Generalised Kontsevich graphs, topological recursion and r-spin intersection numbers

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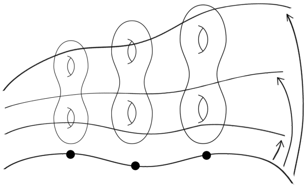
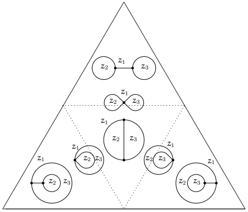
Workshop: Random Geometry





Centre International de Rencontres Mathématiques
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Outline

1. Topological recursion (TR)
2. Witten's conjecture, Kontsevich's theorem
3. Cohomological field theories (CohFT)
4. Witten's r-spin class and the r-KdV hierarchy
   (based on joint work with R. Belliard, S. Charbonnier and B. Eynard, arXiv:2105.08035)
   - Combinatorial model: Generalised Kontsevich graphs
   - Tutte's recursion
   - Topological recursion for ciliated maps
   - From graphs to intersection numbers with Witten's class
5. Topological recursion for fully simple maps
   (based on joint work with G. Borot and S. Charbonnier, arXiv:2106.09002)
   - Disks and cylinders
   - Symplectic invariance and combinatorial interpretation
6. Further consequences: ongoing and future

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Topological recursion (TR, Chekhov–Eynard–Orantin '04-'07)

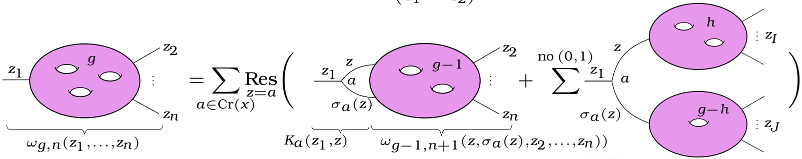
Goal: "Count surfaces $S_{g,n}$ of genus g with n boundaries (topology (g,n))."

Spectral curve

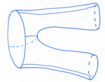
TR : $\begin{cases} \Sigma \text{ Riemann surface} \\ x\colon \Sigma \to \mathbb{CP}^1 \\ \omega_{0,1} = y\, dx \text{ 1-form (discs)} \\ \omega_{0,2} \ (1,1)\text{-form (cylinders)} \end{cases}$ $\rightsquigarrow$ **recursion on** $|\chi(S_{g,n})| = 2g-2+n$ $\rightsquigarrow$ Differential forms $\omega_{g,n}(z_1,\dots,z_n)$, $z_i \in \Sigma$, $\forall g,n \geq 0$.

- x finitely many simple ramification points ($\mathrm{Cr}(x)$) and y holomorphic around $a \in \mathrm{Cr}(x)$ and $dy(a) \neq 0 \Rightarrow$ Local involution σ around every ramification point: $x(z) = x(\sigma(z))$.
- $\omega_{0,2}$ symmetric bi-differential on $\Sigma \times \Sigma$ with only double poles along the diagonal and vanishing residues, that is when $z_1 \to z_2$

$$\omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2} + \overbrace{h(z_1,z_2)}^{\text{holomorphic}}.$$



- Terms in correspondence with the ways of cutting a **pair of pants** $(0,3)$ from $S_{g,n}$.



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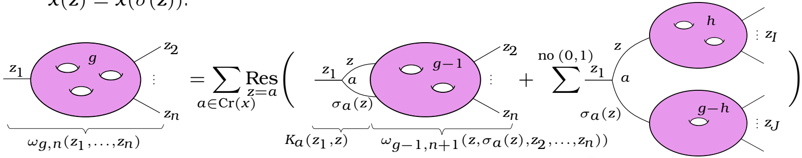

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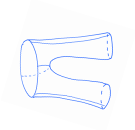


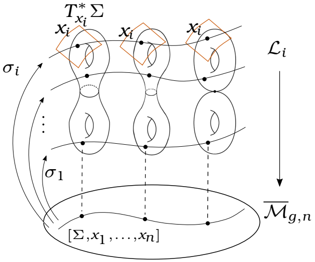
- **Motivations:** nice properties, universality, structure.

Outline

For $g, n \geq 0$, with $2g - 2 + n > 0$, we define the moduli space:

$$\mathcal{M}_{g,n} := \left\{ \begin{array}{l} \text{curves of genus } g \text{ with } n \\ \text{marked points } x_1, \ldots, x_n \end{array} \right\} \Big/ \sim \ .$$

- $\overline{\mathcal{M}}_{g,n} \rightsquigarrow$ Deligne–Mumford compactification (including nodal curves).



$$\psi_i := c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$
$$i = 1, \ldots, n.$$

Intersection numbers or correlators of psi classes:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \in \mathbb{Q},$$

which are zero unless $\sum_{i=1}^n d_i = \dim_{\mathbb{C}}(\overline{\mathcal{M}}_{g,n}) = 3g - 3 + n$.

Geometric information from intersection theory

- Intersection theory of an algebraic variety X: Interesting cycles (algebraic subvarieties)? What cohomology classes do they represent? Interesting vector bundles over X? What are their characteristic classes? Can we describe the full cohomology ring of X and identify the above classes in it? Compute their intersection numbers?

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- Chern classes: topological invariants of vector bundles. $c_1(X \times \mathbb{C}^k) = 0$. How far is the bundle from a trivial bundle? $c_d(E) = 0$, if $d > \mathrm{rk}(E)$.
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- $\dim = 1$: $\overline{\mathcal{M}}_{0,4} = \mathbb{C}P_1$.

$$\int_{\overline{\mathcal{M}}_{0,4}} \psi_i = \#\{\text{generic section of } \mathcal{L}_i \cap \text{zero section}\} = 1, \quad i = 1, \ldots, 4.$$

$\overline{\mathcal{M}}_{1,1} = \mathcal{M}_{1,1} \cup \mathcal{M}_{0,3}$, with $\mathcal{M}_{1,1} = \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$.

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- $\dim = 2$: $\overline{\mathcal{M}}_{0,5} = \mathbb{C}P_1 \times \mathbb{C}P_1 \cup \mathbb{C} \cup \mathbb{C} \cup \mathbb{C}$.

$$\int_{\overline{\mathcal{M}}_{0,5}} \psi_i^2 = 1, \quad \int_{\overline{\mathcal{M}}_{0,5}} \psi_i \psi_j = 2, \quad i \neq j.$$

Witten's conjecture

Generating series of intersection numbers of psi classes:

$$F(t_0, t_1, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(d_1,\ldots,d_n)} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle t_{d_1} \cdots t_{d_n}$$

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Conjecture: The series F satisfies the Korteweg–de Vries (KdV) hierarchy, the first equation of which is the classical KdV equation

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3} \quad \left(U = \frac{\partial^2 F}{\partial t_0^2} \right),$$

and the string equation $\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}$.

- Witten's motivation: Two different models of 2D quantum gravity should coincide $\rightsquigarrow$ Double scaling limit of the generating series of large maps ($=$ KdV tau-function for the KdV hierarchy) $=$ volume of moduli space of Riemann surfaces (modulo holomorphic parametrizations).
- The conjecture uniquely determines F.

One explicit version of Witten's conjecture

Virasoro operators:

$$V_{-1} = -\frac{1}{2}\frac{\partial}{\partial t_0} + \frac{1}{2}\sum_{k=0}^{\infty} t_{k+1}\frac{\partial}{\partial t_k} + \frac{t_0^2}{4}, \quad V_0 = -\frac{3}{2}\frac{\partial}{\partial t_1} + \frac{1}{2}\sum_{k=0}^{\infty}(2k+1)t_k\frac{\partial}{\partial t_k} + \frac{1}{48},$$

and for $n > 0$,

$$V_n = -\frac{(2n+3)!!}{2}\frac{\partial}{\partial t_{n+1}} + \sum_{k=0}^{\infty}\frac{(2k+2n+1)!!}{2(2k-1)!!}t_k\frac{\partial}{\partial t_{k+n}} + \sum_{k_1+k_2=n-1}\frac{(2k_1+1)!!(2k_2+1)!!}{4}\frac{\partial^2}{\partial t_{k_1}\partial t_{k_2}}.$$

They satisfy the Virasoro relations:

$$[V_m, V_n] = (m-n)V_{m+n}.$$

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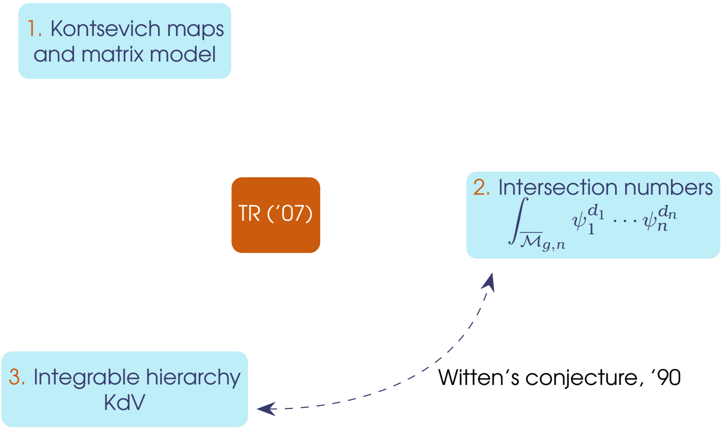
Theorem (equivalent to Witten's conjecture ('91))

For every integer $n \geq -1$, $V_n(\exp F) = 0$.

Witten's conjecture $\rightsquigarrow$ Kontsevich's theorem



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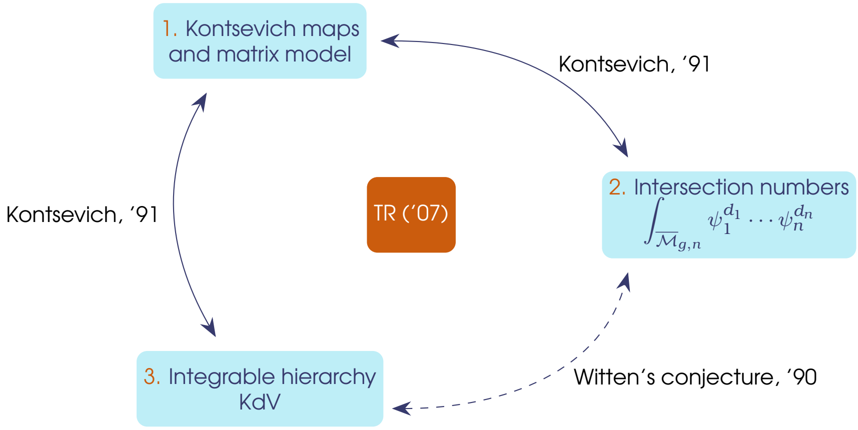

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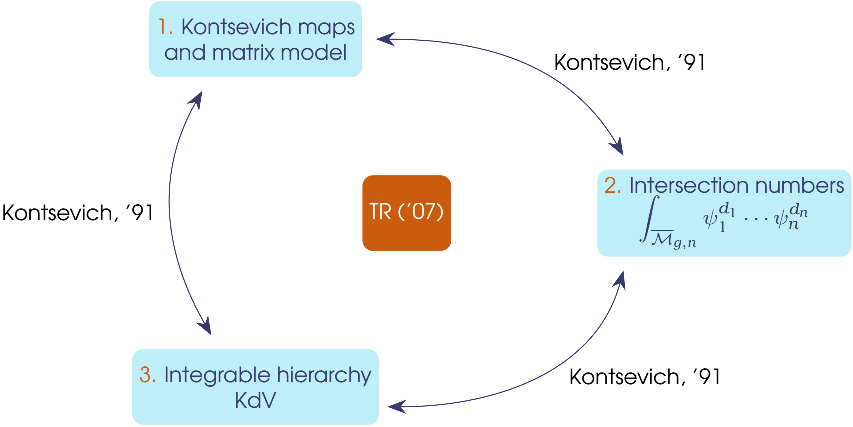

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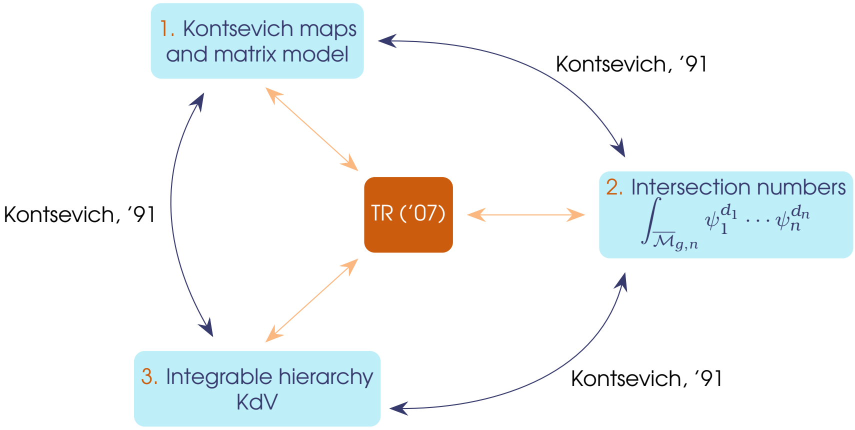

TR applied to the Airy curve $(x, y) = (\frac{z^2}{2}, z)$ produces

$$\omega_{g,n}(z_1, \ldots, z_n) = 2^{2-2g-n} \sum_{\sum_i d_i = 3g-3+n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \prod_{i=1}^{n} \frac{(2d_i+1)!!dz_i}{z_i^{2d_i+2}}.$$

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Cohomological field theories

Definition (cohomological field theory (CohFT))

V vector space with a nondegenerate symmetric bilinear form η. A CohFT $\{\Omega_{g,n}\}_{2g-2+n>0}$ over (V,η) is a collection of $\mathfrak{S}_n$-invariant morphisms

$$\Omega_{g,n}\colon V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}) \quad \text{such that}$$

given the gluing maps

$$q\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$
$$r\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}, \quad g_1+g_2=g,\ n_1+n_2=n,$$

we have

$$q^*\Omega_{g,n}(v_1\otimes\cdots\otimes v_n) = \Omega_{g-1,n+2}(v_1\otimes\cdots\otimes v_n\otimes\eta^\dagger),$$
$$r^*\Omega_{g,n}(v_1\otimes\cdots\otimes v_n) = (\Omega_{g_1,n_1+1}\times\Omega_{g_2,n_2+1})\Big(\bigotimes_{i=1}^{n_1} v_i\otimes\eta^\dagger\otimes\bigotimes_{j=1}^{n_2} v_{n_1+j}\Big),$$

where $\eta^\dagger \in V^{\otimes 2}$ is the bivector dual to η.

Correlators: With $\sum_{i=1}^n d_i \leq \dim_{\mathbb{C}}(\overline{\mathcal{M}}_{g,n}) = 3g-3+n$,

$$\langle \tau_{d_1}(v_1)\cdots\tau_{d_n}(v_n)\rangle_g^{\Omega} := \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(v_1\otimes\cdots\otimes v_n)\prod_{i=1}^n \psi_i^{d_i}.$$

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Examples: $V=\mathbb{Q}$, $\eta(1,1)=1$. Then $\Omega_{g,n}=\Omega_{g,n}(1^{\otimes n})$.

- Trivial CohFT $\Omega_{g,n}=1 \rightsquigarrow$ Witten–Kontsevich intersection numbers.
- $\Omega_{g,n}=\exp(2\pi^2\kappa_1)$, with $\kappa_m := \pi_*(\psi_{n+1}^{m+1}) \in H^{2m}(\overline{\mathcal{M}}_{g,n}) \rightsquigarrow$ Weil–Petersson volumes (hyperbolic geometry).

- A CohFT defines a quantum product $\star$ on V by

$$\eta(v_1 \star v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3).$$

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- A CohFT Ω on (V, η) is semi-simple if (V, η) is a semi-simple algebra ($= \exists$ a basis $\{e_i\}$ such that $e_i e_j = \delta_{ij} e_i$, after extending scalars to $\mathbb{C}$).

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Let Ω be a semi-simple CohFT with flat unit and ω the associated TFT (degree 0 part). Then there exists a unique R-matrix such that

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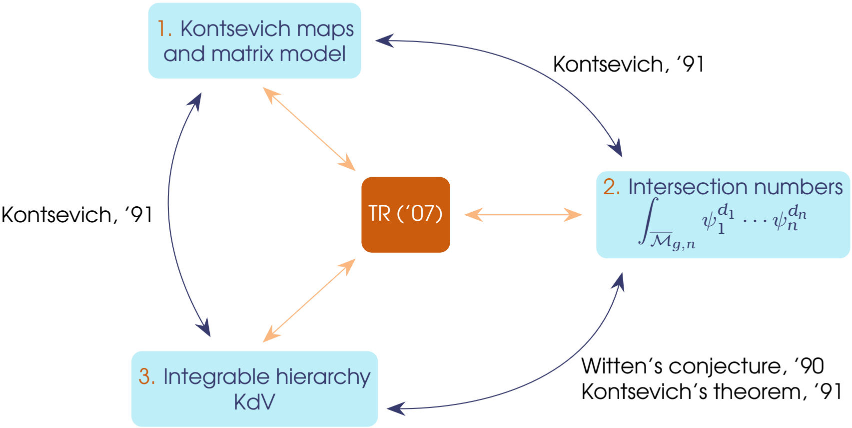
Example (non semi-simple)

$V = \langle e_0, e_1, \ldots, e_{r-2} \rangle_{\mathbb{Q}}$, $\eta(e_a, e_b) = \delta_{a+b,r-2}$. Witten's r-spin CohFT:

$$c_W^r(a_1, \ldots, a_n) = \Omega_{g,n}(e_{a_1}, \ldots, e_{a_n}),$$

of degree $\frac{(r-2)(g-1)+\sum_{i=1}^n a_i}{r}$, with $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$.

Witten's conjecture ⇝ Kontsevich's theorem



Theorem (Eynard '11, Dunin-Barkowski–Orantin–Shadrin–Spitz '14)

TR for spectral curves with simple ramification points $\leftrightarrow$ *Semi-simple CohFTs.*

Outline

1. Topological recursion (TR)
2. Witten's conjecture, Kontsevich's theorem
3. Cohomological field theories (CohFT)
4. Witten's r-spin class and the r-KdV hierarchy
   (based on joint work with R. Belliard, S. Charbonnier and B. Eynard, arXiv:2105.08035)
   - Combinatorial model: Generalised Kontsevich graphs
   - Tutte's recursion
   - Topological recursion for ciliated maps
   - From graphs to intersection numbers with Witten's class
5. Topological recursion for fully simple maps
   (based on joint work with G. Borot and S. Charbonnier, arXiv:2106.09002)
   - Disks and cylinders
   - Symplectic invariance and combinatorial interpretation
6. Further consequences: ongoing and future

r-spin Witten conjecture, '93 (proved by Faber–Shadrin–Zvonkine)

1. Generalised Kontsevich maps and matrix model

Higher TR ('13)

2. Intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} c_W^r(a_1, \ldots, a_n)\psi_1^{d_1} \cdots \psi_n^{d_n}$$

Witten, '93

3. Hierarchy r-KdV

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Kontsevich $r=2$ (Mirzakhani, . . .)
Faber–Shadrin–Zvonkine $r\geq 2$, '10

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Can we complete the picture in the general r case? Combinatorial side?

Generalised Kontsevich graphs

Definition

A *map* of genus g with n *boundaries* is an embedding of a graph Γ into an oriented surface X of genus g such that

$$X \setminus \Gamma \cong \bigsqcup \mathbb{D} \quad \text{(\textit{faces}), with } n \text{ marked faces (\textit{boundaries}).}$$

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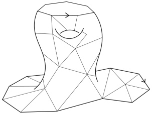

Topology $(g, n) = (1, 2$ boundaries$)$.

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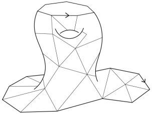

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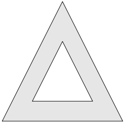

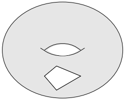

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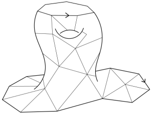

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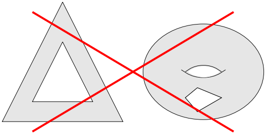

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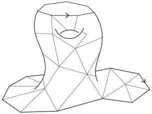

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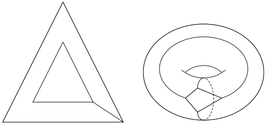

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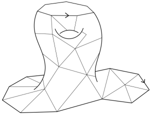

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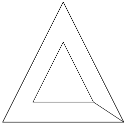
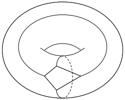
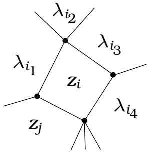
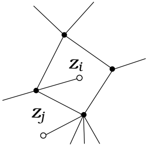


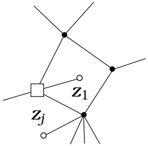


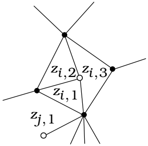
(Ciliated) Kontsevich maps: Degree of black vertices $v \rightsquigarrow 3 \leq d_v \leq r + 1$. Max one white vertex per boundary (uniqueness). $\{\lambda_1, \ldots, \lambda_N\} \rightsquigarrow$ internal faces.

$\mathcal{F}^{[r]}_{g,n}(z_1, \ldots, z_n)$ $\quad$ $\mathcal{W}^{[r]}_{g,n}(z_1, \ldots, z_n)$ $\quad$ $\mathcal{U}^{[r]}_{g,n}(z_1, \ldots, z_n)$ $\quad$ $\mathcal{S}^{[r]}_{g,(k_1,\ldots,k_n)}(S_1, \ldots, S_n)$









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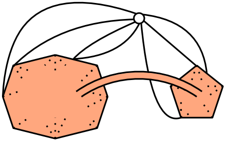
$/ \sim$

White vertices $\rightsquigarrow$ star constraint.

No star constraint $\rightsquigarrow$



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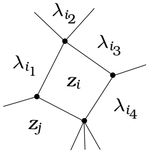
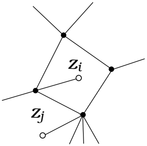
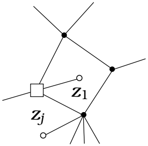


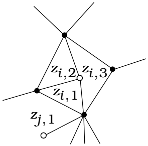
$\mathcal{W}^{[r]}_{g,n}(z_1, \ldots, z_n)$



$\mathcal{U}^{[r]}_{g,n}(z_1, \ldots, z_n)$



$\mathcal{S}^{[r]}_{g,(k_1,\ldots,k_n)}(S_1, \ldots, S_n)$



Map degrees and local weights

- Degree: $\deg G = (r+1)\,(\#\mathcal{E}(G) - \#\mathcal{V}(G)) = (r+1)\,(\#\mathcal{F}(G) - 2 + 2g(G))$.

Fixed a degree $\delta = \deg G/(r+1)$ and a topology (g,n), the sets

$$\mathcal{F}_{g,n}^{[r],\delta}(z_1,\ldots,z_n),\ \mathcal{W}_{g,n}^{[r],\delta}(z_1,\ldots,z_n),\ \mathcal{U}_{g,n}^{[r],\delta}(u;z_1,\ldots,z_n) \ \text{ and } \ \mathcal{S}_{g,\underline{k}}^{[r],\delta}(S_1,\ldots,S_n)$$

are finite.

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The potential of the model is a polynomial $V \in \mathbb{C}[z]$ of degree $r+1$:

$$V(z) = \sum_{j=1}^{r+1} \frac{v_j}{j}\, z^j.$$

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With $a_i \in \{\lambda_1, \ldots, \lambda_N\} \cup \{z_1, \ldots, z_n\}$, we define the weight per:

- **Edge** bounding faces decorated by a_1, a_2

$$\mathcal{P}(a_1, a_2) := \frac{a_1 - a_2}{V'(a_1) - V'(a_2)},$$

  and $\mathcal{P}(a_1, a_1) = \lim_{a_2 \to a_1} \mathcal{P}(a_1, a_2) = \frac{1}{V''(a_1)}$.

- **Black vertex** of degree $3 \le d \le r+1$ adjacent to faces decorated with $a_1, \ldots, a_d$

$$\mathcal{V}_d(a_1, \ldots, a_d) := \sum_{i=1}^{d} \frac{-V'(a_i)}{\prod_{j \neq i}(a_i - a_j)}.$$

- **White vertex**: 1.

Generating series

Weight of a map G:

$$w(G) = \prod_{\substack{e \in \mathcal{E}(G) \\ e=(f_1,f_2)}} \mathcal{P}(a_{f_1}, a_{f_2}) \prod_{\substack{v \in \mathcal{V}(G) \\ \text{black}}} \mathcal{V}_{d_v}(\{a_f\}_{f \mapsto v}) \prod_{\substack{v \in \mathcal{V}(G) \\ \text{square}}} \prod_{f \mapsto v} \frac{1}{u - a_f}.$$

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Generating series of unciliated maps of topology (g,n):

$$F^{[r]}_{g,n}(z_1,\ldots,z_n;\lambda;v_j;\alpha) = \sum_{G\in\mathcal{F}^{[r]}_{g,n}(z_1,\ldots,z_n)} \frac{w(G)}{\#\mathrm{Aut}\,G}\alpha^{-\deg G}$$

$$= \sum_{\delta\geq(2g+n-2)} \alpha^{-(r+1)\delta} \sum_{G\in\mathcal{F}^{[r],\delta}_{g,n}(z_1,\ldots,z_n)} \frac{w(G)}{\#\mathrm{Aut}\,G} \in \mathbb{Q}[\{z_i^{-1},\lambda_j^{-1},v_k\}][[\alpha^{-1}]],$$

$i\in[\![1,n]\!]$, $j\in[\![1,N]\!]$, $k\in[\![1,r+1]\!]$.

- Analogously:

$$W^{[r]}_{g,n}(z_1,\ldots,z_n;\lambda;v_j;\alpha),\ U^{[r]}_{g,n}(z_1,\ldots,z_n;\lambda;v_j;\alpha),\ S^{[r]}_{g,\underline{k}}(S_1,\ldots,S_n;\lambda;v_j;\alpha).$$

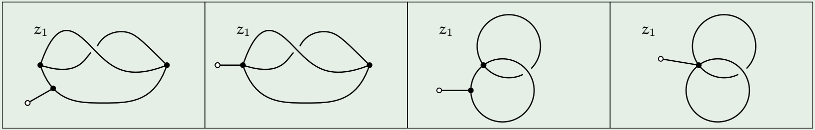
Torus with one boundary

Weight of a map G:

$$w(G) = \prod_{\substack{e\in\mathcal{E}(G)\\ e=(f_1,f_2)}} \mathcal{P}(a_{f_1},a_{f_2}) \prod_{\substack{v\in\mathcal{V}(G)\\ \text{black}}} \mathcal{V}_{d_v}(\{a_f\}_{f\mapsto v}) \prod_{\substack{v\in\mathcal{V}(G)\\ \text{square}}} \prod_{f\mapsto v} \frac{1}{u-a_f}.$$

Example (topology $(1,1)$ and case $\lambda_j = \infty$, i.e. without internal faces:)

$\deg G = (r+1)(2g-2+n) = r+1 \rightsquigarrow \mathcal{W}^{[r]}_{1,1}(z_1)$ has 4 graphs:



$$W^{[r]}_{1,1}(z_1) = \alpha^{-(r+1)} \sum_{G\in\mathcal{W}^{[r],1}_{1,1}(z_1)} \frac{w(G)}{\#\mathrm{Aut}\, G} = \alpha^{-(r+1)}\Big[\mathcal{P}(z_1,z_1)^5\mathcal{V}_3(z_1,z_1,z_1)^3$$

$$+\,2\mathcal{P}(z_1,z_1)^4\mathcal{V}_3(z_1,z_1,z_1)\mathcal{V}_4(z_1,z_1,z_1,z_1) + \mathcal{P}(z_1,z_1)^3\mathcal{V}_5(z_1,z_1,z_1,z_1,z_1)\Big]$$

$$= \alpha^{-(r+1)}\left[\frac{1}{6}\frac{V^{(3)}(z_1)V^{(4)}(z_1)}{V''(z_1)^4} - \frac{1}{8}\frac{V^{(3)}(z_1)^3}{V''(z_1)^5} - \frac{1}{24}\frac{V^{(5)}(z_1)}{V''(z_1)^3}\right],\quad \mathcal{V}_m(z,\ldots,z) = \frac{-V^{(m)}(z)}{(m-1)!}.$$

Theorem

$$W^{[r]}_{g,n}(z_1,\dots,z_n) = \frac{1}{V''(z_1)}\frac{\partial}{\partial z_1}\cdots\frac{1}{V''(z_n)}\frac{\partial}{\partial z_n}F^{[r]}_{g,n}(z_1,\dots,z_n)$$
$$+\,\delta_{g,0}\delta_{n,2}\left(\frac{1}{V''(z_1)V''(z_2)(z_1-z_2)^2}-\frac{1}{(V'(z_1)-V'(z_2))^2}\right)$$
$$+\,\delta_{g,0}\delta_{n,1}\sum_{j=1}^{N}\left(\frac{1}{V''(z_1)(z_1-\lambda_j)}-\frac{1}{(V'(z_1)-V'(\lambda_j))}\right).$$

For $(g,n)\neq(0,1)$*:*

$$S^{[r]}_{g;\underline{k}}(S_1,\dots,S_n) = \sum_{j_1=1}^{k_1}\cdots\sum_{j_n=1}^{k_n}\frac{W^{[r]}_{g,n}(z_{1,j_1},\dots,z_{n,j_n})}{\prod_{m=1}^{n}\alpha^{k_m(r+1)}\prod_{\substack{i_m=1\\ i_m\neq j_m}}^{k_m}\left(V'(z_{m,i_m})-V'(z_{m,j_m})\right)}.$$

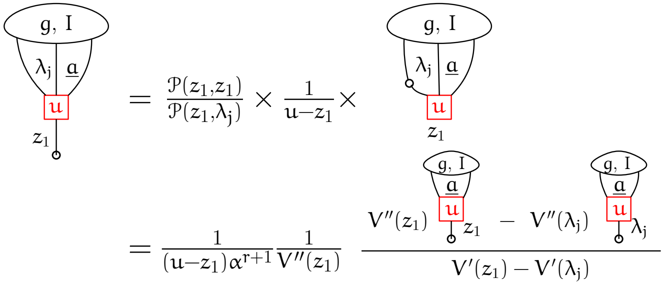
$$-\operatorname*{Res}_{u=\infty} du\; V'(u)\,(u-z_1)U^{[r]}_{g,n}(u;z_1,\dots,z_n) = \frac{V'(z_1)}{V''(z_1)}W^{[r]}_{g,n}(z_1,\dots,z_n)$$
$$+\,\delta_{g,0}\delta_{n,1}\left(\frac{N}{V''(z_1)}\right).$$

Tutte's recursion

Idea: Erase the first cilium from $G \in \mathcal{U}^{[r]}_{g,n}(u; z_1, \ldots, z_n)$ and introduce a bivalent white vertex on the following edge around the square vertex in the clockwise direction $\rightsquigarrow$ Recursion for $U^{[r]}_{g,n}$ and $W^{[r]}_{g,n}$ (à la Tutte).

$(g, n) \neq (0, 1) \rightsquigarrow$ 4 cases. $I = \{z_2, \ldots, z_n\}$, $I_j = I \backslash \{z_j\}$, $J \sqcup J' = I$, $h + h' = g$.

1. Following edge is adjacent to a face decorated with $\lambda_j, j \in \{1, \ldots, N\}$:



Contribution:

$$\frac{\alpha^{-(r+1)}}{u - z_1} \frac{1}{V''(z_1)} \sum_{j=1}^{N} \frac{V''(z_1) U^{[r]}_{g,n}(u; z_1, I) - V''(\lambda_j) U^{[r]}_{g,n}(u; \lambda_j, I)}{V'(z_1) - V'(\lambda_j)}.$$

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Idea: Erase the first cilium from $G \in \mathcal{U}^{[r]}_{g,n}(u; z_1, \dots, z_n)$ and introduce a bivalent white vertex on the following edge around the square vertex in the clockwise direction $\rightsquigarrow$ Recursion for $U^{[r]}_{g,n}$ and $W^{[r]}_{g,n}$ (à la Tutte).

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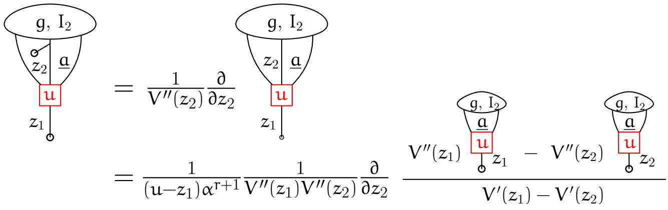
2 Following edge is adjacent to a face decorated with z_m, $m \in \{2, \dots, n\}$:



Contribution:

$$\frac{\alpha^{-(r+1)}}{u - z_1} \frac{1}{V''(z_1)} \sum_{m=2}^{n} \frac{1}{V''(z_m)} \frac{\partial}{\partial z_m} \frac{V''(z_1) U^{[r]}_{g,n-1}(u; z_1, I_m) - V''(z_m) U^{[r]}_{g,n-1}(u; z_m, I_m)}{V'(z_1) - V'(z_m)}.$$

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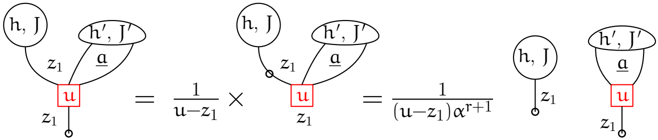
3 Following edge is adjacent to the first marked face:



Contribution:

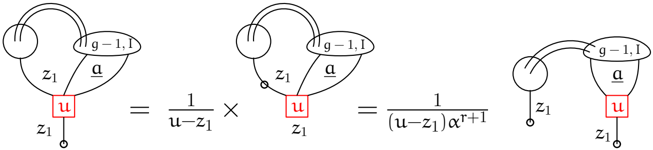
$$\frac{\alpha^{-(r+1)}}{u - z_1} \sum_{\substack{h+h'=g \\ J \sqcup J' = I}} W^{[r]}_{h,1+\#J}(z_1, J) U^{[r]}_{h',1+\#J'}(u; z_1, J').$$

Tutte's recursion

Idea: Erase the first cilium from $G \in \mathcal{U}^{[r]}_{g,n}(u; z_1, \ldots, z_n)$ and introduce a bivalent white vertex on the following edge around the square vertex in the clockwise direction $\rightsquigarrow$ Recursion for $U^{[r]}_{g,n}$ and $W^{[r]}_{g,n}$ (à la Tutte).

$(g, n) \neq (0, 1) \rightsquigarrow$ 4 cases. $I = \{z_2, \ldots, z_n\}$, $I_j = I \backslash \{z_j\}$, $J \sqcup J' = I$, $h + h' = g$.

4 Following edge is adjacent to the first marked face:



Contribution:

$$\frac{\alpha^{-(r+1)}}{u - z_1} U^{[r]}_{g-1,n+1}(u; z_1, z_1, I).$$

Tutte's equation and spectral curve

Tutte's equation $\rightsquigarrow$ recursive relation on $2g+n+\delta$. For $(g,n) \neq (0,1)$:

$$U^{[r]}_{g,n}(u;z_1,I) = \frac{\alpha^{-(r+1)}}{u-z_1}\Bigg(\frac{1}{V''(z_1)}\sum_{j=1}^{N}\frac{V''(z_1)U^{[r]}_{g,n}(u;z_1,I)-V''(\lambda_j)U^{[r]}_{g,n}(u;\lambda_j,I)}{V'(z_1)-V(\lambda_j)}$$

$$+\frac{1}{V''(z_1)}\sum_{m=2}^{n}\frac{1}{V''(z_m)}\frac{\partial}{\partial z_m}\frac{V''(z_1)U^{[r]}_{g,n-1}(u;z_1,I_m)-V''(z_m)U^{[r]}_{g,n-1}(u;z_m,I_m)}{V'(z_1)-V(z_m)}$$

$$+\sum_{\substack{h+h'=g\\ J\sqcup J'=I}} W^{[r]}_{h,1+J}(z_1,J)\,U^{[r]}_{h',1+J'}(u;z_1,J') + U^{[r]}_{g-1,n+1}(u;z_1,z_1,I)\Bigg).$$

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Towards the **spectral curve**:

$$x(z) \coloneqq V'(z),\quad y(z)\coloneqq z+\alpha^{-(r+1)}W^{[r]}_{0,1}(z)+\alpha^{-(r+1)}\sum_{j=1}^{N}\frac{1}{V'(z_1)-V'(\lambda_j)}.$$

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$$x(z) := V'(z),\quad y(z) := z+\alpha^{-(r+1)}W^{[r]}_{0,1}(z)+\alpha^{-(r+1)}\sum_{j=1}^{N}\frac{1}{V'(z_1)-V'(\lambda_j)}.$$

Theorem

$\exists$ polynomial Q of degree r, such that if ζ is the implicit function defined by

$$Q(\zeta)=x(z),\quad \zeta \underset{z\to\infty}{=} z+\mathcal{O}(1),\quad \text{then}\quad y(\zeta)=\zeta+\alpha^{-(r+1)}\sum_{j=1}^{N}\frac{1}{Q'(\xi_j)(\zeta-\xi_j)},$$

where $Q(\xi_i)=V'(\lambda_i)$. Q is a formal power series in $\alpha^{-(r+1)}$ and determined by:

$$V'(y(\zeta))-Q(\zeta) \underset{\zeta\to\infty}{=} \mathcal{O}(1/\zeta).$$

Proof of TR for ciliated maps

1. Recursion for $U_{g,n}^{[r]}$ and $W_{g,n}^{[r]}$ (à la Tutte).

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$$\mathcal{S}: \begin{cases} x(\zeta) = Q(\zeta), \text{ with } Q(\xi_i) = V'(\lambda_i), \\ y(\zeta) = \zeta + \alpha^{-(r+1)} \sum_{i=1}^{N} \frac{1}{Q'(\xi_i)(\zeta - \xi_i)}, \\ \omega_{0,1}^{[r]}(\zeta) = \alpha^{r+1} y(\zeta) \mathrm{d}x(\zeta), \\ \omega_{0,2}^{[r]}(\zeta_1, \zeta_2) = \frac{\mathrm{d}\zeta_1 \, \mathrm{d}\zeta_2}{(\zeta_1 - \zeta_2)^2}. \end{cases}$$

Proof of TR for ciliated maps

1. Recursion for $U_{g,n}^{[r]}$ and $W_{g,n}^{[r]}$ (à la Tutte).
2. Base topologies $(0,1)$ and $(0,2)$ give us the spectral curve.
3. Combinatorial interpretation of certain universal expressions:

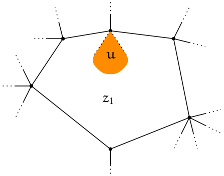
$$\check{H}_{g,n}^{[r]}(u;\zeta_1,I) := v_{r+1}\sum_{k=0}^{r-1}(-1)^k u^{r-1-k}\alpha^{-(k-1)(r+1)}\sum_{\underline{t}\underset{k}{\subseteq}(x^{-1}(x(\zeta_1))\setminus\{\zeta_1\})}\mathcal{E}^{(k)}W_{g,n}^{[r]}(\underline{t};I),$$

$$\check{P}_{g,n}^{[r]}(u;\zeta_1,I) := v_{r+1}\sum_{k=0}^{r}(-1)^k u^{r-k}\alpha^{-(k-1)(r+1)}\sum_{\underline{t}\underset{k}{\subseteq}x^{-1}(x(\zeta_1))}\mathcal{E}^{(k)}W_{g,n}^{[r]}(\underline{t};I),$$

where $I=\{\zeta_2,\ldots,\zeta_n\}$ and

$$\mathcal{E}^{(k)}W_{g,n}^{[r]}(\underline{t};I) := \sum_{\mu\in\mathcal{S}(\underline{t})}\sum_{\bigsqcup_{i=1}^{\ell(\mu)}J_i=I}\sum_{\sum_{i=1}^{\ell(\mu)}g_i=h+\ell(\mu)-k}\left[\prod_{i=1}^{l(\mu)}\widetilde{W}_{g_i,|\mu_i|+|J_i|}^{[r]}(\mu_i,J_i)\right].$$

$$H_{g,n}^{[r]}(u;\zeta_1,I) := V''(\zeta_1)\left[V'(u)U_{g,n}^{[r]}(u;I)\right]_+ = \check{H}_{g,n}^{[r]}(u;\zeta_1,I).$$



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4. Analytic properties: polar structure of $W_{g,n}^{[r]}(z_1,\ldots,z_n)$.
5. 3 $\Rightarrow$ Loop equations. $I=\{\zeta_2,\ldots,\zeta_n\}$. $Q(\zeta)$ polynomial of degree r, so the equation $Q(\zeta)=Q(\zeta_0)$ has r solutions denoted $\zeta_0=\zeta_0^{(0)},\zeta_0^{(1)},\ldots,\zeta_0^{(r-1)}$.

Linear:

$$\sum_{k=0}^{r-1}\omega_{g,n}^{[r]}\left(\zeta_1^{(k)},I\right)=\delta_{g,0}\delta_{n,1}\left(-\frac{v_r\alpha^{r+1}}{v_{r+1}}+\sum_{j=1}^{N}\frac{1}{x(\zeta_1)-x(\xi_j)}\right)\mathrm{d}x(\zeta_1)$$

$$+\,\delta_{g,0}\delta_{n,2}\frac{\mathrm{d}x(\zeta_1)\,\mathrm{d}x(\zeta_2)}{(x(\zeta_1)-x(\zeta_2))^2}.$$

Quadratic:

$$\sum_{k=0}^{r-1}\left[\omega_{g-1,n+1}^{[r]}\left(\zeta_1^{(k)},\zeta_1^{(k)},I\right)+\sum_{\substack{h+h'=g\\ J\sqcup J'=I}}\omega_{h,1+\#J}^{[r]}\left(\zeta_1^{(k)},J\right)\omega_{h',1+\#J'}^{[r]}\left(\zeta_1^{(k)},J'\right)\right]$$

is a differential in $x(\zeta_1)$ without poles at the ramification points of x.

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$$\omega_{g,n}^{[r]}(\zeta_1,\ldots,\zeta_n) = W_{g,n}^{[r]}(z_1,\ldots,z_n)\mathrm{d}x(\zeta_1)\cdots\mathrm{d}x(\zeta_n).$$

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---

7. Consider the family of spectral curves with $V'_\varepsilon(z) = z^r - r\varepsilon^{-r-1}z$, which for $\varepsilon \neq 0$ have $r-1$ simple ramification points. Take the limit $\varepsilon \to 0$ and obtain
   - **topological recursion** (admitting ramification points of higher order) for $\omega^{[r],0}_{g,n}$ with spectral curve with $V'_0(z) = z^r$ (with one ramification point of order $r-1$);
   - $\lim\limits_{\varepsilon\to 0}\omega^{[r],\varepsilon}_{g,n}(\zeta_1, I) = \omega^{[r],0}_{g,n}(\zeta_1, I)$ (needs proof!).

Witten's conjecture r-spin, '93 (proved by Faber–Shadrin–Zvonkine)

1. Generalized maps and matrix model

Higher TR ('13)

2. Intersection numbers
$$\int_{\overline{\mathcal{M}}_{g,n}} c_W^r(a_1,\ldots,a_n)\psi_1^{d_1}\cdots\psi_n^{d_n}$$

3. Hierarchy r-KdV

Faber–Shadrin–Zvonkine, '10

Generalized Kontsevich maps and TR

1. Generalized maps and matrix model

Belliard–Charbonnier–Eynard–G-F, '21

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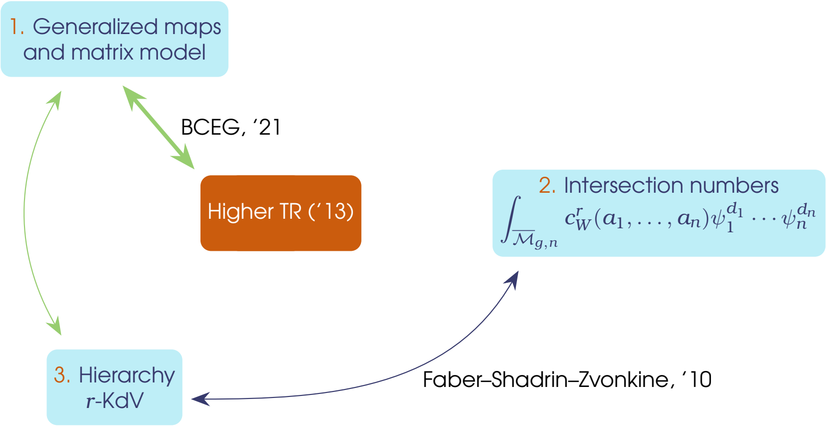
Theorem (Belliard–Charbonnier–Eynard–G-F, '21)

Generalised (Kontsevich) maps satisfy topological recursion.

Generalized Kontsevich maps and integrable hiearchy



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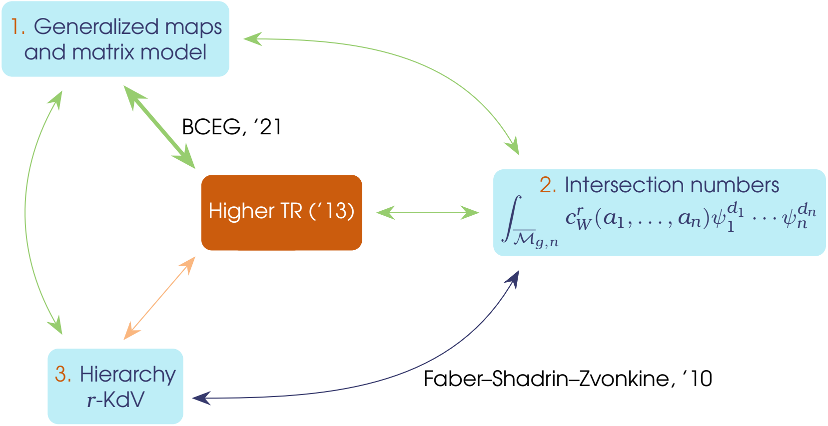
Generalized Kontsevich maps and r-spin intersection numbers



Theorem (Belliard–Charbonnier–Eynard–G-F, '21)

TR (allowing ramification points of higher order) applied to the spectral curve $(x,y)=(z^r,z)$ *produces* r*-spin intersection numbers.*

Generalized Kontsevich matrix model (GKM)

Kontsevich graphs are Feynman graphs of the hermitian matrix model with external field $\lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$:

$$Z = \int_{\mathcal{H}_N + \lambda} \mathrm{d}M \; e^{-N\mathrm{Tr}\left(\frac{M^3}{3} - M\lambda^2\right)}.$$

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Re-writing $M = \lambda + \widetilde{M}$ to eliminate the linear term:

$$C(\lambda) \int_{\mathcal{H}_N} \mathrm{d}\widetilde{M} e^{-N\alpha^{r+1}\left(\frac{1}{2}\sum_{i,j=1}^{N} \widetilde{M}_{i,j}\widetilde{M}_{j,i}\frac{1}{\mathcal{P}(\lambda_i,\lambda_j)} - \sum_{\ell=3}^{r+1}\frac{1}{\ell}\sum_{i_1,\ldots,i_\ell=1}^{N} \widetilde{M}_{i_1,i_2}\widetilde{M}_{i_2,i_3}\ldots\widetilde{M}_{i_\ell,i_1}\mathcal{V}_\ell(\lambda_{i_1},\ldots,\lambda_{i_\ell})\right)}.$$

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$$\log\frac{Z}{Z_0} = \sum_{g\geq 0}\sum_{G\in\mathcal{F}_{g,0}^{[r]}} \frac{N^{-\frac{\deg G}{r+1}}\alpha^{-\deg G}}{\#\mathrm{Aut}\,G} \prod_{\substack{e\in\mathcal{E}(G)\\ e=(f_1,f_2)}} \mathcal{P}(\lambda_{f_1},\lambda_{f_2}) \prod_{v\in\mathcal{V}(G)} \mathcal{V}_{d_v}\left(\{\lambda_f\}_{f\mapsto v}\right).$$

For $i_1 \neq \cdots \neq i_n$, connected correlation functions $\rightsquigarrow$ ciliated maps (1):

$$\langle \widetilde{M}_{i_1,i_1}\cdots\widetilde{M}_{i_n,i_n}\rangle_c = \frac{1}{(N\alpha^{r+1})^n}\frac{\partial}{\partial\Lambda_{i_1}}\cdots\frac{\partial}{\partial\Lambda_{i_n}}\log Z = \sum_{g\geq 0} N^{2-2g-n} W_{g,n}^{[r]}(\lambda_{i_1},\ldots,\lambda_{i_n}).$$

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$\left\langle \mathrm{Tr}\frac{1}{x_1 - M} \cdots \mathrm{Tr}\frac{1}{x_n - M}\right\rangle_c$ admit topological expansions computed by TR applied to the spectral curve (y, x) (Eynard–Orantin,'07,'09).

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From maps to r-spin intersection numbers

r-spin intersection numbers:

$$\langle \tau_{d_1,a_1} \cdots \tau_{d_n,a_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} c_W^r(a_1,\ldots,a_n) \prod_{i=1}^n \psi_i^{d_i}.$$

1. The partition function Z of the GKM provides the only solution $\mathcal{I}_N(t_1, t_2, \ldots)$ of the r-KdV hierarchy that satisfies the string equation, with $t_k = \frac{1}{k}\mathrm{Tr}\big(\alpha N^{\frac{1}{r+1}}\lambda\big)^{-k}$ (using Adler–van Moerbeke, '92).

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3. Using 1, 2 and (1),

$$W_{g,n}^{[r]}(\lambda_{i_1},\ldots,\lambda_{i_n}) = -\frac{(-r)^{g-1}}{\alpha^{(r+1)n}}\frac{\partial}{\partial\Lambda_{i_1}}\cdots\frac{\partial}{\partial\Lambda_{i_n}}F_g^{[r],\mathrm{int}}(\mathbf{t})$$

$$=\frac{(-1)^g r^{g-1+n}}{\alpha^{(r+1)(2g-2+n)}}\sum_{\substack{d_1,\ldots,d_n\geq 0\\ 0\leq j_1,\ldots,j_n\leq r-1}}\prod_{\ell=1}^n\frac{c_{d_\ell+1,j_\ell}}{\Lambda_{i_\ell}^{d_\ell+1+\frac{j_\ell+1}{r}}}\left\langle\prod_{i=1}^n\tau_{d_i,j_i}e^{\sum_{d,j}t_{d,j}\tau_{d,j}}\right\rangle_g,$$

with $t_{d,j}=c_{d,j}\sum_{k=1}^N\Lambda_k^{-d-\frac{j+1}{r}}$ and $c_{d,j}=(-1)^d\frac{\Gamma\left(d+\frac{j+1}{r}\right)}{\Gamma\left(\frac{j+1}{r}\right)}$.

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$$W_{g,n}^{[r]}(\lambda_{i_1},\ldots,\lambda_{i_n}) = -\frac{(-r)^{g-1}}{\alpha^{(r+1)n}} \frac{\partial}{\partial \Lambda_{i_1}} \cdots \frac{\partial}{\partial \Lambda_{i_n}} F_g^{[r],\mathrm{int}}(\mathbf{t})$$

$$= \frac{(-1)^g r^{g-1+n}}{\alpha^{(r+1)(2g-2+n)}} \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ 0 \leq j_1,\ldots,j_n \leq r-1}} \prod_{\ell=1}^n \frac{c_{d_\ell+1,j_\ell}}{\Lambda_{i_\ell}^{d_\ell+1+\frac{j_\ell+1}{r}}} \left\langle \prod_{i=1}^n \tau_{d_i,j_i} e^{\sum_{d,j} t_{d,j}\tau_{d,j}} \right\rangle_g ,$$

with $t_{d,j} = c_{d,j} \sum_{k=1}^N \Lambda_k^{-d-\frac{j+1}{r}}$ and $c_{d,j} = (-1)^d \frac{\Gamma\left(d+\frac{j+1}{r}\right)}{\Gamma\left(\frac{j+1}{r}\right)}$.

Remark (ELSV-type formula)

ELSV-like (Ekedahl–Lando–Shapiro–Vainshtein, '01) formulas relate combinatorial problems with intersection theory over $\overline{\mathcal{M}}_{g,n}$.

r-spin intersection numbers for topology $(1,1)$

From the enumeration of ciliated maps of topology $(1,1)$:

$$W_{1,1}^{[r]}(z_1) = \alpha^{-(r+1)}\left[\frac{1}{6}\frac{V^{(3)}(z_1)V^{(4)}(z_1)}{V''(z_1)^4} - \frac{1}{8}\frac{V^{(3)}(z_1)^3}{V''(z_1)^5} - \frac{1}{24}\frac{V^{(5)}(z_1)}{V''(z_1)^3}\right].$$

In the case $V(z) = \frac{z^{r+1}}{r+1}$, we get

$$W_{1,1}^{[r]}(z_1) = -\frac{\alpha^{-(r+1)}}{24}\frac{r^2-1}{r^2}\frac{1}{z_1^{2r+1}},$$

$$\omega_{1,1}^{[r]}(z_1) = W_{1,1}^{[r]}(z_1)V''(z_1)\mathrm{d}z_1 = -\frac{\alpha^{-(r+1)}}{24}\frac{r^2-1}{r}\frac{\mathrm{d}z_1}{z_1^{r+2}}.$$

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From our ELSV-type formula:

$$\omega_{1,1}^{[r]}(z_1) = -\frac{\alpha^{-(r+1)}}{24} \frac{r^2-1}{r} \frac{\mathrm{d}z_1}{z_1^{r+2}} = -\alpha^{-(r+1)} \frac{r+1}{r} \langle \tau_{1,0} \rangle_1 \frac{\mathrm{d}z_1}{z_1^{r+2}} .$$

Therefore,

$$\boxed{\langle \tau_{1,0} \rangle_1 = \frac{r-1}{24} .}$$

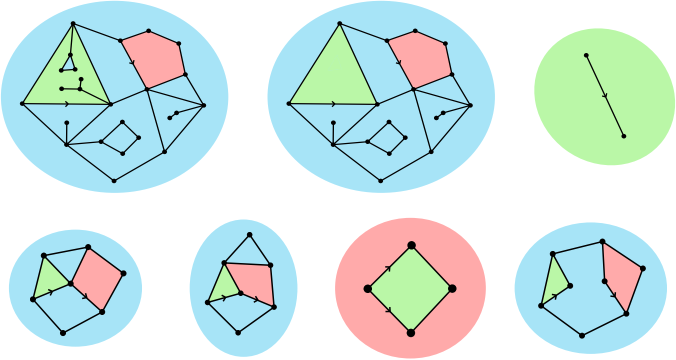
Outline

1. Topological recursion (TR)
2. Witten's conjecture, Kontsevich's theorem
3. Cohomological field theories (CohFT)
4. Witten's r-spin class and the r-KdV hierarchy
   (based on joint work with R. Belliard, S. Charbonnier and B. Eynard, arXiv:2105.08035)
   - Combinatorial model: Generalised Kontsevich graphs
   - Tutte's recursion
   - Topological recursion for ciliated maps
   - From graphs to intersection numbers with Witten's class
5. Topological recursion for fully simple maps
   (based on joint work with G. Borot and S. Charbonnier, arXiv:2106.09002)
   - Disks and cylinders
   - Symplectic invariance and combinatorial interpretation
6. Further consequences: ongoing and future

Simple maps

Definition

Simple:	Boundaries are simple.
Fully simple:	Simple and pairwise disjoint boundaries.



$\mathbb{M}_n^{[g]}(l_1, \ldots, l_n) \rightsquigarrow$ Set of (ordinary) maps of genus g with n boundaries (marked faces) of fixed lengths $l_1, \ldots, l_n$.

Generating series. Disks and cylinders

Generating series of maps of genus g and n boundaries of lengths $l_1, \ldots, l_n$:

$$\mathrm{Map}^{[g]}_{l_1,\ldots,l_n} := \sum_{\mathcal{M} \in \mathbb{M}^{[g]}_n(l_1,\ldots,l_n)} w(\mathcal{M}), \quad \text{with} \quad w(\mathcal{M}) := \frac{1}{|\mathrm{Aut}\,\mathcal{M}|} \prod_{f \in \mathrm{IFaces}(\mathcal{M})} t_{\mathrm{length}(f)}.$$

$\mathrm{FSMap}^{[g]}_{k_1,\ldots,k_n} \rightsquigarrow$ Same for fully simple maps.

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$$W^{[g]}_n(x_1, \ldots, x_n) := \sum_{l_1,\ldots,l_n \geq 0} \frac{\mathrm{Map}^{[g]}_{l_1,\ldots,l_n}}{x_1^{1+l_1} \cdots x_n^{1+l_n}},$$

$$X^{[g]}_n(w_1, \ldots, w_n) := \sum_{k_1,\ldots,k_n \geq 0} \mathrm{FSMap}^{[g]}_{k_1,\ldots,k_n} w_1^{k_1-1} \cdots w_n^{k_n-1}.$$

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Theorem (Borot, G-F, '17)

- ***Disks:*** $X^{[0]}_1(W^{[0]}_1(x)) = x$.
- ***Cylinders:*** *Setting* $x_i = X^{[0]}_1(w_i)$*, or equivalently* $w_i = W^{[0]}_1(x_i)$*,*

$$\left(W^{[0]}_2(x_1, x_2) + \frac{1}{(x_1 - x_2)^2}\right) \mathrm{d}x_1 \mathrm{d}x_2 = \left(X^{[0]}_2(w_1, w_2) + \frac{1}{(w_1 - w_2)^2}\right) \mathrm{d}w_1 \mathrm{d}w_2.$$

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Generating series of maps of genus g and n boundaries of lengths $l_1, \ldots, l_n$:

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Example (quadrangulations):

$$\mathrm{Map}^{[0]}_4 = 2 + 9t_4 + 54t_4^2 + 378t_4^2 + \ldots, \qquad \mathrm{FSMap}^{[0]}_4 = t_4 + 10t_4^2 + 90t_4^3 + \ldots$$

Topological recursion for maps

Initial data:
$$\begin{cases} \Sigma = \mathbb{C}P^1, \\ (x(z) = \alpha + \gamma(z + \frac{1}{z}), y = W_1^{[0]}(x(z))), \\ \omega_{0,1}(z) = y(z)\,\mathrm{d}x(z), \\ \omega_{0,2}(z_1, z_2) = B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}. \end{cases}$$

TR output: For $n \geq 1$,

$$\omega_{g,n}(z_1, \ldots, z_n) = \operatorname*{Res}_{z \to \pm 1} \frac{\int_{1/z}^{z} \omega_{0,2}(z_1, \cdot)}{2(\omega_{0,1}(z) - \omega_{0,1}(1/z))} \Bigg(\omega_{g-1,n+1}(z, 1/z, z_{[\![2,n]\!]}) + \sum_{\substack{0 \leq h \leq g \\ I \sqcup J = [\![2,n]\!]}}^{\text{no disk}} \omega_{g-h,1+|I|}(z, z_I)\omega_{h,1+|J|}(1/z, z_J) \Bigg).$$

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Example $\rightsquigarrow$ Quadrangulations: All internal faces are quadrangles, i.e. $t_j = \delta_{j,4} t_j$, with t_4 the weight per internal quadrangle. Spectral curve:

$$x(z) = \gamma\left(z + \frac{1}{z}\right), \quad y(z) = W_1^{[0]}(x(z)) = \frac{1}{\gamma z} - \frac{t_4 \gamma^3}{z^3},$$

with $\gamma = \sqrt{\frac{1 - \sqrt{1 - 12t_4}}{6t_4}} = 1 + \frac{3t_4}{2} + \frac{63}{8} t_4^2 + \frac{891}{16} t_4^3 + \frac{57915}{128} t_4^4 + O(t_4^4)$.

Initial data: $\begin{cases} (x(z) = \alpha + \gamma(z + \frac{1}{z}), y = W_1^{[0]}(x(z))), \omega_{0,1}(z) = y(z)\,\mathrm{d}x(z), \\ \omega_{0,2}(z_1, z_2) = B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}. \end{cases}$

TR output: $\omega_{g,n}(z_1, \ldots, z_n)$, for all $g, n \geq 0$.

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with $\gamma = \sqrt{\frac{1 - \sqrt{1 - 12t_4}}{6t_4}} = 1 + \frac{3t_4}{2} + \frac{63}{8} t_4^2 + \frac{891}{16} t_4^3 + \frac{57915}{128} t_4^4 + O(t_4^4)$. The zeros of $x'(z)$ are at $z = \pm 1$ and the deck transformation is $\sigma(z) = \frac{1}{z}$. One can compute, for example:

$$\frac{\omega_{1,1}(z)}{dx(z)} = \frac{z^3(t_4\gamma^4 z^4 + z^2(1 - 5t_4\gamma^4) + t_4\gamma^4)}{\gamma(z^2 - 1)^5(1 - 3t_4\gamma^4)^2} \mathrm{d}z = W_1^{[1]}(x(z)) \Rightarrow$$

$$W_1^{[1]}(x_1) = (t_4 + 15t_4^2 + 198t_4^3 + \ldots)\frac{1}{x_1^3} + (1 + 15t_4 + 198t_4^2 + 2511t_4^3 \ldots)\frac{1}{x_1^5}$$

$$+(10+150t_4+1980t_4^2+25110t_4^3 \ldots)\frac{1}{x_1^7}+(70+1190t_4+16590t_4^2+216720t_4^3 \ldots)\frac{1}{x_1^9}+\ldots$$

Symplectic invariance

$$(\Sigma, (x, y)) \overset{\text{TR}}{\rightsquigarrow} \omega_{g,n}(z_1, \ldots, z_n) \; (\omega_{g,0} = \mathfrak{F}_g \in \mathbb{C})$$

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$$\Phi \text{ preserving } |\mathrm{d}x \wedge \mathrm{d}y| \; \big\downarrow$$

$$(\Sigma, (\check{x}, \check{y}))$$

Symplectic invariance

$$
\begin{array}{lcl}
(\Sigma,(x,y)) & \overset{\mathrm{TR}}{\rightsquigarrow} & \omega_{g,n}(z_1,\ldots,z_n)\ \ (\omega_{g,0}=\mathfrak{F}_g\in\mathbb{C}) \\
\left.\begin{array}{r}\Phi\\ \text{preserving}\\ |\mathrm{d}x\wedge\mathrm{d}y|\end{array}\right\downarrow & & \\
(\Sigma,(\check{x},\check{y})) & \overset{\mathrm{TR}}{\rightsquigarrow} & \check{\omega}_{g,n}(z_1,\ldots,z_n)\ \ (\check{\omega}_{g,0}=\check{\mathfrak{F}}_g)
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$\Phi = \mathcal{E} : (x, y) \mapsto (y, x)$
not well understood.

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Let $x(z) = \alpha + \gamma(z + \frac{1}{z})$.

Theorem (Eynard, '05)

$$(\mathbb{CP}^1, (x, y = W_1^{[0]}(x)), \omega_{0,2} = B)$$

$\wr$ TR

$$\frac{\omega_{g,n}(z_1,\ldots,z_n)}{dx_1 \cdots dx_n} = W_n^{[g]}(x_1,\ldots,x_n),$$

$$\forall 2g - 2 + n > 0,\ z_i \to \infty.$$

Maps $\overset{\mathcal{E}}{\longleftrightarrow}$

Symplectic invariance

$$(\Sigma,(x,y)) \overset{\text{TR}}{\rightsquigarrow} \omega_{g,n}(z_1,\ldots,z_n) \quad (\omega_{g,0}=\mathfrak{F}_g\in\mathbb{C})$$

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$$\forall 2g-2+n>0,\ z_i\to\infty.$$

Maps

$\overset{\mathcal{E}}{\longleftrightarrow}$

Theorem (Borot–Charbonnier–G-F, '21)

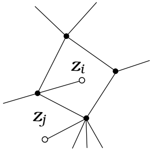
$$(\mathbb{CP}^1,(y,x),\omega_{0,2}=B)$$
$$\downarrow \text{TR}$$
$$\frac{\check{\omega}_{g,n}(z_1,\ldots,z_n)}{dy_1\cdots dy_n}=X_n^{[g]}(y_1,\ldots,y_n),$$
$$\forall 2g-2+n>0,\ z_i\to\infty.$$

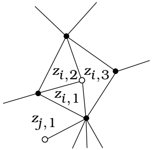
Fully simple maps

- Our proof: combinatorial, via ciliated maps.
- Proof by Bychkov–Dunin-Barkowsi–Kazarian–Shadrin, '21: via semi-infinite wedge formalism.

Sketch of the proof (Borot–Charbonnier–G-F, '21)

$\mathcal{C}^{[r]}_{g,n}(z_1,\ldots,z_n) = \mathcal{S}^{[r]}_{g,(1,\ldots,1)}(z_1,\ldots,z_n)$ $\qquad$ $\mathcal{S}^{[r]}_{g,(k_1,\ldots,k_n)}(S_1,\ldots,S_n)$





1. $\mathcal{M} \in \mathbb{M}^{[g]}_n(k_1,\ldots,k_n)$ fully simple $\Leftrightarrow$ dual $\mathcal{M}^* \in S_{g,(k_1,\ldots,k_n)}$ multi-ciliated.
   - Simplicity $\Leftrightarrow$ star constraint.
   - Full simplicity $\Leftrightarrow$ star constraint and uniqueness constraint.

Sketch of the proof (Borot–Charbonnier–G-F, '21)

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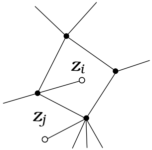

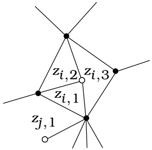

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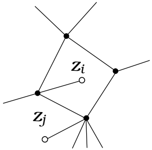
$$V(u) = \frac{u^2}{2} - \sum_{j=3}^{r} \frac{t_j}{j} u^j \quad \text{and} \quad \mathcal{V}_d(a_1,\ldots,a_d) = \operatorname*{Res}_{u=\infty} \prod_{i=1}^{d} \frac{\mathrm{d}u\, V'(u)}{u-a_i}$$

$$\Rightarrow \mathcal{V}_d(0,\ldots,0) = t_d \quad \text{and} \quad \mathcal{P}(a_1,a_2) = \frac{a_1-a_2}{V'(a_1)-V'(a_2)} \underset{a_1,a_2\to 0}{\longrightarrow} V''(0) = 1.$$

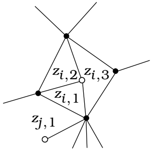
(Weights of generalised Kontsevich graphs are just deformations of the weights for maps)

Sketch of the proof (Borot–Charbonnier–G-F, '21)

$\mathcal{C}^{[r]}_{g,n}(z_1,\dots,z_n) = \mathcal{S}^{[r]}_{g,(1,\dots,1)}(z_1,\dots,z_n)$



$\mathcal{S}^{[r]}_{g,(k_1,\dots,k_n)}(S_1,\dots,S_n)$



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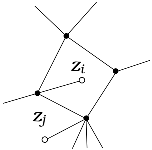
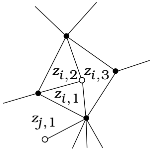
3. $S_{g,(k_1,\dots,k_n)}([0^{k_1}],\dots,[0^{k_n}]) =$

$$\frac{1}{(k_1-1)!\dots(k_n-1)!}\frac{\partial^{k_1-1}}{\partial V'(z_1)^{k_1-1}}\cdots\frac{\partial^{k_n-1}}{\partial V'(z_1)^{k_n-1}}(C_{g,n}(z_1,\dots,z_n))\Big|_{z_j=0,\lambda_1=0}.$$

Sketch of the proof (Borot–Charbonnier–G-F, '21)

$$\mathcal{C}^{[r]}_{g,n}(z_1,\dots,z_n) = \mathcal{S}^{[r]}_{g,(1,\dots,1)}(z_1,\dots,z_n) \qquad \mathcal{S}^{[r]}_{g,(k_1,\dots,k_n)}(S_1,\dots,S_n)$$





1. $\mathcal{M} \in \mathbb{M}^{[g]}_n(k_1,\dots,k_n)$ fully simple $\Leftrightarrow$ dual $\mathcal{M}^* \in S_{g,(k_1,\dots,k_n)}$ multi-ciliated.
2. $\mathrm{FSMap}^{[g]}_{k_1,\dots,k_n} = S_{g,(k_1,\dots,k_n)}([0^{k_1}],\dots,[0^{k_n}])\big|_{\lambda_1=0}\,.$
3. $S_{g,(k_1,\dots,k_n)}([0^{k_1}],\dots,[0^{k_n}]) =$
$$\frac{1}{(k_1-1)!\dots(k_n-1)!}\frac{\partial^{k_1-1}}{\partial V'(z_1)^{k_1-1}}\cdots\frac{\partial^{k_n-1}}{\partial V'(z_1)^{k_n-1}}(C_{g,n}(z_1,\dots,z_n))\Big|_{z_j=0,\lambda_1=0}\,.$$
4. $X_{g,n}(w_1,\dots,w_n) = C_{g,n}(z_1,\dots,z_n)$, with $w_j = V'(z_j)$.

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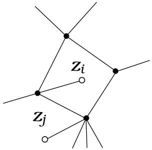
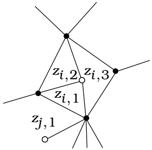


$$\mathcal{S}^{[r]}_{g,(k_1,\dots,k_n)}(S_1,\dots,S_n)$$



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4. $X_{g,n}(w_1,\dots,w_n) = C_{g,n}(z_1,\dots,z_n)$, with $w_j = V'(z_j)$.
5. **Theorem** (Belliard–Charbonnier–Eynard–G-F, '21). $C_{g,n}(z_1,\dots,z_n)$ satisfy TR for a spectral curve $(\tilde{x},\tilde{y})$.
6. If (x,y) is the spectral curve for ordinary maps, then we prove
$$(\tilde{x},\tilde{y}) = (y,x).$$

TR for fully simple maps and motivation

1. $\mathcal{M} \in \mathbb{M}_n^{[g]}(k_1, \ldots, k_n)$ fully simple $\Leftrightarrow$ dual $\mathcal{M}^* \in S_{g,(k_1,\ldots,k_n)}$ multi-ciliated.
2. $\mathrm{FSMap}^{[g]}_{k_1,\ldots,k_n} = S_{g,(k_1,\ldots,k_n)}([0^{k_1}], \ldots, [0^{k_n}])$.
3. $S_{g,(k_1,\ldots,k_n)}([0^{k_1}], \ldots, [0^{k_n}]) =$
$$\frac{1}{(k_1-1)!\ldots(k_n-1)!} \frac{\partial^{k_1-1}}{\partial V'(z_1)^{k_1-1}} \cdots \frac{\partial^{k_n-1}}{\partial V'(z_1)^{k_n-1}} (C_{g,n}(z_1, \ldots, z_n))\Big|_{z_j=0}.$$
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6. If (x, y) is the spectral curve for ordinary maps, then
$$(\tilde{x}, \tilde{y}) = (y, x).$$

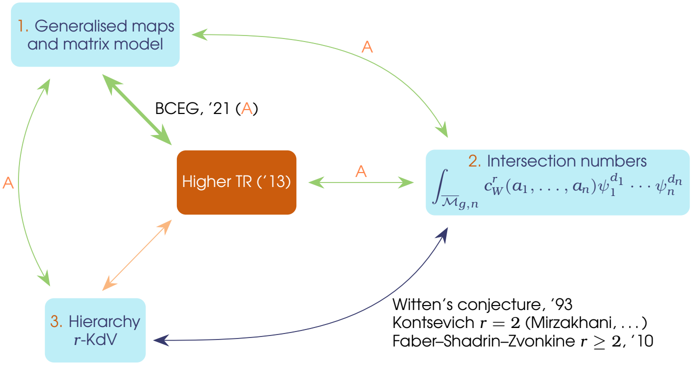
3-fold motivation:

- In **TR**, fully simple maps implement the **symplectic invariance dual** of ordinary maps.
- In **free probability**, fully simple maps are identified with (certain) free cumulants, the **free dual** of (certain) moments identified with maps (see Gaëtan's talk).
- TR solves the **enumeration of fully simple maps** (so far only achieved for planar triangulations (Krikun, '07), planar quadrangulations with even boundary lengths (Bernardi–Fusy, '18)).

Outline

1. Topological recursion (TR)
2. Witten's conjecture, Kontsevich's theorem
3. Cohomological field theories (CohFT)
4. Witten's r-spin class and the r-KdV hierarchy
   (based on joint work with R. Belliard, S. Charbonnier and B. Eynard, arXiv:2105.08035)
   - Combinatorial model: Generalised Kontsevich graphs
   - Tutte's recursion
   - Topological recursion for ciliated maps
   - From graphs to intersection numbers with Witten's class
5. Topological recursion for fully simple maps
   (based on joint work with G. Borot and S. Charbonnier, arXiv:2106.09002)
   - Disks and cylinders
   - Symplectic invariance and combinatorial interpretation
6. Further consequences: ongoing and future

Work so far and further consequences ⇝ in progress and future

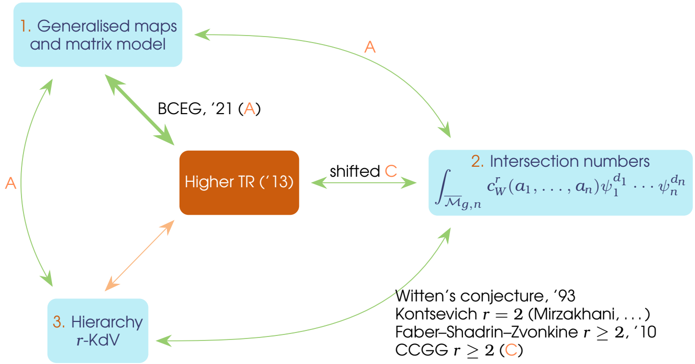


So far:

A *Topological recursion for generalised Kontsevich graphs and r-spin intersection numbers*, with R. Belliard, S. Charbonnier and B. Eynard, arXiv:2105.08035.

B *Topological recursion for fully simple maps from ciliated maps*, with G. Borot and S. Charbonnier, arXiv:2106.09002.

Work so far and further consequences ⇝ in progress and future

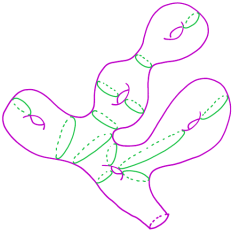
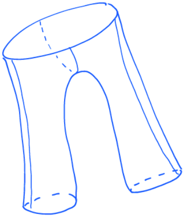


Work in progress ⇝

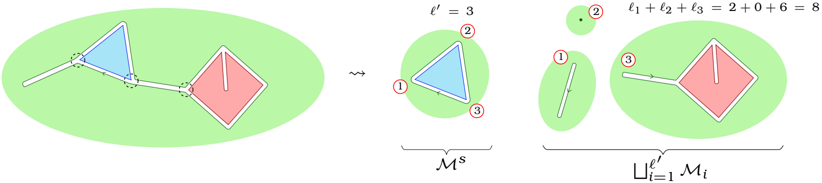
C Use the power of TR to study the intersection of Witten's class when varying the spectral curve (via Eynard–DOSS) (with S. Charbonnier, N. Chidambaran and A. Giacchetto).

Future work ⇝

- Better understanding of Witten's class making use of our graphs? Can we establish the relation 1. ↔ 2. directly?
- Symplectic invariance for a large class of spectral curves (tuning the potential V and the λ's)?

Merci beaucoup pour votre attention !



Recursion for multi-ciliated maps

Lemma

$$\mathcal{V}_{m+1}(a_1, a_2, a_3, \dots, a_{m+1}) = \frac{\mathcal{V}_m(a_1, a_3, \dots, a_{m+1}) - \mathcal{V}_m(a_2, a_3, \dots, a_{m+1})}{a_1 - a_2}.$$

Lemma

Let $\underline{k}' = (k_1 - 1, k_2, \dots, k_n)$ *and* $\underline{k}'' = (1, k_2, \dots, k_n)$.

$$S^{[r]}_{g;\underline{k}}(S_1, \dots, S_n) = \delta_{g,0}\delta_{n,1}\delta_{k_1,2}\mathcal{P}(z_{1,1}, z_{1,2})$$

$$+ \frac{1}{\alpha^{r+1}} \frac{S^{[r]}_{g;\underline{k}'}\left([z_{1,1}, z_{1,3}, \dots, z_{1,k_1}], S_2, \dots, S_n\right) - S^{[r]}_{g;\underline{k}'}\left([z_{1,2}, \dots, z_{1,k_1}], S_2, \dots, S_n\right)}{V'(z_{1,1}) - V'(z_{1,2})}.$$

Applying this formula $k_1 - 1$ *times,*

$$S^{[r]}_{g;\underline{k}}(S_1, \dots, S_n) = \frac{1}{\alpha^{(k_1-1)(r+1)}} \sum_{j=1}^{k_1} \frac{S^{[r]}_{g;\underline{k}''}([z_{1,j}], S_2, \dots, S_n) + \delta_{g,0}\delta_{n,1}\alpha z_{1,j}}{\prod_{\substack{i=1 \\ i \neq j}}^{k_1} (V'(z_{1,j}) - V'(z_{1,i}))}.$$

Remark

Local property valid for the weight of a single vertex extends to a similar property at the macroscopic level, that is to say at the level of generating functions.

Witten's class

$V = \langle e_0, e_1, \ldots, e_{r-2}\rangle_{\mathbb{Q}}$, $\eta(e_a, e_b) = \delta_{a+b,r-2}$. Witten's r-spin CohFT :

$$c_W^r(a_1, \ldots, a_n) = \Omega_{g,n}(e_{a_1}, \ldots, e_{a_n}),$$

of degree $D^r_{g,n} := \frac{(r-2)(g-1)+\sum_{i=1}^n a_i}{r}$, with $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$.

For $[\Sigma, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n}$, $\exists\, \mathcal{T}$ line bundle over Σ such that

$$\mathcal{T}^{\otimes r} \cong \omega_\Sigma\Big(-\sum_{i=1}^n a_i x_i\Big), \text{ with } [\Sigma, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n},$$

with ω_Σ the canonical bundle. Every r-th root of this fiber (r-spin structure) $\rightsquigarrow$ point in $\overline{\mathcal{M}}^{1/r}_{g,n}(a_1, \ldots, a_n)$:

$$\pi\colon \overline{\mathcal{M}}^{1/r}_{g,n}(a_1, \ldots, a_n) \to \overline{\mathcal{M}}_{g,n}.$$

- **Genus** $0 \rightsquigarrow$ Witten. For $[\Sigma, x_1, \ldots, x_n, \mathcal{T}] \in \overline{\mathcal{M}}^{1/r}_{0,n}(a_1, \ldots, a_n)$, $U = H^1(\Sigma, \mathcal{T}) \rightsquigarrow$ vector bundle $\mathcal{U} \to \overline{\mathcal{M}}^{1/r}_{0,n}(a_1, \ldots, a_n)$ (U has constant dimension, since $H^0(\Sigma, \mathcal{T}) = 0$).

$$c_W^r(a_1, \ldots, a_n) := \pi_* e(\mathcal{U}^*) \in H^{2D^r_{0,n}}(\overline{\mathcal{M}}_{0,n}).$$

- **For** $g > 0$, existence non-trivial and construction complicated (Polishchuk–Vaintrob '04, Chiodo '06, Fan–Jarvis–Ruan '13...).