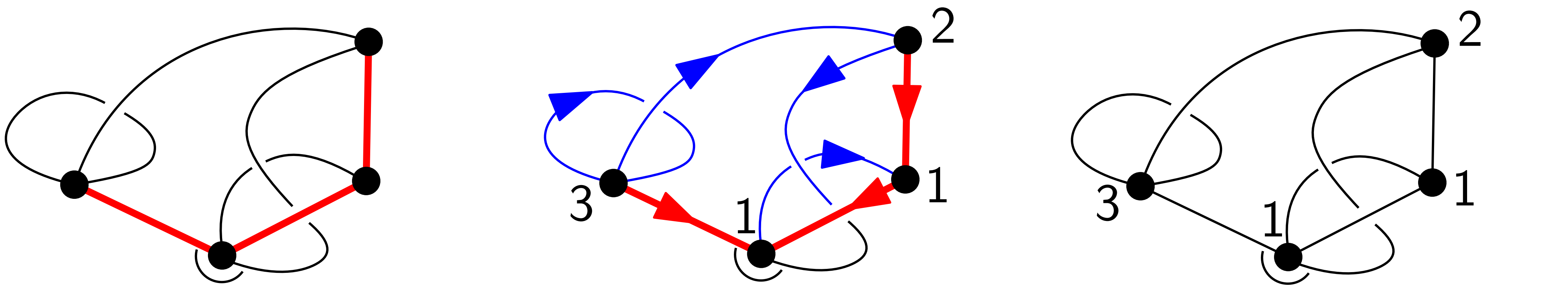

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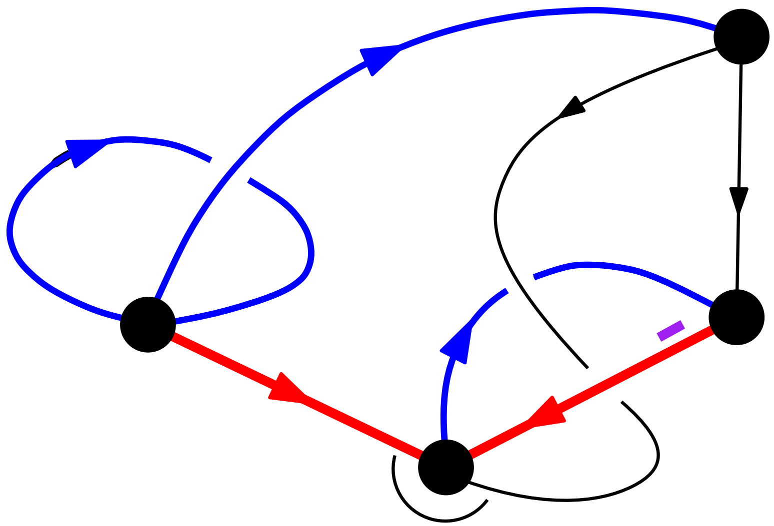

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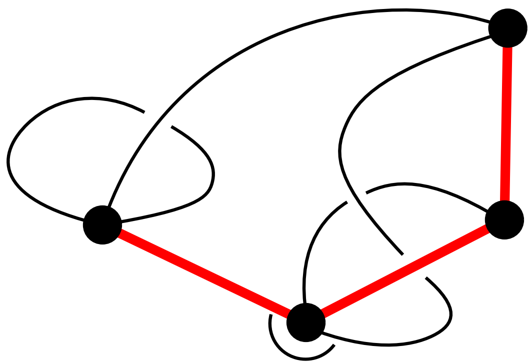

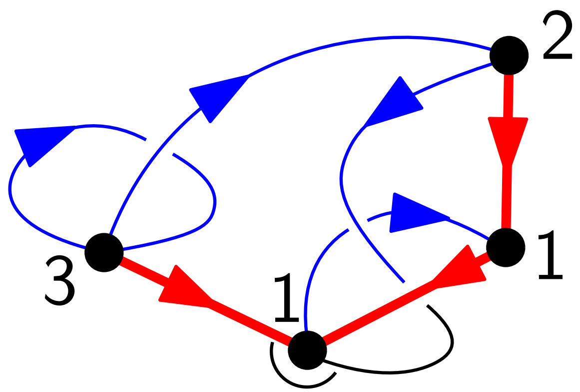

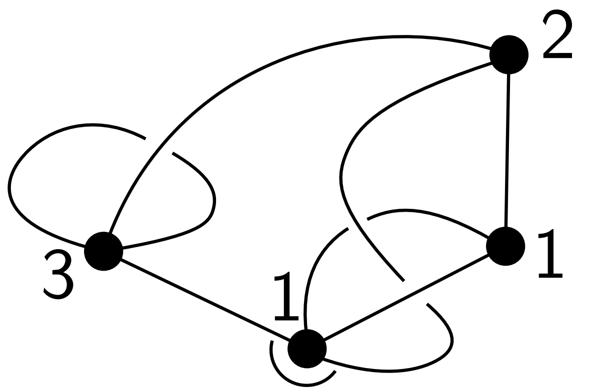

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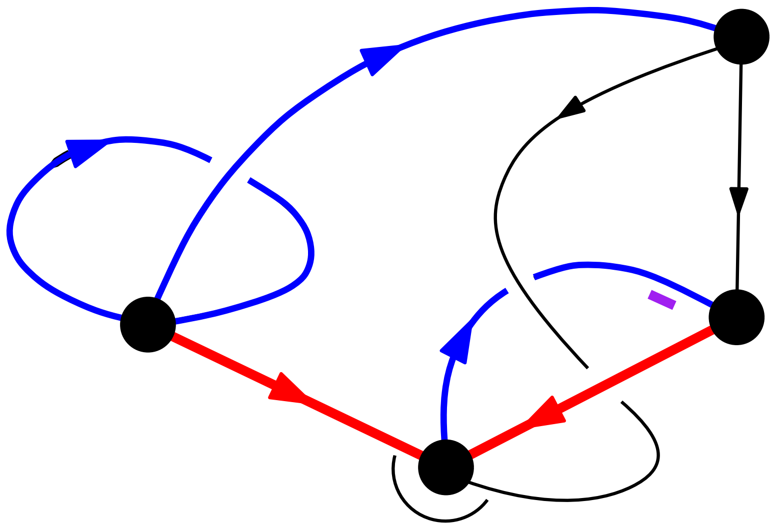

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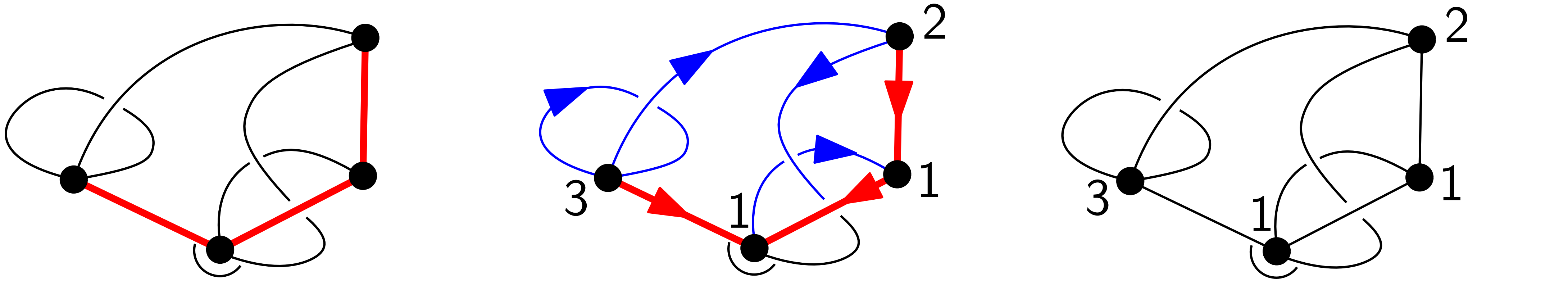

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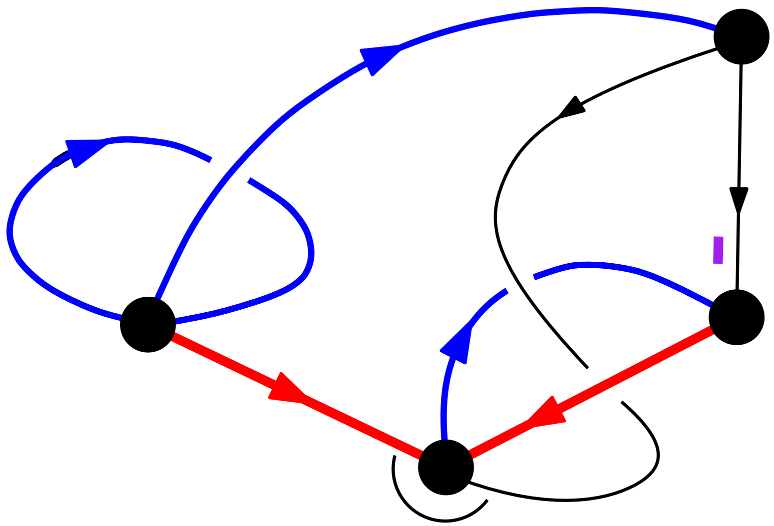

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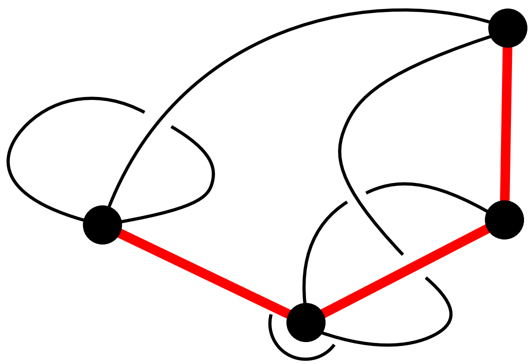

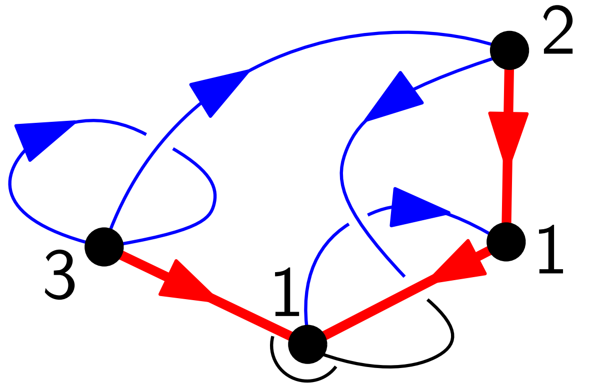

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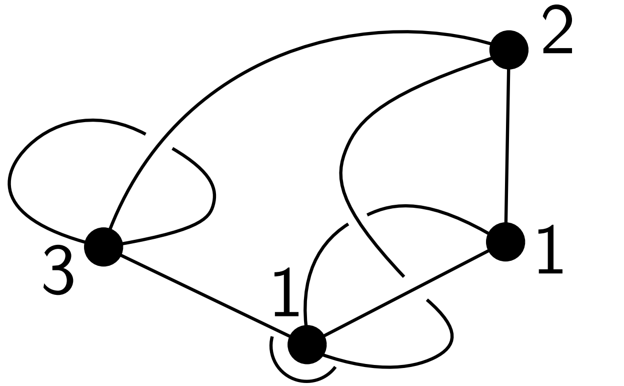

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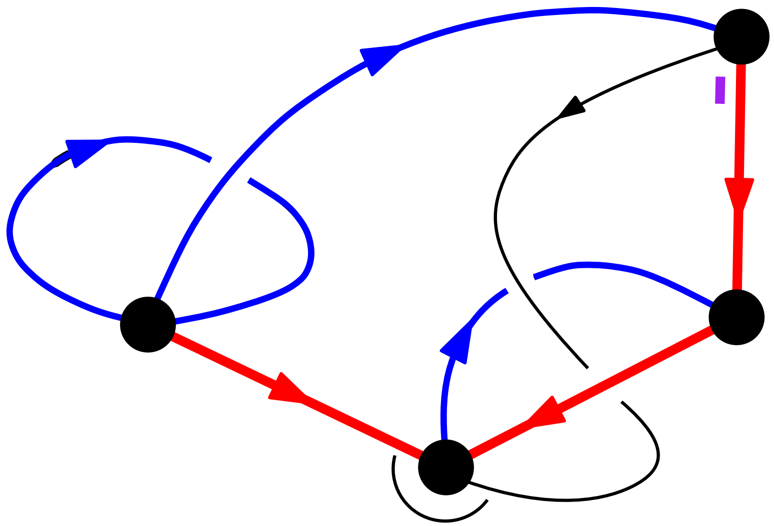

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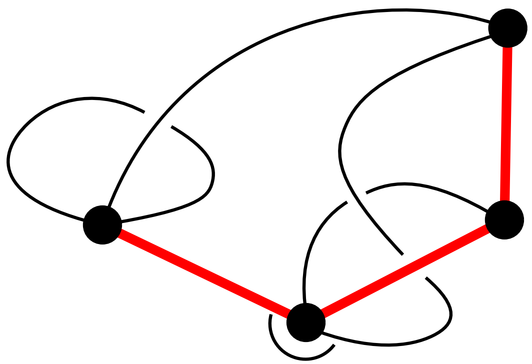

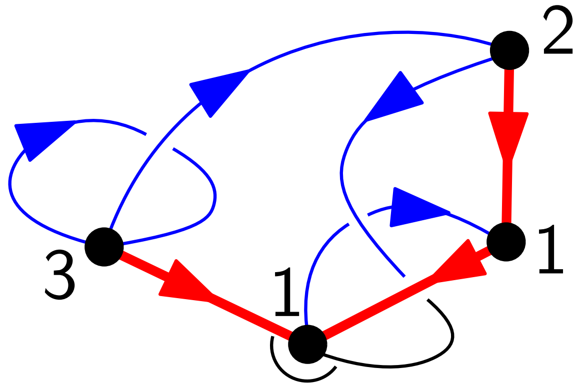

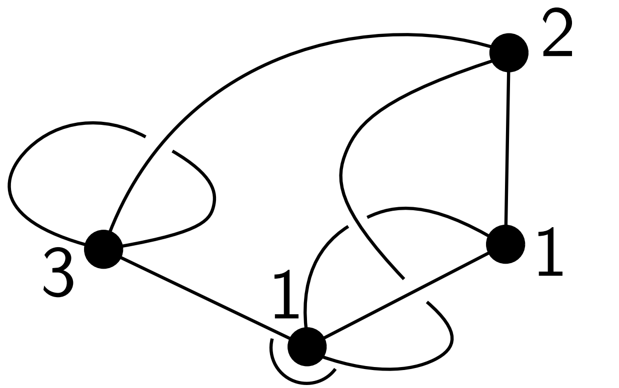

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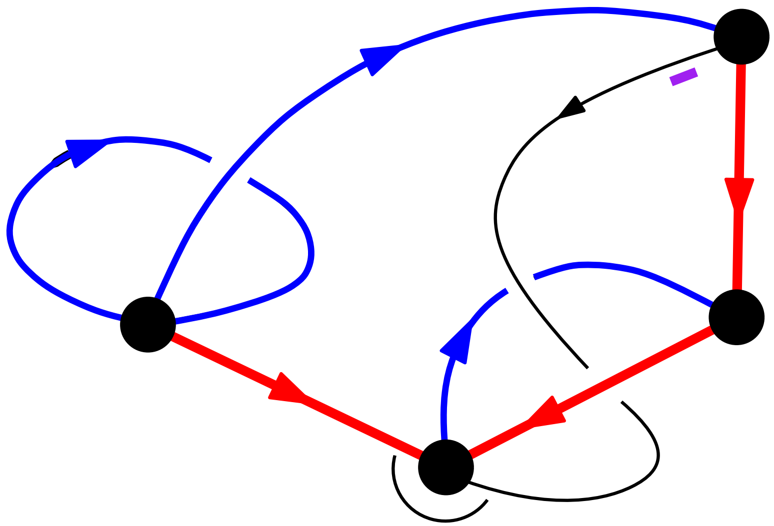

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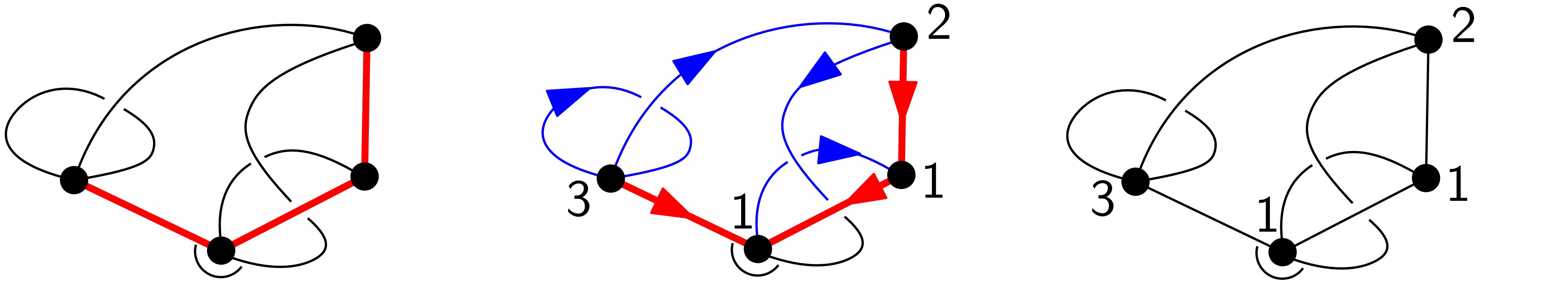

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(spanning tree property)

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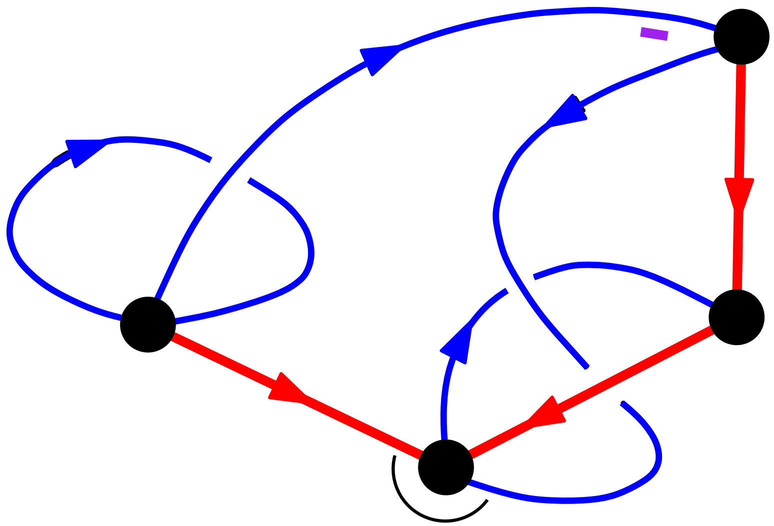

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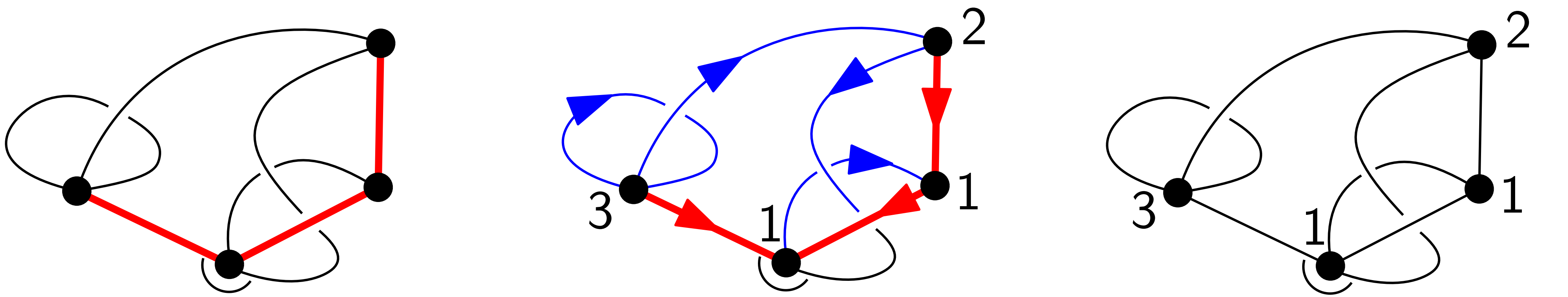

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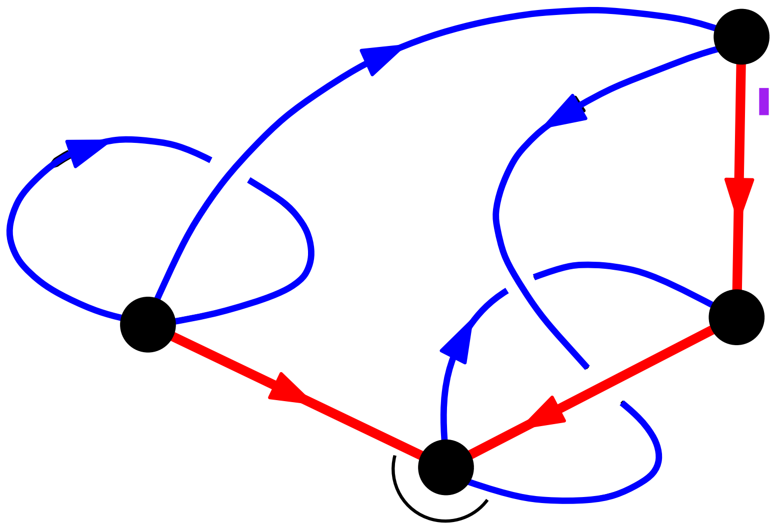

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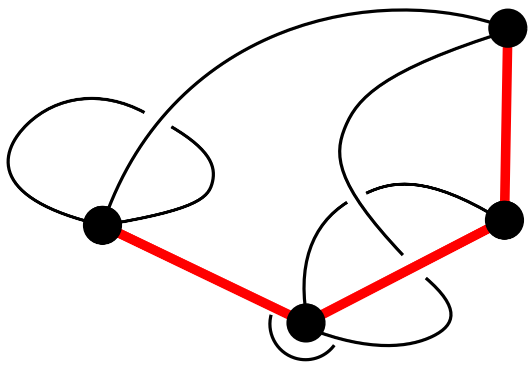

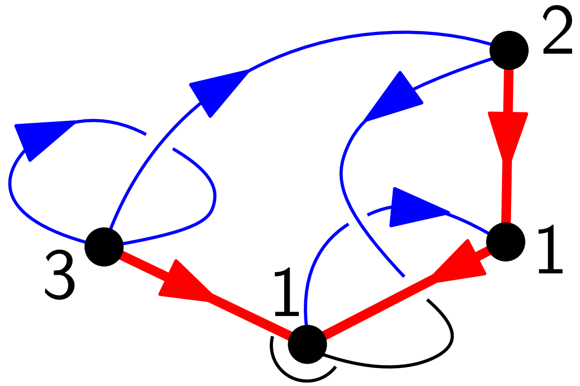

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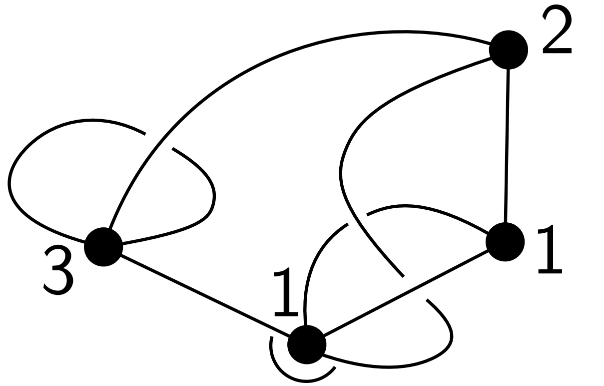

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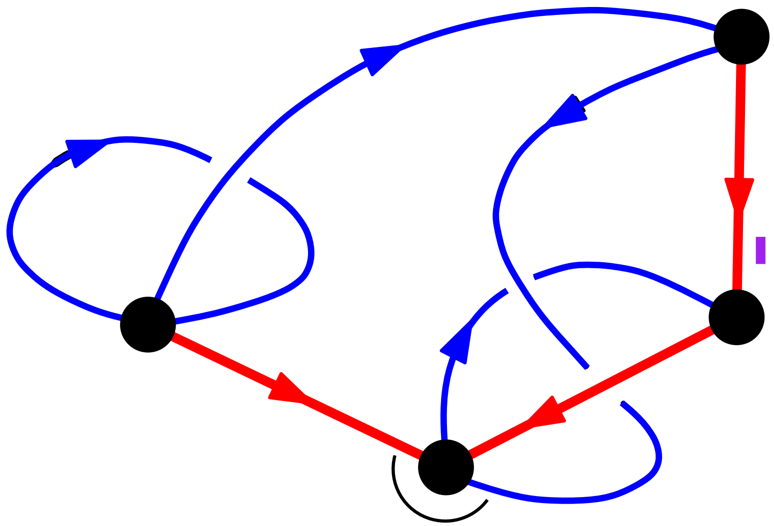

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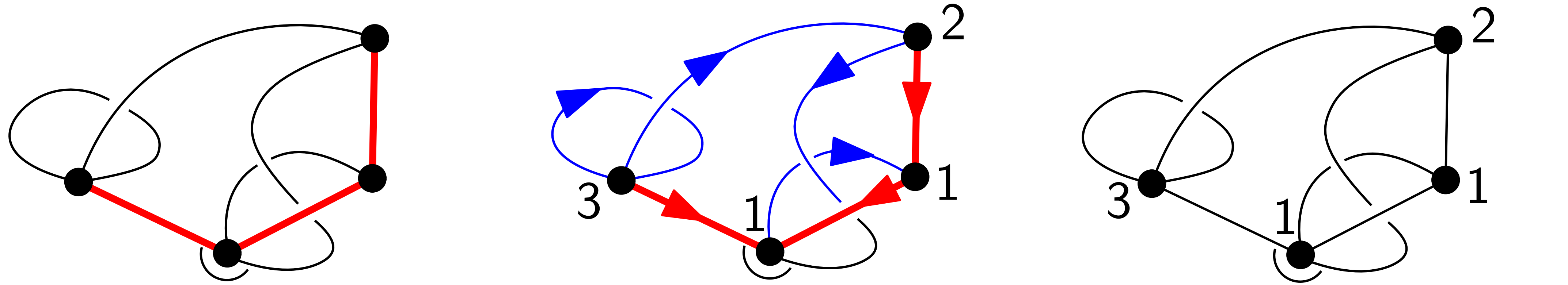

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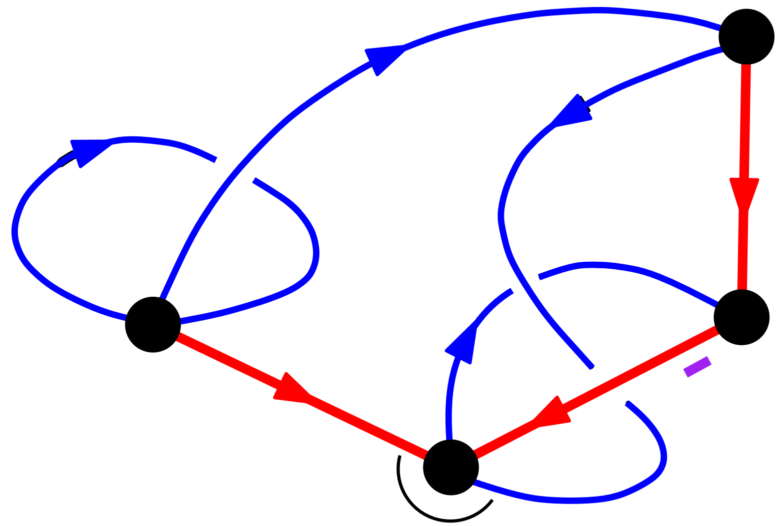

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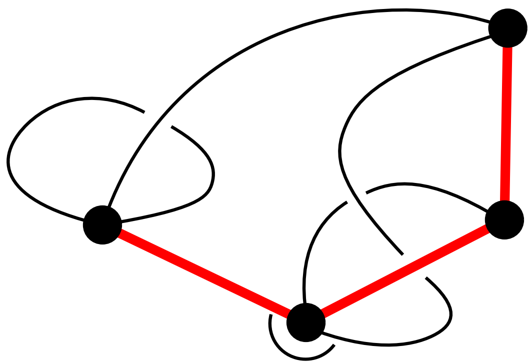

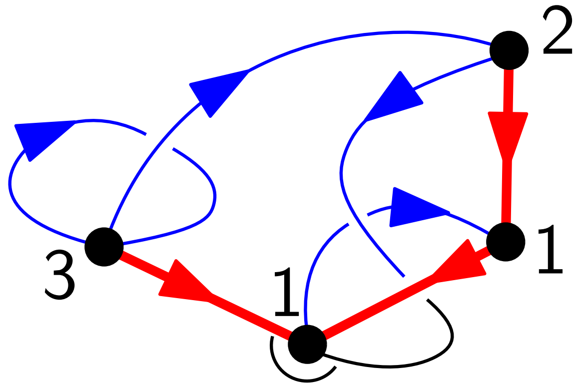

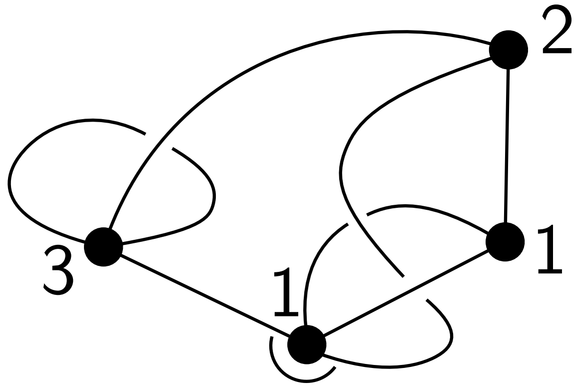

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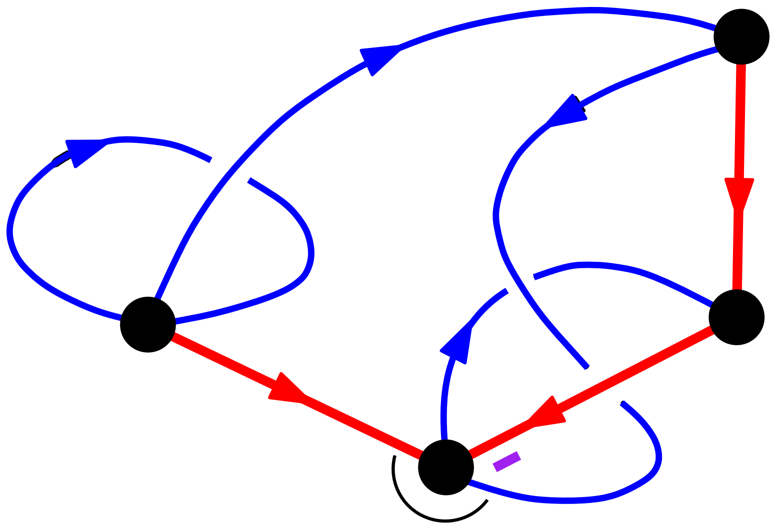

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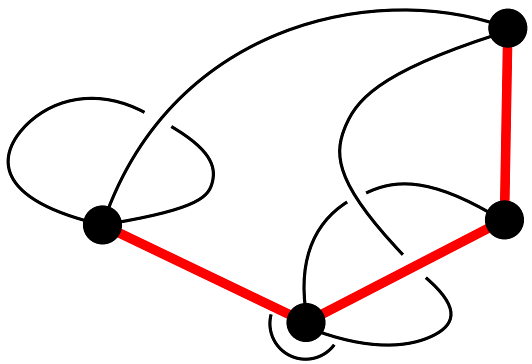

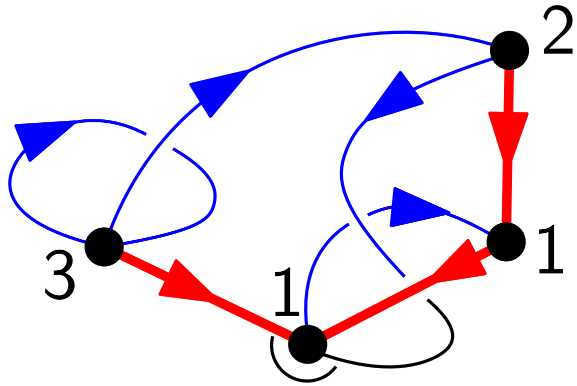

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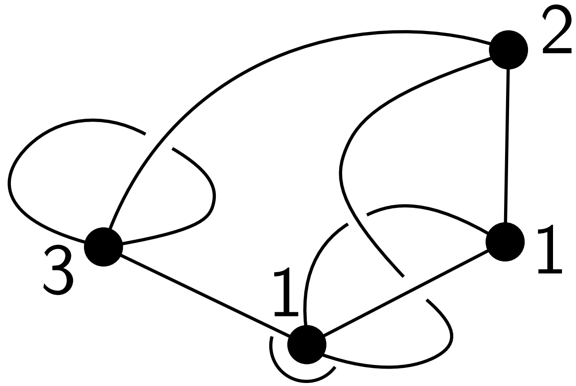

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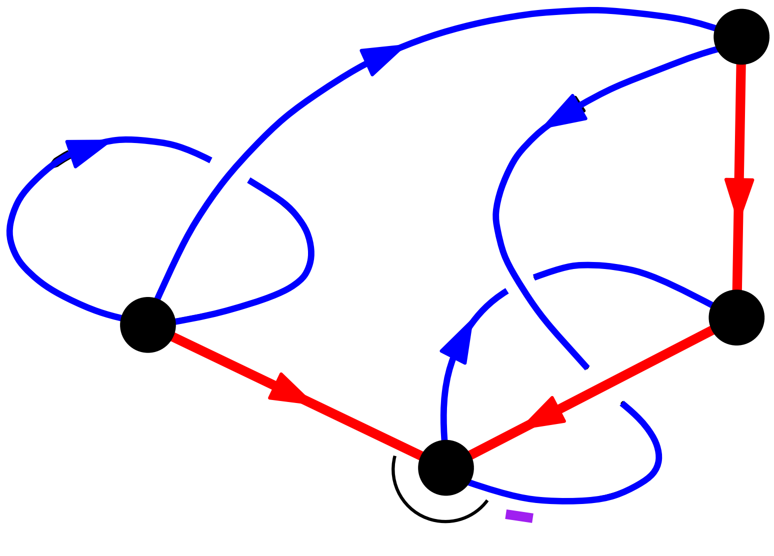

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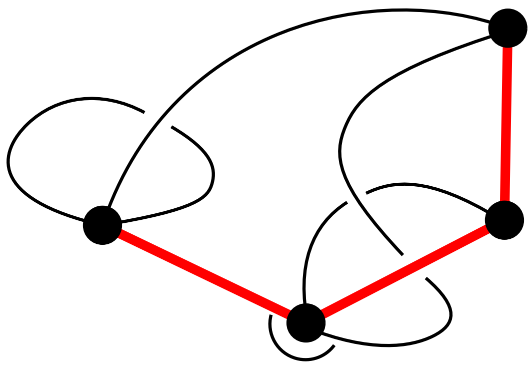

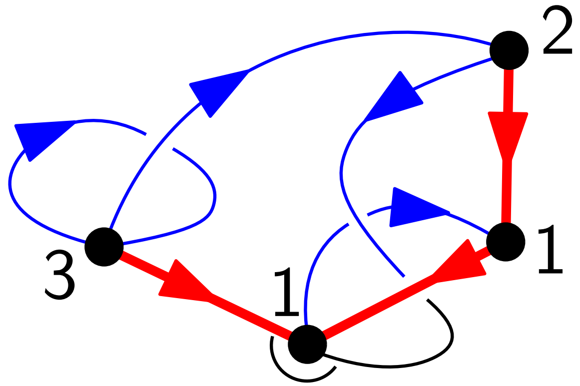

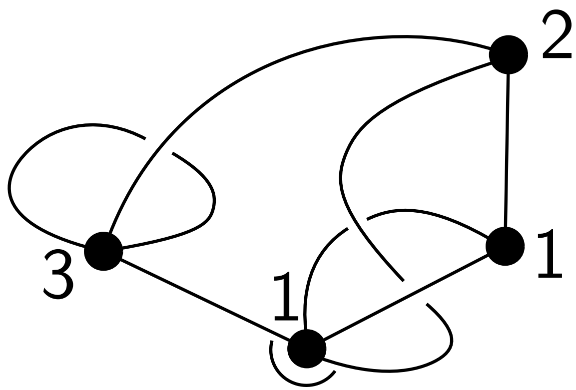

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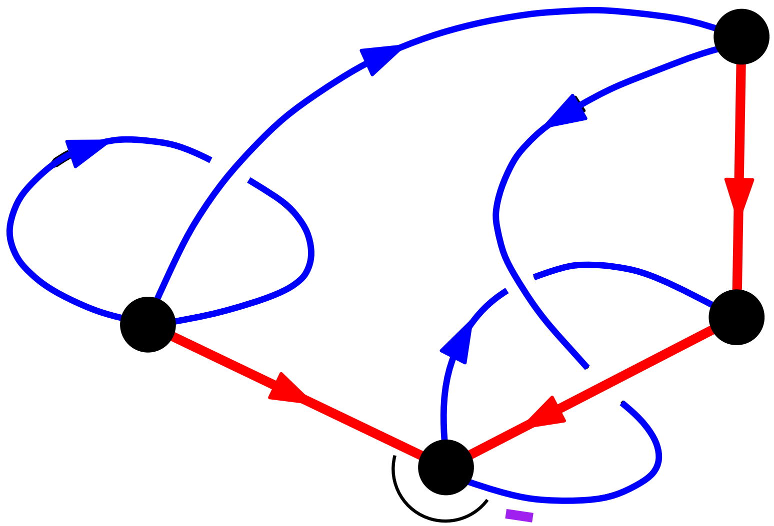
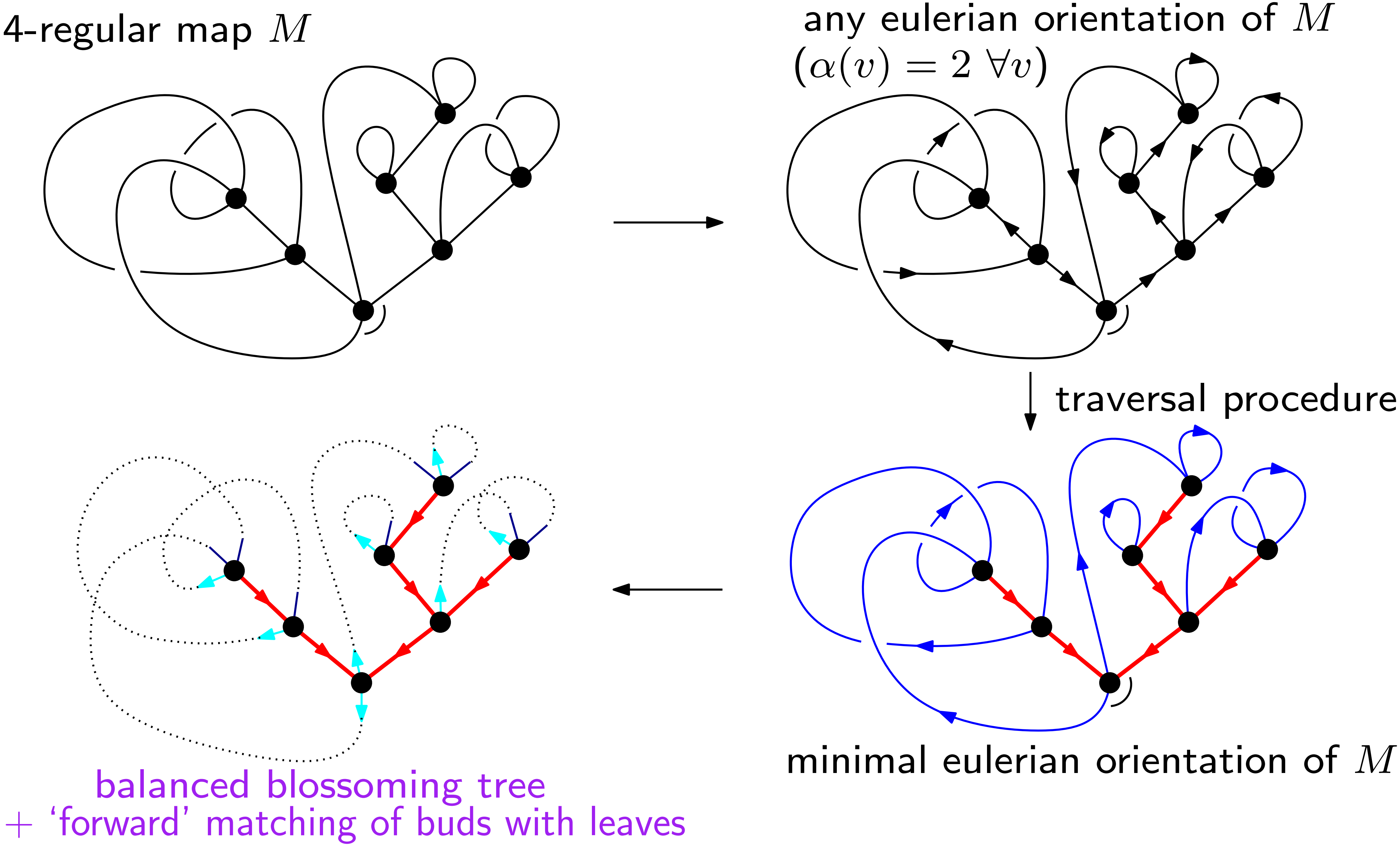


Application to 4-regular maps



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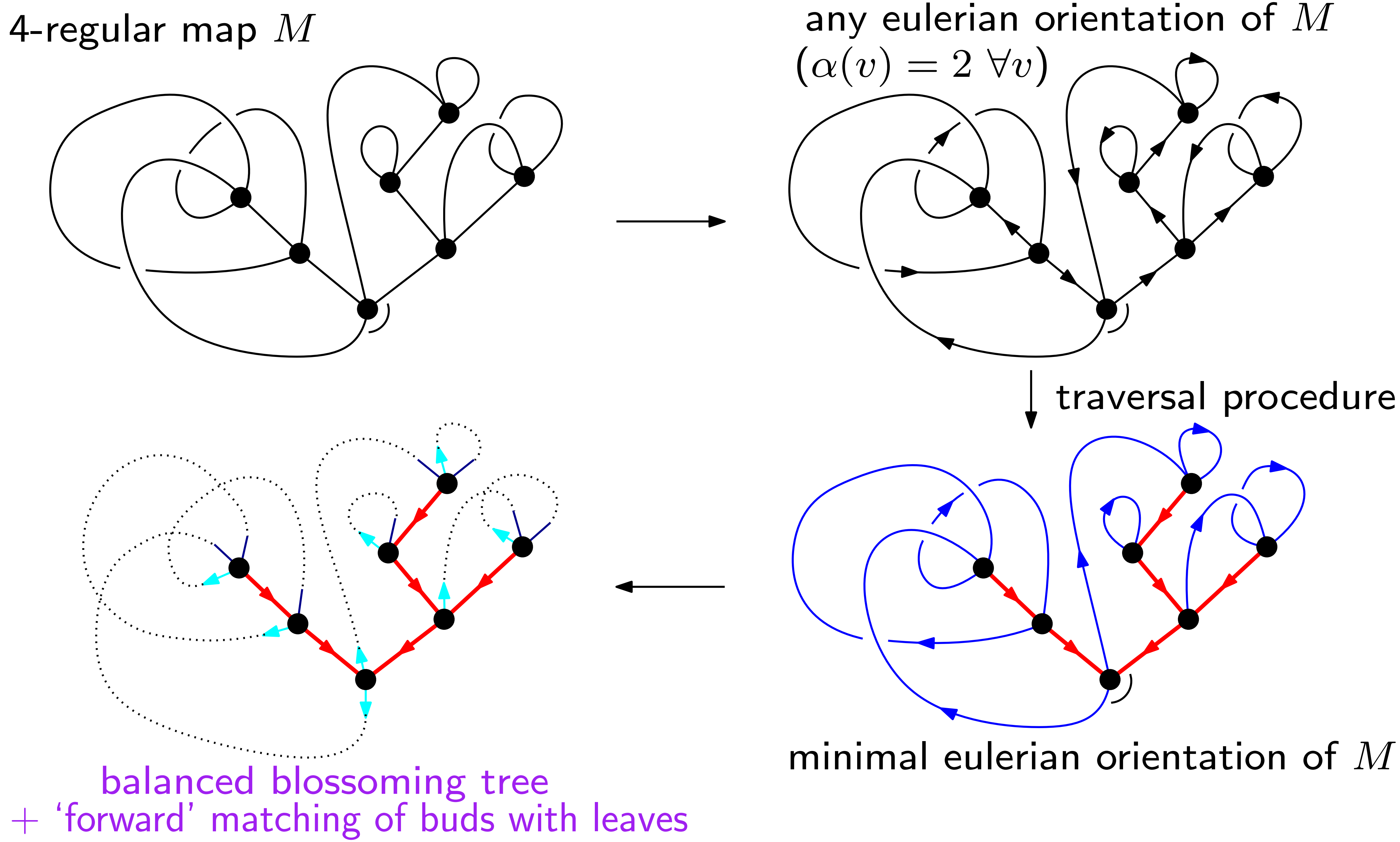
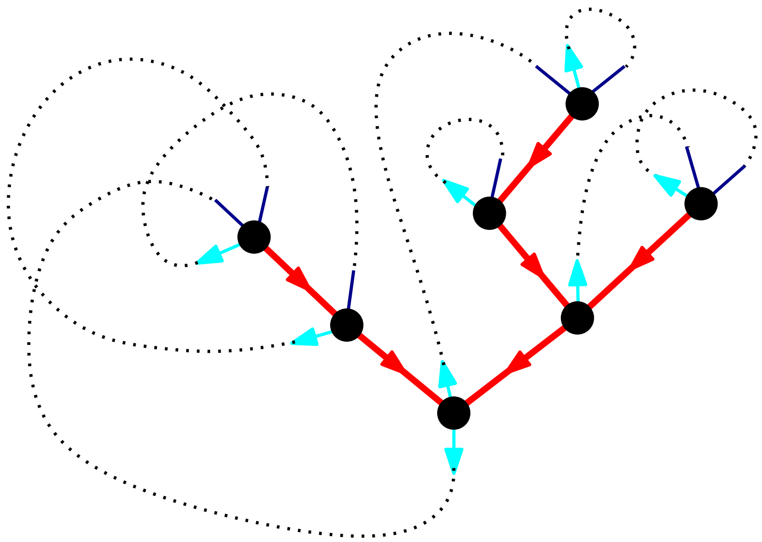
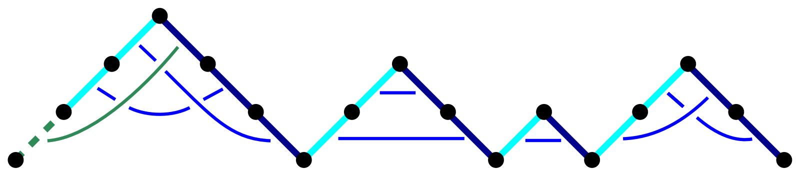


Rk: different from extensions of Schaeffer's bijection with control on the genus
[Lepoutre'19, Albenque-Lepoutre'20, Lepoutre-Dołęga'20]

Label-encoding of matching-decorated blossoming trees



leaf-path



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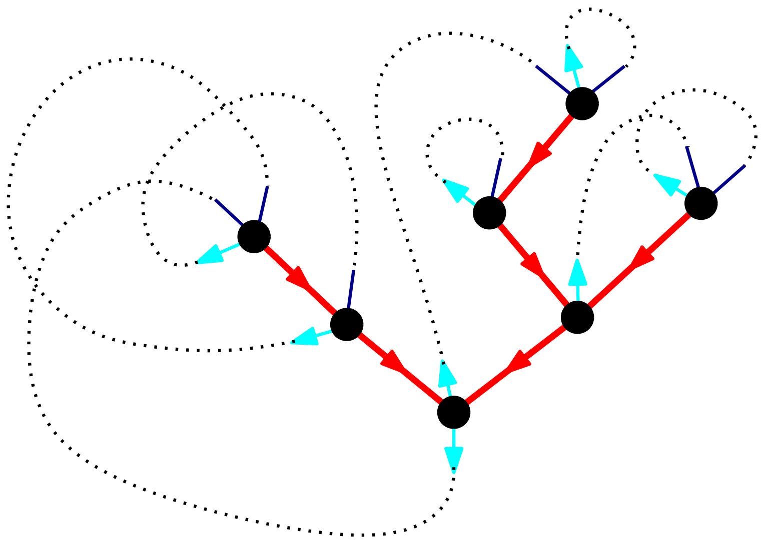

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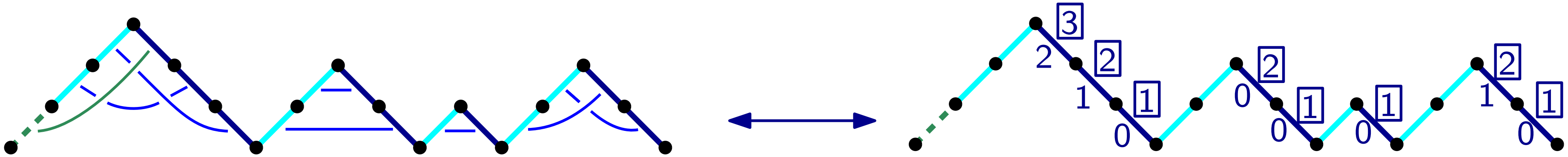
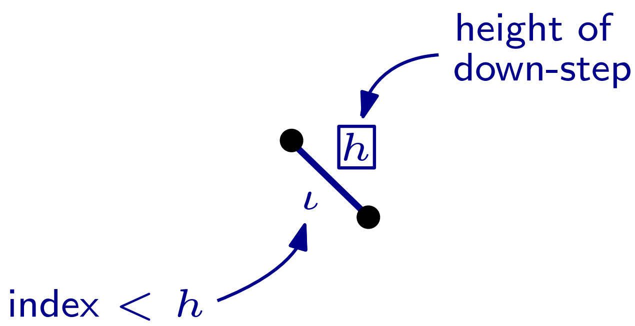


[Flajolet'80]
[de Médicis-Viennot'94]



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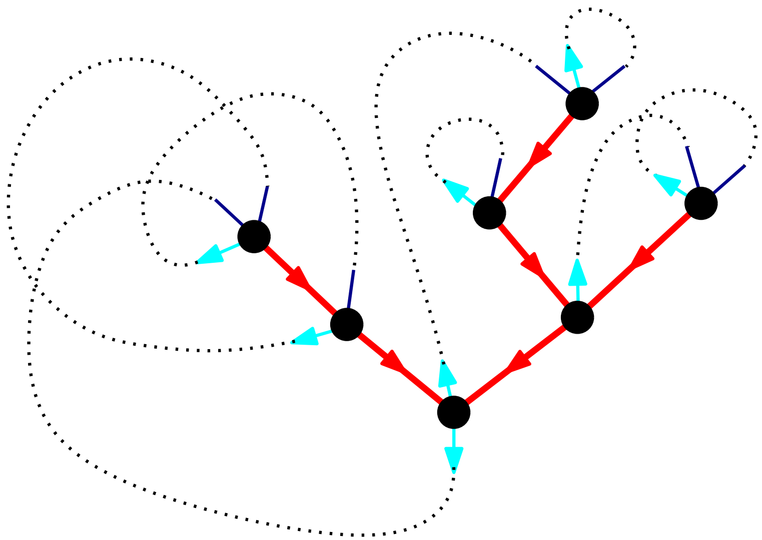

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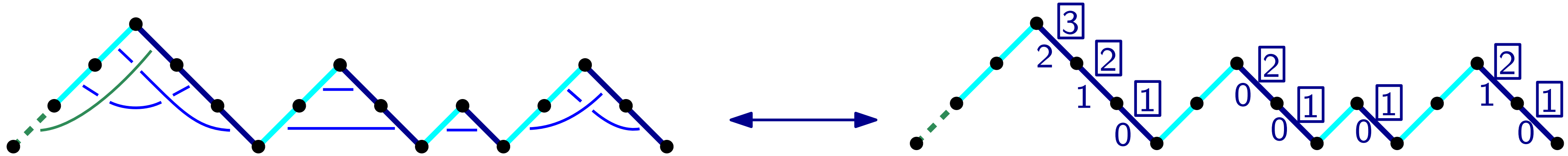

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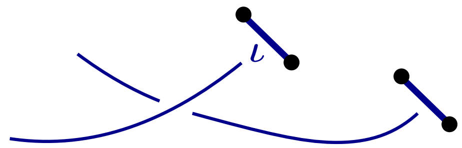
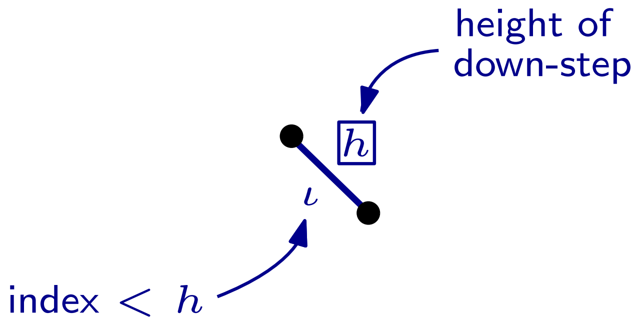
$\sum$ indices = # crossings





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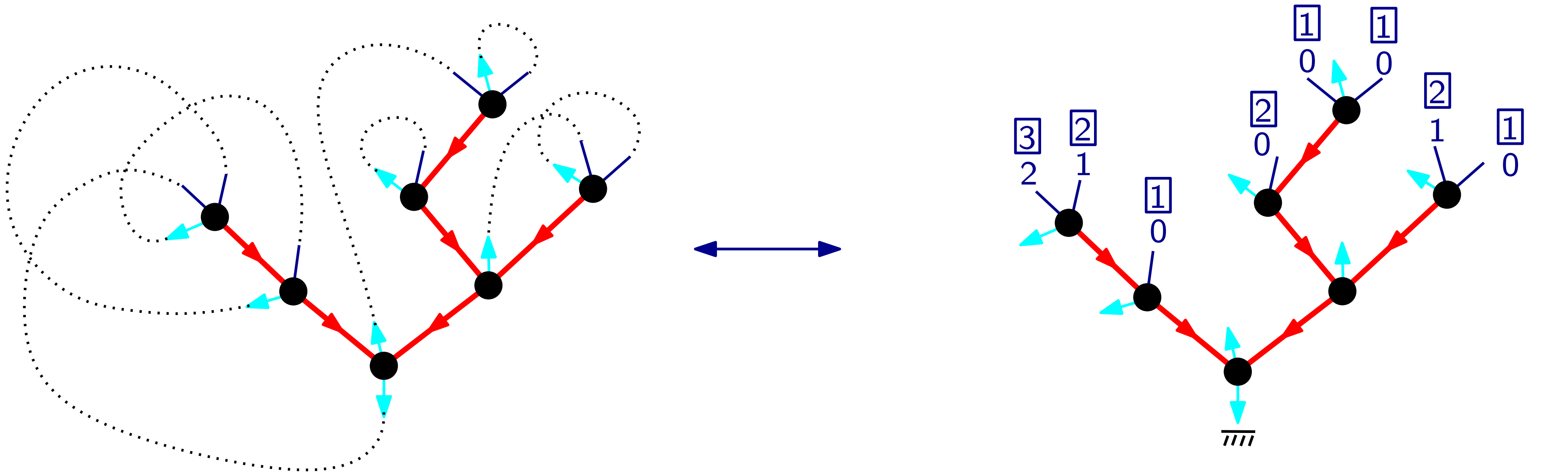

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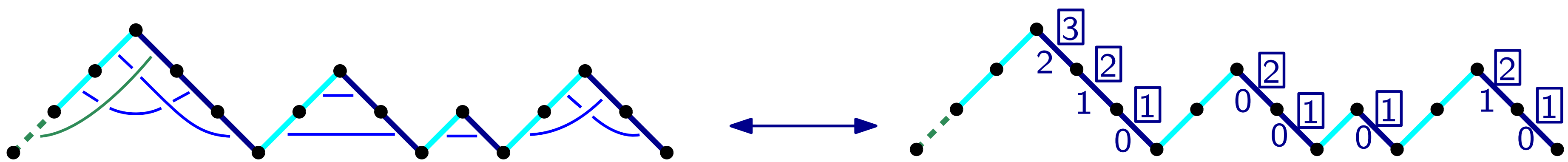

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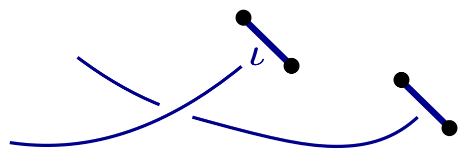
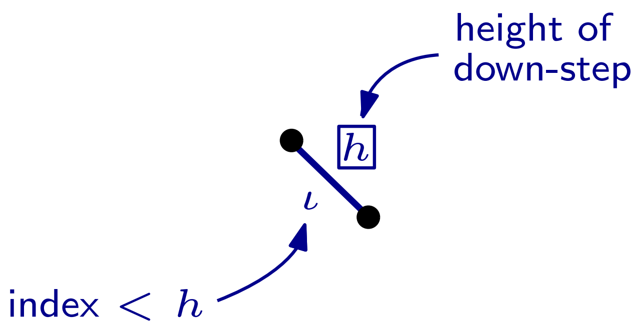
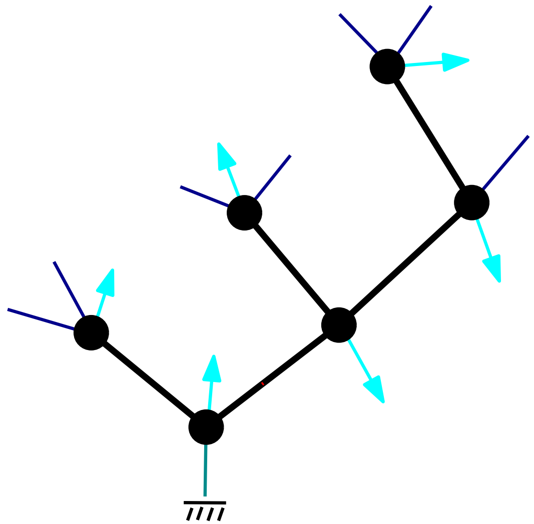
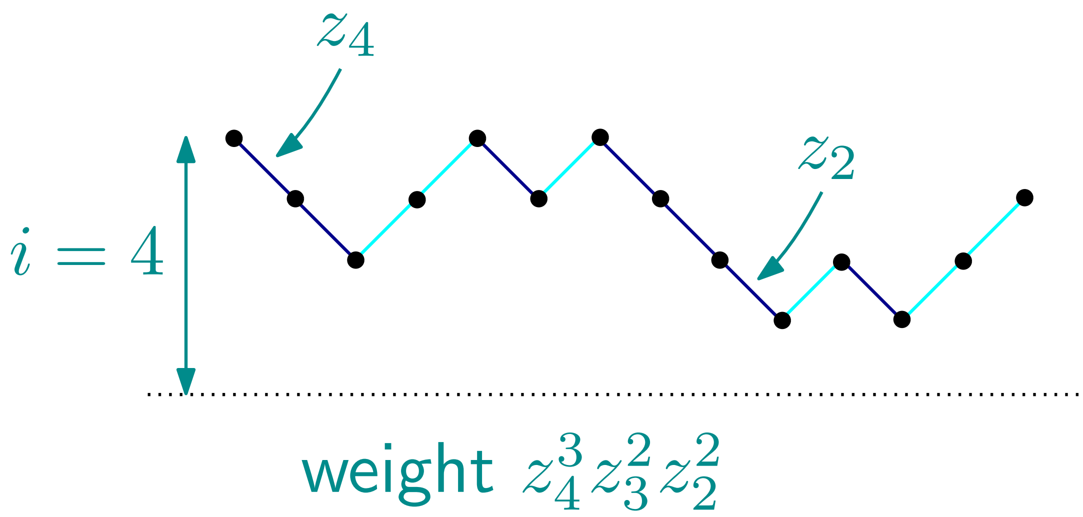


$\iota = \#$ such crossings



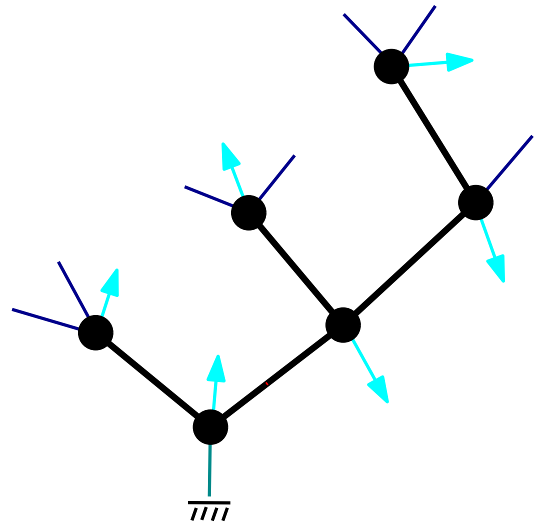
Generating function expressions

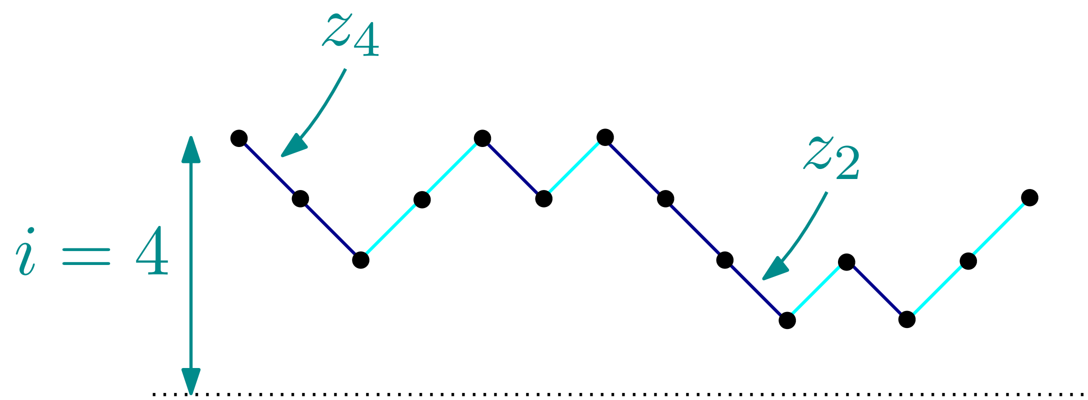




Let $\hat{r}_i(t)$ = GF i-balanced blossoming trees
with weight z_h per leaf of height h

Generating function expressions





weight $z_4^3 z_3^2 z_2^2$

Let $\hat{r}_i(t)$ = GF i-balanced blossoming trees
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bijection
$\Downarrow$
$M(t) = \hat{r}_1(t)|_{z_h=h}$

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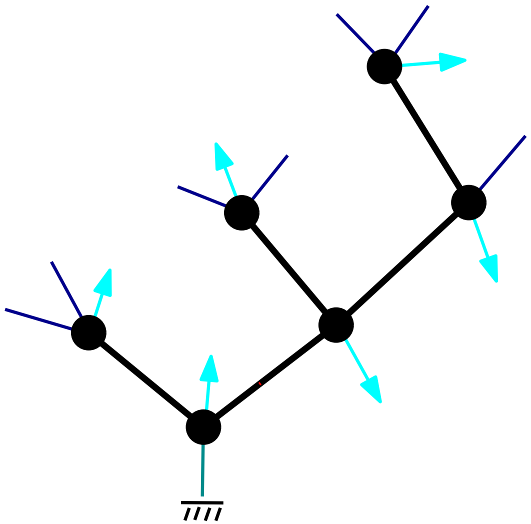

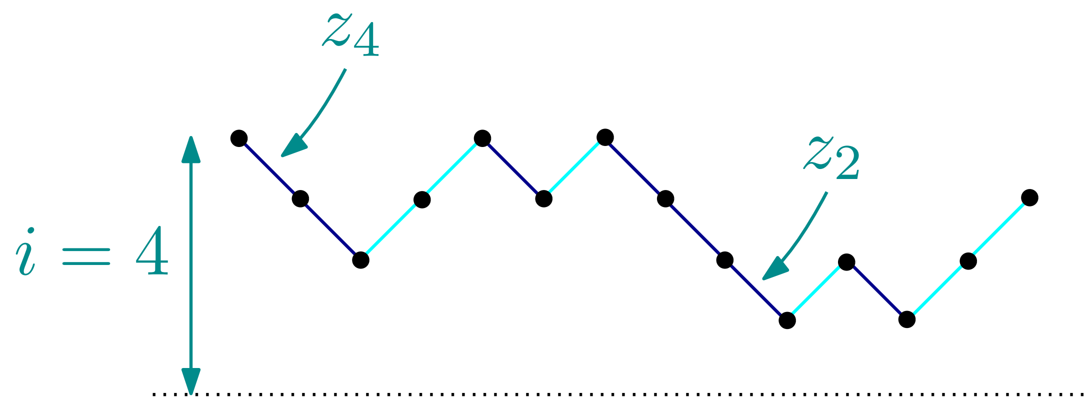

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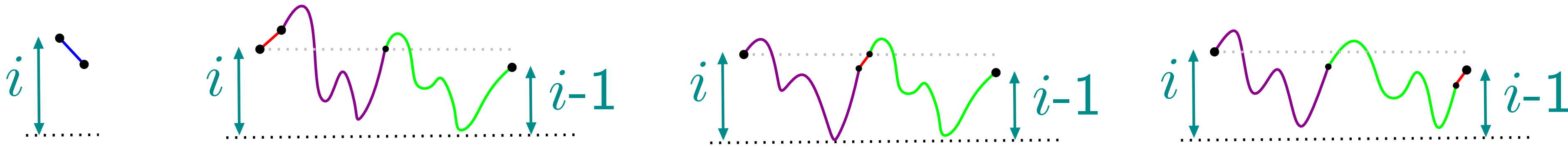
Equation:

$$\hat{r}_i(t) \quad = \quad z_i \quad + \quad t\,\hat{r}_{i+1}(t)\hat{r}_i(t) \quad + \quad t\,\hat{r}_i(t)\hat{r}_i(t) \quad + \quad t\,\hat{r}_i(t)\hat{r}_{i-1}(t)$$

leaf-path

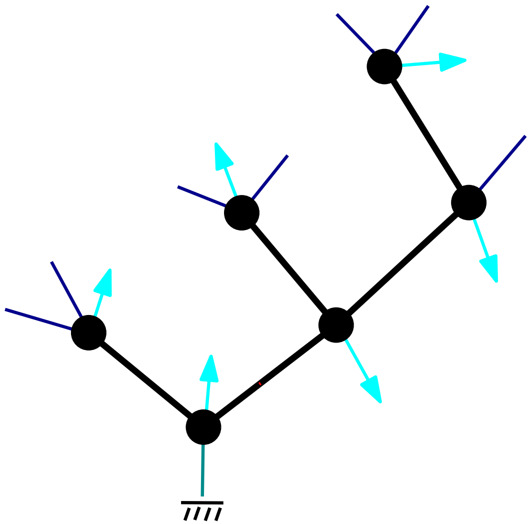


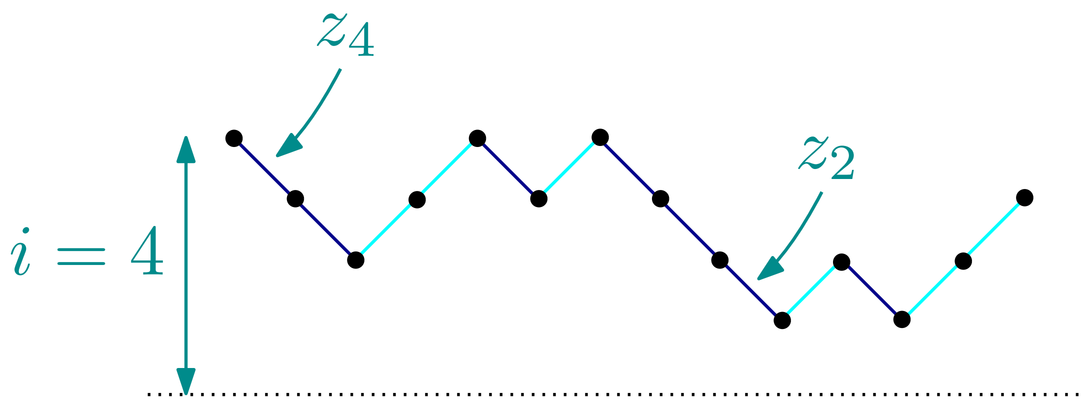
$\hat{r}_0 = 0$

$$\boxed{\hat{r}_i(t) = z_i + t\hat{r}_i(t) \cdot (\hat{r}_{i-1}(t) + \hat{r}_i(t) + \hat{r}_{i+1}(t))}$$

for $i \geq 1$

Generating function expressions





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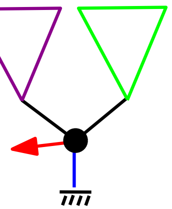

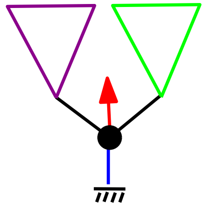

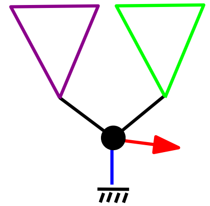

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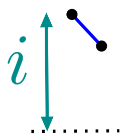
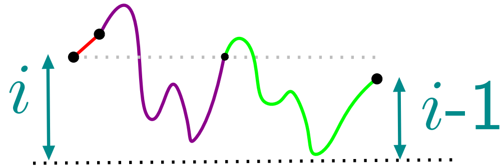
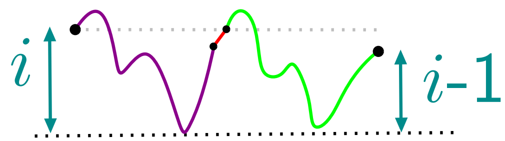
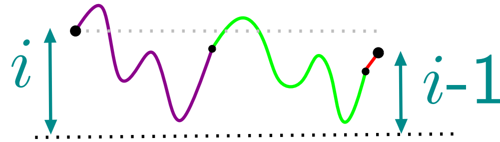
leaf-path







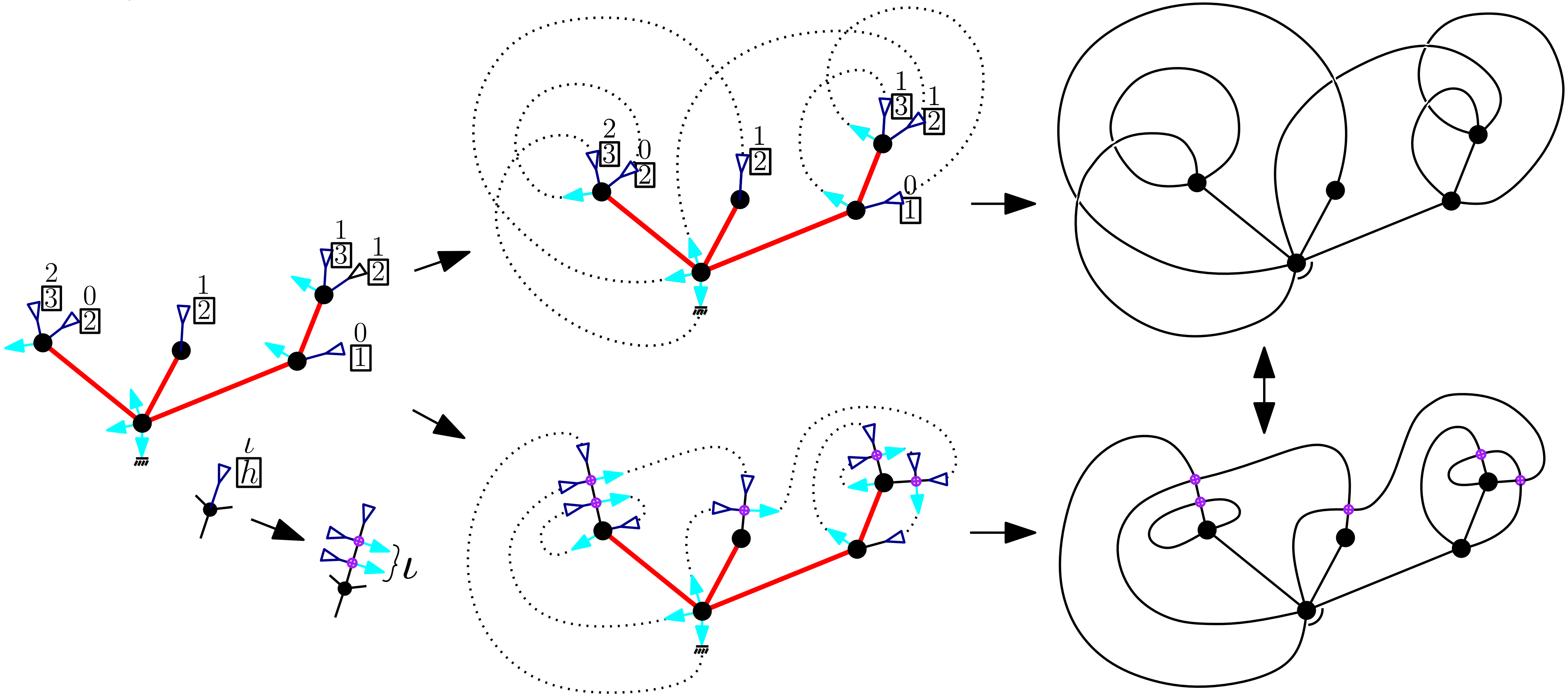


$\hat{r}_0 = 0$ $\quad \boxed{\hat{r}_i(t) = z_i + t\hat{r}_i(t) \cdot (\hat{r}_{i-1}(t) + \hat{r}_i(t) + \hat{r}_{i+1}(t))}$ $\quad$ for $i \geq 1$

We have $R_i(t) = \hat{r}_i(t)|_{z_h=1}$ $\qquad r_i(t) = \hat{r}_i(t)|_{z_h=h}$ combinatorial interpretation

Planarized version of the bijection

Bijection $\Leftrightarrow$ leaf-extensions + Schaeffer's planar construction



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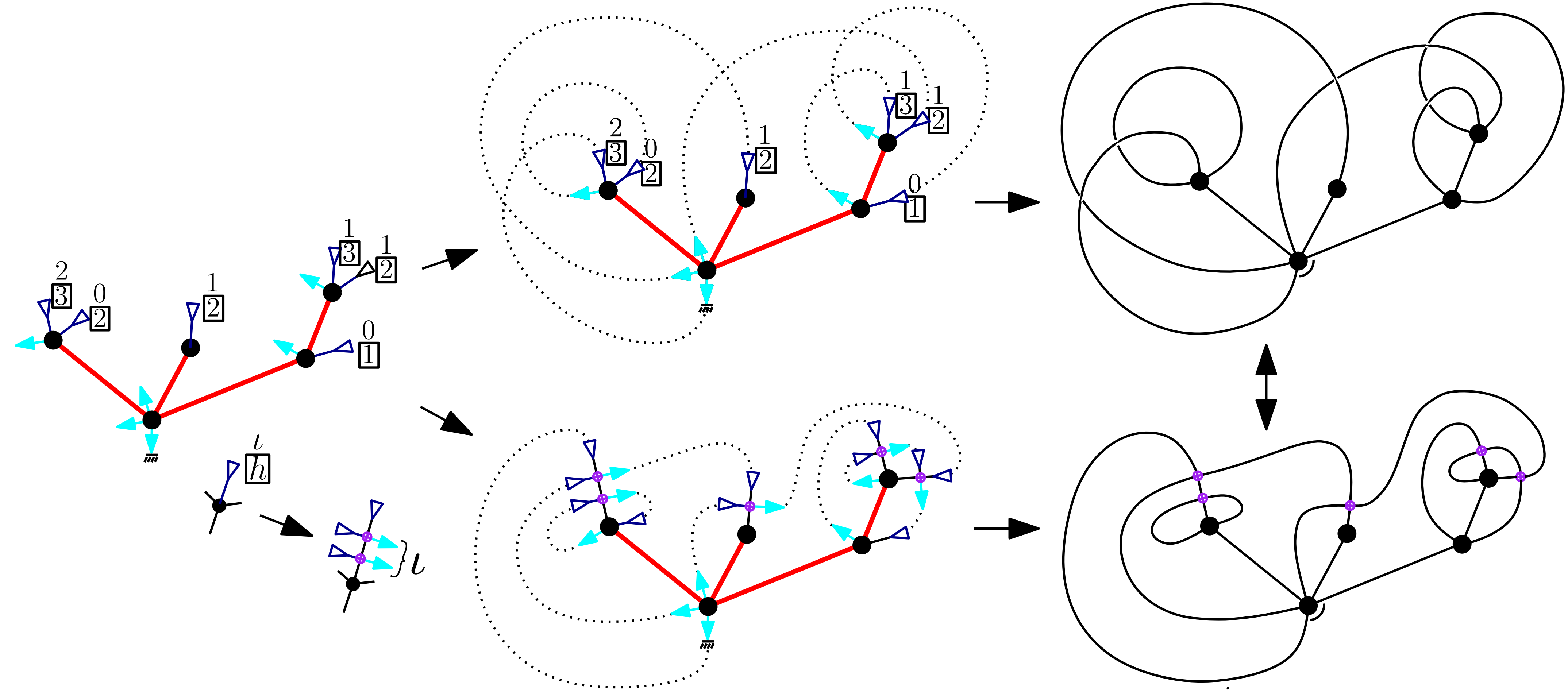


Rk: Let $r_i(t,q) := \hat{r}_i|_{z_j=[j]}$, where $[j] := 1 + \cdots + q^{j-1} = \dfrac{1-q^j}{1-q}$

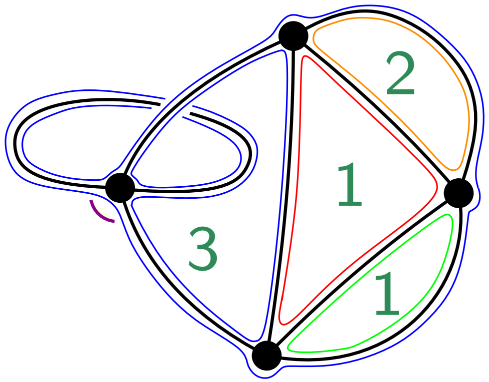
Then $r_1(t,q) =$ GF of Eulerian maps with q conjugate to crossing-number

$R_i(t) = r_i(t,0)$ $\qquad$ $r_i(t) = r_i(t,1)$

w.r.t canonical spanning tree

Bijection for $N \geq 2$?

Let $\widehat{M}(t, N)$ = GF of **fully-N-colored** 4-regular maps
(every color $\in [1..N]$ is used by at least one face)



a fully N-colored map, for $N = 3$

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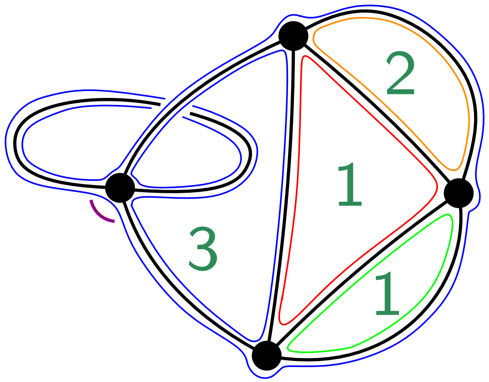
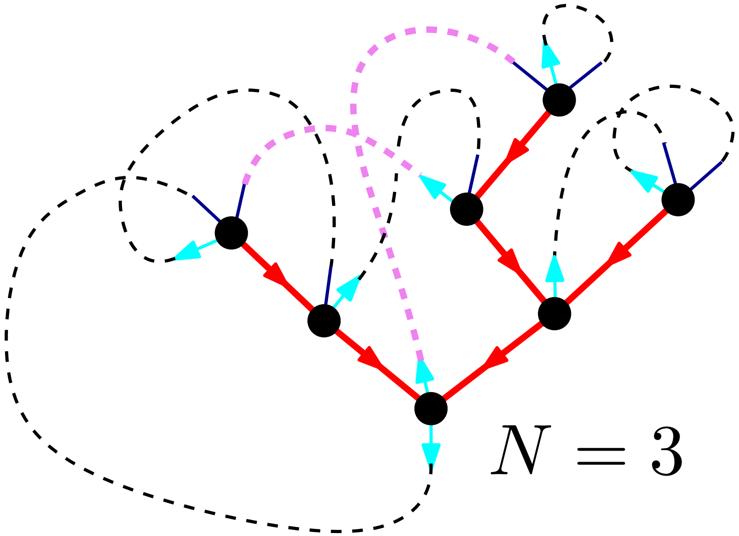
Let $\widehat{M}(t,N)$ = GF of **fully-N-colored** 4-regular maps

(every color $\in [1..N]$ is used by at least one face)



a fully N-colored map, for $N = 3$

$\widehat{M}(t,N)$ = GF of

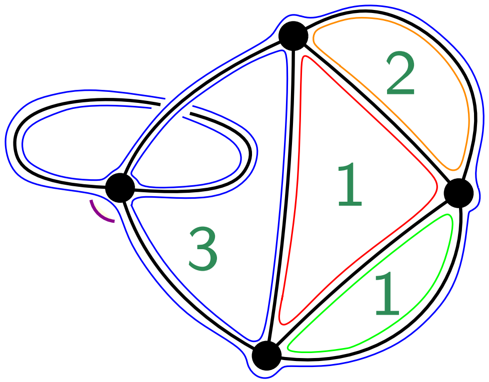


$N-1$ marked matched pairs allowed not to be forward

bijection ?

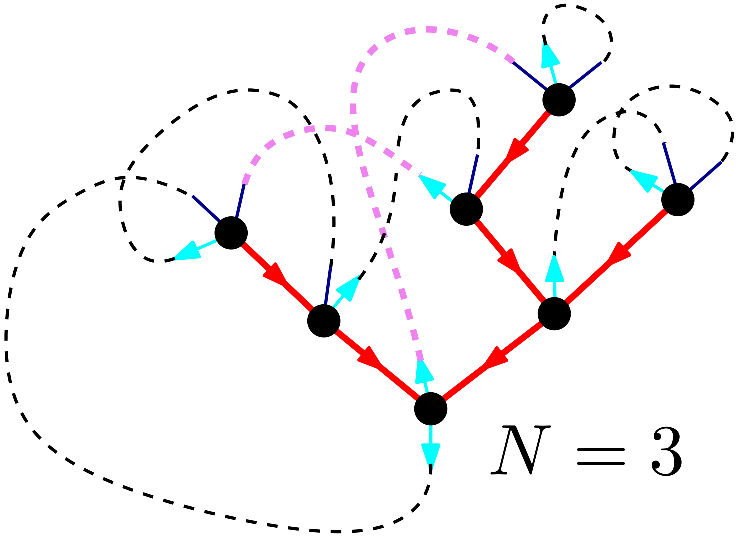
Bijection for $N \geq 2$?

Let $\widehat{M}(t,N)$ = GF of **fully-N-colored** 4-regular maps
(every color $\in [1..N]$ is used by at least one face)



a fully N-colored map, for $N = 3$

$\widehat{M}(t,N)$ = GF of



$N-1$ marked matched pairs
allowed not to be forward

bijection ?

Rk: approach extends to Eulerian maps with prescribed vertex-degrees
1-vertex case: Harer-Zagier formula (has bijective proofs)