Multicritical Schur measures

Jérémie Bouttier
Based on joint work with Dan Betea and Harriet Walsh
arXiv:2012.01995 [math.CO], longer version to appear soon

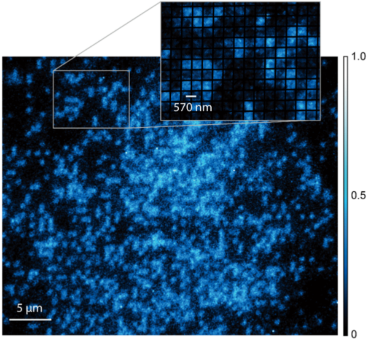
Institut de physique théorique
CEA Paris-Saclay

Random Geometry
CIRM, 18 January 2022

Outline

Outline

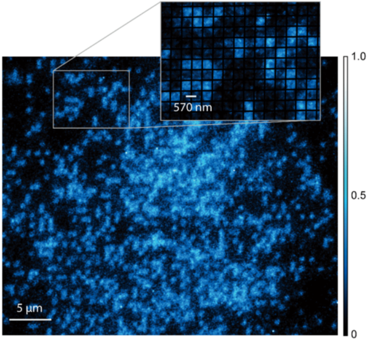
Introduction



Parsons *et al.*, PRL114, 213002 (2015)

Driven by progress in experiments on cold atoms, much attention has been recently devoted to systems of trapped fermions.

Introduction

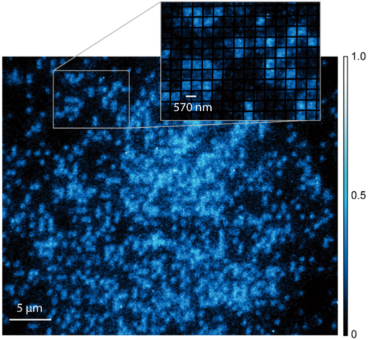


Parsons *et al.*, PRL114, 213002 (2015)

Driven by progress in experiments on cold atoms, much attention has been recently devoted to systems of trapped fermions.

The simplest model of noninteracting 1D fermions in a harmonic potential is known to be equivalent to the Gaussian Unitary Ensemble of random matrix theory.

Introduction



Parsons *et al.*, PRL114, 213002 (2015)

Driven by progress in experiments on cold atoms, much attention has been recently devoted to systems of trapped fermions.

The simplest model of noninteracting 1D fermions in a harmonic potential is known to be equivalent to the Gaussian Unitary Ensemble of random matrix theory.

For a large number of fermions, universal asymptotic behaviours are observed both in the bulk and at the edge. The latter is described by the Airy ensemble and the Tracy-Widom $\beta = 2$ distribution (or their finite-temperature analogues), which are closely related with the Kardar-Parisi-Zhang equation.

Universality of the Airy ensemble

For noninteracting 1D trapped fermions, the one-particle hamiltonian is of the form

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

where $V(x)$ is the confining potential.

Universality of the Airy ensemble

For noninteracting 1D trapped fermions, the one-particle hamiltonian is of the form

$$H = - \quad \frac{d^2}{dx^2} + V(x)$$

where $V(x)$ is the confining potential. At low temperatures, close to the edge and Fermi level, $V(x)$ can be approximated by a linear potential so

$$H \approx -\frac{d^2}{dx^2} + x$$

Universality of the Airy ensemble

For noninteracting 1D trapped fermions, the one-particle hamiltonian is of the form

$$H = - \frac{d^2}{dx^2} + V(x)$$

where $V(x)$ is the confining potential. At low temperatures, close to the edge and Fermi level, $V(x)$ can be approximated by a linear potential so

$$H \approx -\frac{d^2}{dx^2} + x$$

and the eigenfunctions are given by Airy functions:

$$\psi_\lambda(x) := \mathrm{Ai}(x+\lambda), \qquad H\psi_\lambda = -\lambda\psi_\lambda.$$

Universality of the Airy ensemble

For noninteracting 1D trapped fermions, the one-particle hamiltonian is of the form

$$H = - \frac{d^2}{dx^2} + V(x)$$

where $V(x)$ is the confining potential. At low temperatures, close to the edge and Fermi level, $V(x)$ can be approximated by a linear potential so

$$H \approx -\frac{d^2}{dx^2} + x$$

and the eigenfunctions are given by Airy functions:

$$\psi_\lambda(x) := \mathrm{Ai}(x+\lambda), \qquad H\psi_\lambda = -\lambda\psi_\lambda.$$

At zero temperature we must fill all negative energy states: the propagator (or correlation kernel) is given by the Airy kernel

$$K_{\mathrm{Ai}}(x,x') := \int_0^\infty \mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(x'+\lambda)d\lambda.$$

See e.g. J.-M. Stéphan, SciPost Phys. 6, 057 (2019) for more details.

Multicritical edge behaviour

Le Doussal, Majumdar and Schehr (PRL121, 030603, 2018) considered "flat traps" for which

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + gx^{2n}, \qquad g > 0,\ n = 1, 2, 3, \ldots$$

Multicritical edge behaviour

Le Doussal, Majumdar and Schehr (PRL121, 030603, 2018) considered "flat traps" for which

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + gx^{2n}, \qquad g > 0,\ n = 1, 2, 3, \ldots$$

In momentum space it reads

$$H = (-1)^n \hbar^{2n} g \frac{d^{2n}}{dp^{2n}} + \frac{p^2}{2m}$$

Multicritical edge behaviour

Le Doussal, Majumdar and Schehr (PRL121, 030603, 2018) considered "flat traps" for which

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + gx^{2n}, \qquad g > 0,\ n = 1, 2, 3, \ldots$$

In momentum space it reads

$$H = (-1)^n \hbar^{2n} g \frac{d^{2n}}{dp^{2n}} + \frac{p^2}{2m} \approx (-1)^n \hbar^{2n} g \frac{d^{2n}}{dp^{2n}} + p_e p.$$

Multicritical edge behaviour

Le Doussal, Majumdar and Schehr (PRL121, 030603, 2018) considered "flat traps" for which

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + gx^{2n}, \qquad g > 0,\ n = 1, 2, 3, \dots$$

In momentum space it reads

$$H = (-1)^n \hbar^{2n} g \frac{d^{2n}}{dp^{2n}} + \frac{p^2}{2m} \approx (-1)^n \hbar^{2n} g \frac{d^{2n}}{dp^{2n}} + p_e p.$$

In dimensionless units the eigenfunctions of $(-1)^n \frac{d^{2n}}{dp^{2n}} + p$ are given by $\psi_\lambda^{(n)}(p) := \mathrm{Ai}_{2n+1}(p + \lambda)$ with the generalized Airy functions

$$\mathrm{Ai}_{2n+1}(p) := \frac{1}{2i\pi} \int_{i\mathbb{R}+\epsilon} e^{(-1)^{n+1} \frac{z^{2n+1}}{2n+1} - pz} dz.$$

Multicritical edge behaviour

At zero temperature, the propagator/correlation kernel is given by

$$\mathcal{A}_{2n+1}(p,p') = \int_0^\infty \mathrm{Ai}_{2n+1}(p+\lambda)\,\mathrm{Ai}_{2n+1}(p'+\lambda)d\lambda$$

and the distribution of the highest momentum is given by the Fredholm determinant

$$F(2n+1;s) = \det(I - A_{2n+1})_{L^2(s,\infty)}.$$

which generalizes the Tracy-Widom $\beta = 2$ distribution ($n = 1$).

Multicritical edge behaviour

At zero temperature, the propagator/correlation kernel is given by

$$\mathcal{A}_{2n+1}(p,p') = \int_0^\infty \mathrm{Ai}_{2n+1}(p+\lambda)\,\mathrm{Ai}_{2n+1}(p'+\lambda)d\lambda$$

and the distribution of the highest momentum is given by the Fredholm determinant

$$F(2n+1;s) = \det(I - A_{2n+1})_{L^2(s,\infty)}.$$

which generalizes the Tracy-Widom $\beta = 2$ distribution ($n = 1$).

These Fredholm determinants were studied by Cafasso, Claeys and Girotti (IMRN, 2021) who showed that they are related with solutions of the Painlevé II hierarchy.

Multicritical edge behaviour

At zero temperature, the propagator/correlation kernel is given by

$$\mathcal{A}_{2n+1}(p, p') = \int_0^\infty \mathrm{Ai}_{2n+1}(p+\lambda)\,\mathrm{Ai}_{2n+1}(p'+\lambda)d\lambda$$

and the distribution of the highest momentum is given by the Fredholm determinant

$$F(2n+1; s) = \det(I - A_{2n+1})_{L^2(s,\infty)}.$$

which generalizes the Tracy-Widom $\beta = 2$ distribution ($n = 1$).

These Fredholm determinants were studied by Cafasso, Claeys and Girotti (IMRN, 2021) who showed that they are related with solutions of the Painlevé II hierarchy. As noted by Le Doussal *et al.*, this hierarchy also appears in the context of multicritical unitary random matrix models, introduced by Periwal and Shevitz (1990) and generalizing the Gross-Witten-Wadia model.

Multicritical edge behaviour

At zero temperature, the propagator/correlation kernel is given by

$$\mathcal{A}_{2n+1}(p,p') = \int_0^\infty \mathrm{Ai}_{2n+1}(p+\lambda)\,\mathrm{Ai}_{2n+1}(p'+\lambda) d\lambda$$

and the distribution of the highest momentum is given by the Fredholm determinant

$$F(2n+1;s) = \det(I - A_{2n+1})_{L^2(s,\infty)}.$$

which generalizes the Tracy-Widom $\beta = 2$ distribution ($n = 1$).

These Fredholm determinants were studied by Cafasso, Claeys and Girotti (IMRN, 2021) who showed that they are related with solutions of the Painlevé II hierarchy. As noted by Le Doussal *et al.*, this hierarchy also appears in the context of multicritical unitary random matrix models, introduced by Periwal and Shevitz (1990) and generalizing the Gross-Witten-Wadia model. Can we explain this connection?

Outline

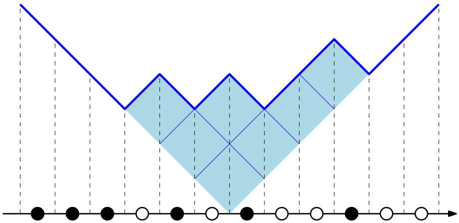
Our work

We introduce multicritical Schur measures which are probability distributions over integer partitions:

- which have the same edge behaviour as the LDMS model,
- which admit an exact mapping to the Periwal-Shevitz unitary matrix models.

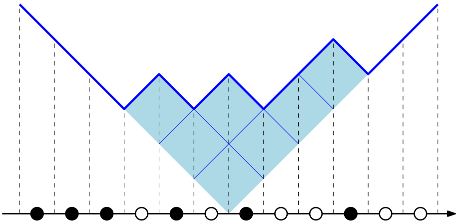
Schur measures (Okounkov 2001)

An integer partition is a finite noncreasing sequence of positive integers. It is usually represented as a Young diagram. They are in correspondence with 1D fermionic configurations ("Maya diagrams").



Schur measures (Okounkov 2001)

An integer partition is a finite noncreasing sequence of positive integers. It is usually represented as a Young diagram. They are in correspondence with 1D fermionic configurations ("Maya diagrams").



A Schur measure is a probability measure on partitions of the form

$$\mathrm{Prob}(\lambda) = \frac{1}{Z_{X,Y}} s_\lambda(X) s_\lambda(Y)$$

where s_λ is the Schur function associated with the partition λ and X, Y are two sets of variables.

Fermionic formulation

We consider fermions on a 1D lattice with a one-particle hamiltonian of the form

$$(H\psi)_k = k\psi_k + t_1(\psi_{k-1} + \psi_{k+1}) + \frac{t_2}{2}(\psi_{k-2} + \psi_{k+2}) + \cdots$$

where $t_1, t_2, \ldots$ are real parameters.

Fermionic formulation

We consider fermions on a 1D lattice with a one-particle hamiltonian of the form

$$(H\psi)_k = k\psi_k + t_1(\psi_{k-1} + \psi_{k+1}) + \frac{t_2}{2}(\psi_{k-2} + \psi_{k+2}) + \cdots$$

where $t_1, t_2, \ldots$ are real parameters. The eigenfuctions are given by the generalized Bessel functions:

$$\psi_k^{(\ell)} = J_{k-\ell}(t_1, t_2, \ldots) = \frac{1}{2i\pi} \oint \frac{dz}{z^{k-\ell+1}} e^{t_1(z-z^{-1}) + \frac{t_2}{2}(z^2-z^{-2}) + \cdots}.$$

Fermionic formulation

We consider fermions on a 1D lattice with a one-particle hamiltonian of the form

$$(H\psi)_k = k\psi_k + t_1(\psi_{k-1} + \psi_{k+1}) + \frac{t_2}{2}(\psi_{k-2} + \psi_{k+2}) + \cdots$$

where $t_1, t_2, \ldots$ are real parameters. The eigenfuctions are given by the generalized Bessel functions:

$$\psi_k^{(\ell)} = J_{k-\ell}(t_1, t_2, \ldots) = \frac{1}{2i\pi} \oint \frac{dz}{z^{k-\ell+1}} e^{t_1(z-z^{-1}) + \frac{t_2}{2}(z^2-z^{-2})+\cdots}.$$

The ground state is obtained by filling all states with negative energy so that the correlation kernel is given by

$$K(k, k') = \sum_{\ell>0} J_{k+\ell}(t_1, t_2, \ldots) J_{k'+\ell}(t_1, t_2, \ldots)$$

which matches the Schur measure with $t_i = p_i(X) = p_i(Y)$ for all i.

Asymptotic behaviour

Now suppose that we take $t_i = \theta\gamma_i$, with γ_i fixed and $\theta \to \infty$. We also take $k = a\theta$ for some fixed a.

Asymptotic behaviour

Now suppose that we take $t_i = \theta\gamma_i$, with γ_i fixed and $\theta \to \infty$. We also take $k = a\theta$ for some fixed a. Then, we have

$$J_k(t_1, t_2, \dots) = \frac{1}{2i\pi} \oint e^{\theta S(z)} dz$$

$$S(z) := -a \log z + \gamma_1(z - z^{-1}) + \frac{\gamma_2}{2}(z^2 - z^{-2}) + \cdots$$

Asymptotic behaviour

Now suppose that we take $t_i = \theta\gamma_i$, with γ_i fixed and $\theta \to \infty$. We also take $k = a\theta$ for some fixed a. Then, we have

$$J_k(t_1, t_2, \dots) = \frac{1}{2i\pi} \oint e^{\theta S(z)} dz$$

$$S(z) := -a \log z + \gamma_1(z - z^{-1}) + \frac{\gamma_2}{2}(z^2 - z^{-2}) + \cdots$$

As argued by Okounkov, asymptotics follow from the saddle-point method. The saddle points are the roots of $S'(z) = 0$ and there are generically two dominant saddle points $z_+ = z_-^{-1}$ (depending on a) with

- $z_\pm$ real: "frozen region"
- $z_\pm$ complex: "bulk"
- $z_+ = z_- = \pm 1$: "edge".

Asymptotic behaviour

Now suppose that we take $t_i = \theta\gamma_i$, with γ_i fixed and $\theta \to \infty$. We also take $k = a\theta$ for some fixed a. Then, we have

$$J_k(t_1, t_2, \ldots) = \frac{1}{2i\pi} \oint e^{\theta S(z)} dz$$

$$S(z) := -a \log z + \gamma_1(z - z^{-1}) + \frac{\gamma_2}{2}(z^2 - z^{-2}) + \cdots$$

As argued by Okounkov, asymptotics follow from the saddle-point method. The saddle points are the roots of $S'(z) = 0$ and there are generically two dominant saddle points $z_+ = z_-^{-1}$ (depending on a) with

- $z_\pm$ real: "frozen region"
- $z_\pm$ complex: "bulk"
- $z_+ = z_- = \pm 1$: "edge".

At the edge we have a double critical point $S'(1) = S''(1) = 0$, with generically $S'''(1) \neq 0$: this leads to the Airy kernel.

Multicritical behaviour

By tuning the γ_i appropriately, it is possible to have a critical point of higher order!

$$S'(1) = S''(1) = \cdots = S^{(2n)}(1) = 0, \qquad S^{(2n+1)}(1) \neq 0.$$

Multicritical behaviour

By tuning the γ_i appropriately, it is possible to have a critical point of higher order!

$$S'(1) = S''(1) = \cdots = S^{(2n)}(1) = 0, \qquad S^{(2n+1)}(1) \neq 0.$$

This choice is unique if we restrict the support of $(\gamma_i)_{i \geq 1}$ to a set with n elements:

- "odd-even measure": only $\gamma_1, \gamma_2, \ldots, \gamma_n$ nonzero
- "odd measure": only $\gamma_1, \gamma_3, \ldots, \gamma_{2n-1}$ nonzero

Multicritical behaviour

By tuning the γ_i appropriately, it is possible to have a critical point of higher order!

$$S'(1) = S''(1) = \cdots = S^{(2n)}(1) = 0, \qquad S^{(2n+1)}(1) \neq 0.$$

This choice is unique if we restrict the support of $(\gamma_i)_{i \geq 1}$ to a set with n elements:

- "odd-even measure": only $\gamma_1, \gamma_2, \ldots, \gamma_n$ nonzero

$$\gamma_i = (-1)^{i+1} \frac{(n-1)!(n+1)!}{(n-i)!(n+i)!}$$

- "odd measure": only $\gamma_1, \gamma_3, \ldots, \gamma_{2n-1}$ nonzero

$$\gamma_{2i-1} = (-1)^{i+1} \frac{(n-1)!n!}{(2i-1)(n-i)!(n+i-1)!}$$

Theorem [Betea-B.-Walsh 2021]

Both multicritical Schur measures $\mathbb{P}_{n,\theta}$ satisfy

$$\lim_{\theta\to\infty} \mathbb{P}_{n,\theta}\left[\frac{\lambda_1 - b\theta}{(\theta d)^{\frac{1}{2n+1}}} < s\right] = F(2n+1; s)$$

with b, d explicit. Furthermore, the odd measure is invariant under conjugation of partitions hence we can replace λ_1 by λ_1' in the statement. (For the odd-even measure λ_1' has generic edge fluctuations as for $n = 1$.)

Theorem [Betea-B.-Walsh 2021]

Both multicritical Schur measures $\mathbb{P}_{n,\theta}$ satisfy

$$\lim_{\theta\to\infty} \mathbb{P}_{n,\theta}\left[\frac{\lambda_1 - b\theta}{(\theta d)^{\frac{1}{2n+1}}} < s\right] = F(2n+1; s)$$

with b, d explicit. Furthermore, the odd measure is invariant under conjugation of partitions hence we can replace λ_1 by λ_1' in the statement. (For the odd-even measure λ_1' has generic edge fluctuations as for $n = 1$.)

Proof is done by the saddle-point method.

Theorem [Betea-B.-Walsh 2021]

Both multicritical Schur measures $\mathbb{P}_{n,\theta}$ satisfy

$$\lim_{\theta\to\infty} \mathbb{P}_{n,\theta}\left[\frac{\lambda_1 - b\theta}{(\theta d)^{\frac{1}{2n+1}}} < s\right] = F(2n+1; s)$$

with b, d explicit. Furthermore, the odd measure is invariant under conjugation of partitions hence we can replace λ_1 by λ_1' in the statement. (For the odd-even measure λ_1' has generic edge fluctuations as for $n = 1$.)

Proof is done by the saddle-point method. By slightly perturbing the parameters γ_i, we can in fact obtain the more general Fredholm determinants considered by Cafasso, Claeys and Girotti.

Theorem [Betea-B.-Walsh 2021]

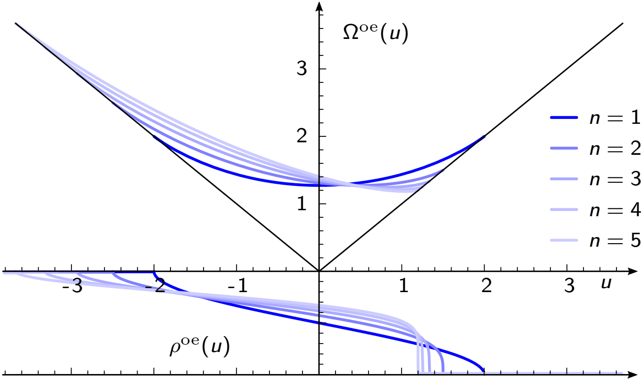
Both multicritical Schur measures $\mathbb{P}_{n,\theta}$ satisfy

$$\lim_{\theta \to \infty} \mathbb{P}_{n,\theta} \left[\frac{\lambda_1 - b\theta}{(\theta d)^{\frac{1}{2n+1}}} < s \right] = F(2n+1; s)$$

with b, d explicit. Furthermore, the odd measure is invariant under conjugation of partitions hence we can replace λ_1 by λ_1' in the statement. (For the odd-even measure λ_1' has generic edge fluctuations as for $n = 1$.)

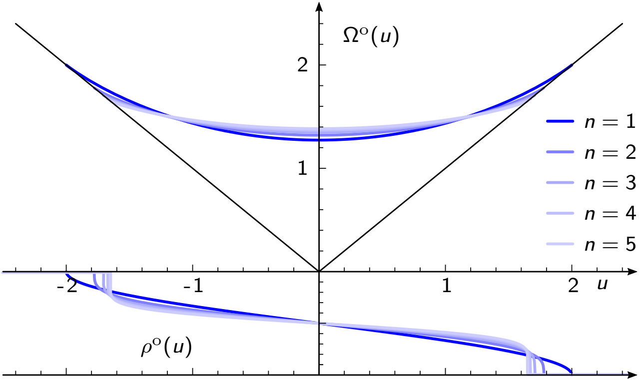
Proof is done by the saddle-point method. By slightly perturbing the parameters γ_i, we can in fact obtain the more general Fredholm determinants considered by Cafasso, Claeys and Girotti. Also, it is possible to get the finite-temperature version easily (using results from Betea-B. 2019), for which the Fredholm determinant is related to an integro-differential generalization of the Painlevé II hierarchy (Krajenbrink 2021).

Limit shapes (odd-even case)



At the right edge, the density (one-point function) vanishes as $(b - u)^{\frac{1}{2n}}$.

Limit shapes (odd case)



We obtain the same exponent $\frac{1}{2n}$ at both edges due to symmetry.

Outline

Connection with unitary random matrix models

Proposition (Gessel/Heine/...)

Consider the Schur measure $\mathbb{P}_{t_1,t_2,\dots}$ and set

$$V(z) := \sum_{i \geq 1} \frac{t_i}{i} z^i, \qquad f_k := [z^k] e^{V(z)+V(z^{-1})}.$$

Then, we have

$$e^{\sum_i \frac{t_i^2}{i}} \mathbb{P}_{t_1,t_2,\dots}(\lambda'_1 \leq N) = \det_{1 \leq i,j \leq N} f_{i-j} = \mathbb{E}_{U(N)} \left(\exp \mathrm{Tr}(V(U) + V(U^\dagger)) \right)$$

where U is distributed according to the Haar measure on $U(N)$ and $\mathbb{E}_{U(N)}$ is the corresponding expectation. We can replace λ'_1 by λ_1 upon changing $t_i \mapsto (-1)^{i+1} t_i$.

Connection with unitary random matrix models

Proposition (Gessel/Heine/...)

Consider the Schur measure $\mathbb{P}_{t_1,t_2,\dots}$ and set

$$V(z) := \sum_{i \geq 1} \frac{t_i}{i} z^i, \qquad f_k := [z^k] e^{V(z)+V(z^{-1})}.$$

Then, we have

$$e^{\sum_i \frac{t_i^2}{i}} \mathbb{P}_{t_1,t_2,\dots}(\lambda'_1 \leq N) = \det_{1 \leq i,j \leq N} f_{i-j} = \mathbb{E}_{U(N)} \left(\exp \mathrm{Tr}(V(U) + V(U^\dagger)) \right)$$

where U is distributed according to the Haar measure on $U(N)$ and $\mathbb{E}_{U(N)}$ is the corresponding expectation. We can replace λ'_1 by λ_1 upon changing $t_i \mapsto (-1)^{i+1} t_i$.

When we take the t_i as in our odd-even multicritical measure, we recover precisely the unitary matrix models of Periwal and Shevitz (1990)!

Multicritical unitary random matrix models

The Periwal-Shevitz models (1990) are multicritical generalizations of the Gross-Witten-Wadia model (1980), whose connection with the Plancherel measure on partitions was observed by Johansson (1998). We thus extend this observation to the multicritical setting.

Multicritical unitary random matrix models

The Periwal-Shevitz models (1990) are multicritical generalizations of the Gross-Witten-Wadia model (1980), whose connection with the Plancherel measure on partitions was observed by Johansson (1998). We thus extend this observation to the multicritical setting.

By the Weyl integration formula

$$\mathbb{E}_{U(N)}\left(\exp \mathrm{Tr}(V(U)+V(U^\dagger))\right) = \frac{1}{(2\pi)^N N!}\int_{[-\pi,\pi]^N} \prod_{k<\ell} \left|e^{i\alpha_k} - e^{i\alpha_\ell}\right|^2 \prod_{k=1}^{N} e^{V(e^{i\alpha_k})+V(e^{-i\alpha_k})} d\alpha_k.$$

Multicritical unitary random matrix models

The Periwal-Shevitz models (1990) are multicritical generalizations of the Gross-Witten-Wadia model (1980), whose connection with the Plancherel measure on partitions was observed by Johansson (1998). We thus extend this observation to the multicritical setting.

By the Weyl integration formula

$$\mathbb{E}_{U(N)}\left(\exp \mathrm{Tr}(V(U) + V(U^\dagger))\right) = \frac{1}{(2\pi)^N N!} \int_{[-\pi,\pi]^N} \prod_{k<\ell} \left|e^{i\alpha_k} - e^{i\alpha_\ell}\right|^2 \prod_{k=1}^{N} e^{V(e^{i\alpha_k})+V(e^{-i\alpha_k})} d\alpha_k.$$

As N gets large, the empirical measure on eigenvalues converges to an equilibrium measure on the unit circle, whose support can be the full circle (e.g. for $V = 0$) or a strict subset therefore ("V largely varying").

Multicritical unitary random matrix models

The Periwal-Shevitz models (1990) are multicritical generalizations of the Gross-Witten-Wadia model (1980), whose connection with the Plancherel measure on partitions was observed by Johansson (1998). We thus extend this observation to the multicritical setting.

By the Weyl integration formula

$$\mathbb{E}_{U(N)}\left(\exp \mathrm{Tr}(V(U)+V(U^\dagger))\right) = \frac{1}{(2\pi)^N N!}\int_{[-\pi,\pi]^N} \prod_{k<\ell}\left|e^{i\alpha_k}-e^{i\alpha_\ell}\right|^2 \prod_{k=1}^{N} e^{V(e^{i\alpha_k})+V(e^{-i\alpha_k})} d\alpha_k.$$

As N gets large, the empirical measure on eigenvalues converges to an equilibrium measure on the unit circle, whose support can be the full circle (e.g. for $V=0$) or a strict subset therefore ("V largely varying"). The GWW model corresponds to $V(U)=gNU$ and there is a phase transition between the two situations at $g=1$.

Multicritical unitary random matrix models

The Periwal-Shevitz models (1990) are multicritical generalizations of the Gross-Witten-Wadia model (1980), whose connection with the Plancherel measure on partitions was observed by Johansson (1998). We thus extend this observation to the multicritical setting.

By the Weyl integration formula

$$\mathbb{E}_{U(N)}\left(\exp \mathrm{Tr}(V(U)+V(U^\dagger))\right) =$$

$$\frac{1}{(2\pi)^N N!}\int_{[-\pi,\pi]^N}\prod_{k<\ell}\left|e^{i\alpha_k}-e^{i\alpha_\ell}\right|^2\prod_{k=1}^{N} e^{V(e^{i\alpha_k})+V(e^{-i\alpha_k})}d\alpha_k.$$

As N gets large, the empirical measure on eigenvalues converges to an equilibrium measure on the unit circle, whose support can be the full circle (e.g. for $V=0$) or a strict subset therefore ("V largely varying"). The GWW model corresponds to $V(U)=gNU$ and there is a phase transition between the two situations at $g=1$. The PS models correspond to V higher degree polynomials, with fine-tuned coefficients to modify the critical exponents of the phase transition.

Outline

Conclusion

We introduced random partitions with a non generic $\frac{1}{2n+1}$ exponent for the fluctuations of the largest parts. These are described by the same generalized Airy kernel $\mathcal{A}_{2n+1}$ (and corresponding determinantal point process, Fredholm determinant, etc.) as the multicritical fermionic models of Le Doussal, Majumdar and Schehr. We also explain the connection with unitary random matrix models by an exact mapping.

Conclusion

We introduced random partitions with a non generic $\frac{1}{2n+1}$ exponent for the fluctuations of the largest parts. These are described by the same generalized Airy kernel $\mathcal{A}_{2n+1}$ (and corresponding determinantal point process, Fredholm determinant, etc.) as the multicritical fermionic models of Le Doussal, Majumdar and Schehr. We also explain the connection with unitary random matrix models by an exact mapping.

The same models have been considered recently by Kimura and Zahabi (2021+) in connection with supersymmetric gauge theories. Their work suggests to consider the limit $n \to \infty$ (hard edge?).

Future directions

In the future, we would like to investigate the case where the order n of multicriticality is not an integer: we still get a meaningful measure on partitions, equivalent to a unitary random matrix model.

Future directions

In the future, we would like to investigate the case where the order n of multicriticality is not an integer: we still get a meaningful measure on partitions, equivalent to a unitary random matrix model. Recall for instance the "odd-even measure" which was defined in terms of the $(\gamma_i)_{i \geq 1}$ with

$$\gamma_i = (-1)^{i+1} \frac{(n-1)!(n+1)!}{(n-i)!(n+i)!}$$

Future directions

In the future, we would like to investigate the case where the order n of multicriticality is not an integer: we still get a meaningful measure on partitions, equivalent to a unitary random matrix model. Recall for instance the "odd-even measure" which was defined in terms of the $(\gamma_i)_{i\geq 1}$ with

$$\gamma_i = (-1)^{i+1}\frac{(n-1)!(n+1)!}{(n-i)!(n+i)!} = (-1)^{i+1}\frac{\Gamma(n)\Gamma(n+2)}{\Gamma(n-i+1)\Gamma(n+i-1)}.$$

Future directions

In the future, we would like to investigate the case where the order n of multicriticality is not an integer: we still get a meaningful measure on partitions, equivalent to a unitary random matrix model. Recall for instance the “odd-even measure” which was defined in terms of the $(\gamma_i)_{i\geq 1}$ with

$$\gamma_i = (-1)^{i+1} \frac{(n-1)!(n+1)!}{(n-i)!(n+i)!} = (-1)^{i+1} \frac{\Gamma(n)\Gamma(n+2)}{\Gamma(n-i+1)\Gamma(n+i-1)}.$$

Challenge: for n non integer, the family $(\gamma_i)_{i\geq 1}$ does not have finite support. Polynomials are replaced by series and convergence issues arise.

Future directions

In the future, we would like to investigate the case where the order n of multicriticality is not an integer: we still get a meaningful measure on partitions, equivalent to a unitary random matrix model. Recall for instance the "odd-even measure" which was defined in terms of the $(\gamma_i)_{i\geq 1}$ with

$$\gamma_i = (-1)^{i+1}\frac{(n-1)!(n+1)!}{(n-i)!(n+i)!} = (-1)^{i+1}\frac{\Gamma(n)\Gamma(n+2)}{\Gamma(n-i+1)\Gamma(n+i-1)}.$$

Challenge: for n non integer, the family $(\gamma_i)_{i\geq 1}$ does not have finite support. Polynomials are replaced by series and convergence issues arise.

This (as well as the case $n \to \infty$ mentioned before) is work in progress by Harriet Walsh.

An analogous Hermitian random matrix model

Consider the usual Hermitian one-matrix model

$$Z = \int dM e^{-N \mathrm{Tr} V(M)}$$

An analogous Hermitian random matrix model

Consider the usual Hermitian one-matrix model

$$Z = \int dM e^{-N \mathrm{Tr} V(M)}$$

Since Brézin-Itzykson-Parisi-Zuber (1978) it is known that this model is closely related with random maps (perturb $V(M) = -\frac{1}{2}M^2 + \sum \frac{g_k}{k} M^k$).

An analogous Hermitian random matrix model

Consider the usual Hermitian one-matrix model

$$Z = \int dM e^{-N \mathrm{Tr} V(M)}$$

Since Brézin-Itzykson-Parisi-Zuber (1978) it is known that this model is closely related with random maps (perturb $V(M) = -\frac{1}{2}M^2 + \sum \frac{g_k}{k} M^k$). Kazakov (1989) showed that, for certain fine-tuned polynomials V, it is possible to obtain non generic multicritical exponents:

$$V(M) = \sum_{k=1}^{m} v_k M^{2k}, \qquad v_k \propto \frac{\Gamma(k-m)}{\Gamma(k+\frac{1}{2})k}, \qquad m = 1, 2, 3, \ldots$$

An analogous Hermitian random matrix model

Consider the usual Hermitian one-matrix model

$$Z = \int dM e^{-N \mathrm{Tr} V(M)}$$

Since Brézin-Itzykson-Parisi-Zuber (1978) it is known that this model is closely related with random maps (perturb $V(M) = -\frac{1}{2}M^2 + \sum \frac{g_k}{k} M^k$). Kazakov (1989) showed that, for certain fine-tuned polynomials V, it is possible to obtain non generic multicritical exponents:

$$V(M) = \sum_{k=1}^{m} v_k M^{2k}, \qquad v_k \propto \frac{\Gamma(k-m)}{\Gamma(k+\frac{1}{2})k}, \qquad m = 1, 2, 3, \dots$$

Issue: except for $m = 1$, this leads to a signed measure on maps, whose probabilistic interpretation is unclear. Still, lots is known in physics!

An analogous Hermitian random matrix model

Consider the usual Hermitian one-matrix model

$$Z = \int dM e^{-N \mathrm{Tr} V(M)}$$

Since Brézin-Itzykson-Parisi-Zuber (1978) it is known that this model is closely related with random maps (perturb $V(M) = -\frac{1}{2}M^2 + \sum \frac{g_k}{k} M^k$). Kazakov (1989) showed that, for certain fine-tuned polynomials V, it is possible to obtain non generic multicritical exponents:

$$V(M) = \sum_{k=1}^{m} v_k M^{2k}, \qquad v_k \propto \frac{\Gamma(k-m)}{\Gamma(k+\frac{1}{2})k}, \qquad m = 1, 2, 3, \ldots$$

Issue: except for $m = 1$, this leads to a signed measure on maps, whose probabilistic interpretation is unclear. Still, lots is known in physics! Ambjørn, Budd and Makeenko (2016) investigated the case where n noninteger. For $n = s - \frac{1}{2}$ and $s \in]\frac{3}{2}, \frac{5}{2}]$, we get a "large faces" probability measure on maps!

Future directions

My (crazy?) hope is that, in the analogous unitary matrix model, there could also exist an interesting range of the multicriticality parameter n with "nice" probabilistic interpretation (connection with a growth model, etc).

Future directions

My (crazy?) hope is that, in the analogous unitary matrix model, there could also exist an interesting range of the multicriticality parameter n with "nice" probabilistic interpretation (connection with a growth model, etc).

It would also be worthwhile to investigate if unitary matrix models have a nice graphical/topological expansion in terms of maps or related objects. Periwal and Shevitz make some arguments in that direction (using $U = e^{iH}$) but I wonder if this can be put on a rigorous footing.

Future directions

My (crazy?) hope is that, in the analogous unitary matrix model, there could also exist an interesting range of the multicriticality parameter n with "nice" probabilistic interpretation (connection with a growth model, etc).

It would also be worthwhile to investigate if unitary matrix models have a nice graphical/topological expansion in terms of maps or related objects. Periwal and Shevitz make some arguments in that direction (using $U = e^{iH}$) but I wonder if this can be put on a rigorous footing.

Finally, all this is in the realm of determinantal point processes / noninteracting fermions. Could we lift these considerations to interacting models?

- Schur measures $\rightarrow$ Macdonald/Jack measures
- CUE/GUE $\rightarrow$ β-ensembles
- connection with Gaussian multiplicative chaos and Liouville quantum gravity? See e.g. Chhaibi-Najnudel (2019) on GMC^{γ} vs $C\beta E$, etc.

Thanks for your attention!