Resolution of Liouville CFT: Segal axioms and bootstrap

Colin Guillarmou

CNRS and Univ. Paris Saclay

January 20, 2022

Classical mechanics in dimension 2

- M space of dimension 2 (surface) : points x
- $M \times \mathbb{R}^2$ phase space : points x and velocity $\xi \in \mathbb{R}^2$ (in fact T^*M)
- Observables : $a \in C^\infty(M \times \mathbb{R}^2)$. ex: the Hamiltonian h=kinetic + potential energy

$$h(x,\xi) = |\xi|^2 + V(x)$$

- Dynamics: Hamilton equation for $x(t) = (x_1(t), x_2(t))$, $\xi(t) = (\xi_1(t), \xi_2(t))$

$$\boxed{\dot{x}_j(t) = (\partial_{\xi_j} h)(x(t), \xi(t)), \quad \dot{\xi}_j(t) = -(\partial_{x_j} h)(x(t), \xi(t)), \qquad (x(0), \xi(0)) = (x_0, \xi_0)}$$

- Gives a flow on $(M \times \mathbb{R}^2)$ (Hamiltonian flow of a vector field X_h)

$$\boxed{(x(t), \xi(t)) = e^{tX_h}(x_0, \xi_0)}$$

Quantum mechanics (in dim 2)

- $M \times \mathbb{R}^2$ phase space : points x and velocity $\xi \in \mathbb{R}^2$
- Hilbert space $\mathcal{H} = L^2(M)$: points become probability density $f \in L^2(M)$
- Observables become linear operator :

$$a \in C^\infty(M \times \mathbb{R}^2) \Longrightarrow A = \mathrm{Op}(a) : \mathcal{H} \to \mathcal{H}$$

for example the energy (classical Hamiltonian)

$$h(x,\xi) = |\xi|^2 + V(x) \Longrightarrow H := \mathrm{Op}(|\xi|^2 + V(x)) : f \mapsto (-\partial_x^2 + V(x))f$$

- Dynamics: Schrödinger equation, for initial data $f_0 \in \mathcal{H}$

$$\boxed{i\partial_t f(t,x) = (Hf)(t,x), \qquad f(0,x) = f_0(x)}$$

- Quantum evolution flow

$$\boxed{f_t = e^{itH} f_0}$$

Quantum field theory / gravity (in dim 2)

- M surface, considered as a space time, dim $= 1 + 1$. Example: $\Sigma :=$ a cylinder

$$\theta = \text{ space variable}, \quad t = \text{time variable}$$

- Space of fields $E(\Sigma)$: typically a Sobolev space $H^{-s}(\Sigma)$, $s > 0$. Singular functions (distributions). Points are replaced by fields Φ.
- Restriction of fields Φ to embedded circles (for example at fixed t) produces family of fields φ_t on $\mathbb{S}^1$ evolving

$$\varphi \in H^{-s}(\mathbb{S}^1) \iff \varphi = \sum_{n\in\mathbb{Z}} \varphi_n e^{in\theta}, \ \sum_n |\varphi_n|^2(1+|n|)^{-s} < \infty$$

- Probability density become functionals $F : H^{-s}(\mathbb{S}^1) \to \mathbb{R}$.
- Quantization: $\mathcal{H} = L^2(H^{-s}(\mathbb{S}^1), \mu)$, need a measure μ on $E(\mathbb{S}^1) = H^{-s}(\mathbb{S}^1)$ $\Longrightarrow$ real mathematical difficulty $\Longrightarrow$ probability !!

- Dynamics : for $F : E(\mathbb{S}^1) \to \mathbb{R}$ in $\mathcal{H}$,

$$U(t)F = e^{itH}F$$

for some operator called Hamiltonian $H : \mathcal{H} \to \mathcal{H}$

- probabilistic approach: if (Σ, g) is Riemannian (instead of Lorentzian), the dynamics= Markov process, a contraction semi-group on $\mathcal{H} = L^2(H^{-s}(\mathbb{S}^1, \mu))$

$$\boxed{U(t) = e^{-tH}F}$$

generating some Hamiltonian $H : \mathcal{H} \to \mathcal{H}$.

Liouville action

Liouville action on Riemannian surface (Σ, g) is

$$\boxed{S_\Sigma(\varphi, g) = \frac{1}{4\pi} \int_\Sigma (|d\Phi|_g^2 + QK_g\Phi + e^{\gamma\Phi}) \mathrm{dv}_g}$$

with $Q = 2/\gamma + \gamma/2$ and $\gamma \in (0, 2)$, $K_g = 2\times$ Gauss curvature of g

- Critical points of $S_\Sigma(g, \Phi)$ are related to finding Φ_0 s.t. $K_{e^{\gamma\Phi_0}g} =$ negative constant.

Liouville field theory

Correlation and partition functions:

Partition fct: the mass of the formal measure $e^{-S_\Sigma(\Phi,g)}D\Phi$ on space of fields $E(\Sigma)$

$$\langle 1\rangle_{\Sigma,g} := \int_{E(\Sigma)} e^{-\frac{1}{4\pi}\int_\Sigma(|d\Phi|_g^2+QK_g\Phi+e^{\gamma\Phi})\mathrm{dv}_g} D\Phi$$

physics def / formal

$$\boxed{\langle 1\rangle_{\Sigma,g} \overset{def}{=} \frac{\sqrt{\mathrm{Vol}(\Sigma)}}{\sqrt{\det'(\Delta_g)}} \int_{\mathbb{R}} \mathbb{E}\Big[\prod_{j=1}^{n} e^{-\frac{1}{4\pi}\int_\Sigma(QK_g(c+X_g)+e^{\gamma(c+X_g)})\mathrm{dv}_g}\Big] dc}$$

math def

X_g = Gaussian Free Field on Σ with covariance Green's function G_g for Laplacian Δ_g.

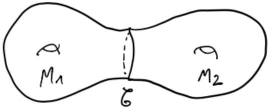
Correlation fct: $x_1, \dots, x_n \in \Sigma$ some points, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ some weights

$$\langle \prod_{j=1}^{n} V_{\alpha_j}(x_j) \rangle_{\Sigma,g} = \int_{E(\Sigma)} e^{\alpha_1 \Phi(x_1)} \dots e^{\alpha_n \Phi(x_n)} e^{-S_\Sigma(\Phi,g)} D\Phi \qquad \text{physics def / formal}$$

and the math definition

$$\langle \prod_{j=1}^{n} V_{\alpha_j}(x_j) \rangle_{\Sigma,g} \overset{def}{=} \frac{\sqrt{\mathrm{Vol}(\Sigma)}}{\sqrt{\det'(\Delta_g)}} \int_{\mathbb{R}} \mathbb{E}\Big[\prod_{j=1}^{n} e^{\alpha_j (c + X_g(x_i))} e^{-\frac{1}{4\pi} \int_\Sigma (Q K_g (c + X_g) + e^{\gamma(c+X_g)}) \mathrm{dv}_g} \Big] dc$$

Segal axioms (physics heuristics)



Desintegration of path integral using conditionning on $\mathcal{C} = \partial\Sigma_1 = \partial\Sigma_2$: if

$$S_\Sigma(\Phi, g) = S_{\Sigma_1}(\Phi|_{\Sigma_1}, g) + S_{\Sigma_2}(\Phi|_{\Sigma_2}, g)$$

one should have

$$\int_{E(\Sigma)} e^{-S_\Sigma(\Phi,g)} D\Phi = \int_{E(\mathcal{C})} \Big(\int_{\substack{E(\Sigma_1),\\ \Phi|_{\mathcal{C}}=\varphi}} e^{-S_{\Sigma_1}(\Phi|_{M_1},g)} D\Phi \Big) \Big(\int_{\substack{E(\Sigma_2),\\ \Phi|_{\mathcal{C}}=\varphi}} e^{-S_{\Sigma_2}(\Phi|_{\Sigma_2},g)} D\Phi \Big) D\varphi$$

$$= \int_{E(\mathcal{C})} \mathcal{A}_{\Sigma_1}(\varphi) \mathcal{A}_{\Sigma_2}(\varphi) D\varphi$$

A_{Σ_j} is called amplitude of Σ_j.

Segal axioms

A Conformal Field Theory is

- Object: $\mathcal{H}$ a Hilbert space attached to $\mathbb{S}^1$ (for us: $\mathcal{H} = L^2(H^{-s}(\mathbb{S}^1), \mu)$)
- Morphism: to each Riemannian surface (Σ, g) with parametrized boundary $\partial\Sigma = \sqcup_{i=1}^b \mathcal{C}_i$, we associate an amplitude

$$\mathcal{A}_{\Sigma,g} \in L^2(H^{-s}(\mathbb{S}^1)^b) = \otimes^b \mathcal{H}$$

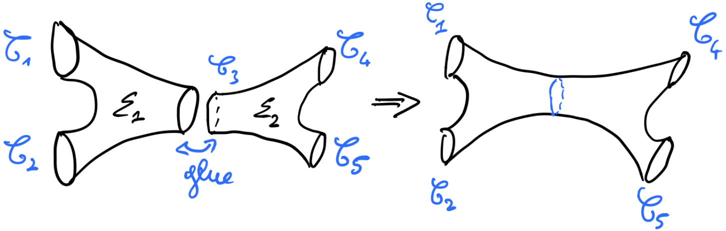
- Conformal invariance: for $\omega \in C^\infty(\Sigma)$ with $\omega = 0$ on $\partial\Sigma$

$$\boxed{\mathcal{A}_{\Sigma,e^\omega g}(\varphi) = e^{\frac{c}{96\pi}\int_\Sigma |d\omega|_g^2 + 2K_g\omega} \mathcal{A}_{\Sigma,g}(\varphi)}$$

- Gluing rules: if we glue (Σ_1, g_1) with (Σ_2, g_2) by identifying $\mathcal{C}_{j_1} \sim \mathcal{C}_{j_2}$ $(\partial\Sigma_1 = \sqcup_{j=1}^{b_1}\mathcal{C}_j$, and $\partial\Sigma_2 = \sqcup_{j=b_1+1}^{b_1+b_2}\mathcal{C}_j)$, for $(\Sigma, g) := (\Sigma_1 \sharp \Sigma_2, g_1 \sharp g_2)$

$$\boxed{\mathcal{A}_{\Sigma,g} = \mathcal{A}_{\Sigma_1,g_1} \circ_{j_1 \to j_2} \mathcal{A}_{\Sigma_2,g_2}}$$

integrate out the j_1 component of A_{Σ_1,g_1} against the j_2 component of A_{Σ_2,g_2}



$$\mathcal{A}_{\Sigma,g}(\varphi_1\,\varphi_2, \varphi_4, \varphi_5) = \int_{H^{-s}(\mathbb{S}^1)} A_{\Sigma_1,g}(\varphi_1, \varphi_2, \varphi_3) A_{\Sigma_2,g}(\varphi_3, \varphi_4, \varphi_5) d\mu(\varphi_3)$$

Hilbert space of Liouville CFT

Hilbert space: if $\Omega := (\mathbb{R}^2)^{\mathbb{N}^*}$ and $\mathbb{P} = \prod_{n\geq 1} \frac{1}{2\pi} e^{-\frac{1}{2}(x_n^2+y_n^2)} dx_n dy_n$,

$$\boxed{\mathcal{H} := L^2(\mathbb{R}_c \times \Omega, dc \otimes \mathbb{P}) = L^2(H^{-\varepsilon}(\mathbb{S}^1), d\mu)}$$

where μ is pushfoward of $dc \otimes \mathbb{P}$ by the real random field

$$(*) \qquad \varphi = c + \sum_{n\neq 0} \varphi_n e^{in\theta}, \quad \varphi_n = \frac{1}{2}\frac{x_n + iy_n}{\sqrt{n}}, \; n > 0$$

If b disjoint circles, $\mathcal{H}^{\otimes b} = L^2(H^{-\varepsilon}(\mathbb{S}^1)^b, d\mu^b)$, take b independent fields $(\varphi^1, \dots, \varphi^b)$ distributed as in $(*)$

Definition of amplitudes and conditional expectations

Let (Σ, g) with b parametrized boundary circles and n weighted marked points (x_i, α_i):

$$\mathcal{A}_{\Sigma,g,x,\alpha}(\varphi) = \int_{\substack{E(\Sigma),\\ \Phi|_{\mathcal{C}}=\varphi}} \prod_{j=1}^{n} e^{\alpha_j \Phi(x_j)} e^{-S_\Sigma(\Phi,g)} D\Phi \qquad \text{physics def / formal}$$

and the rigorous probabilistic definition

$$\boxed{\mathcal{A}_{\Sigma,g,x,\alpha}(\varphi) \stackrel{def}{=} \mathbb{E}\Big[\prod_{i=1}^{n} e^{\alpha_i (X_D(x_i)+P\varphi(x_i))} e^{-\frac{1}{4\pi}\int_\Sigma (QK_g(X_D+P\varphi)+e^{\gamma(X_D+P\varphi)})\mathrm{dv}_g}\Big] \mathcal{A}^0_{\Sigma,g}(\varphi)},$$

- $\Phi = X_D + P\varphi$ with $\varphi = (\varphi^1, \dots, \varphi^b) \in H^{-\varepsilon}(\mathbb{S}^1)^b$,
- X_D = GFF with Dirichlet condition, $\mathbb{E}$ = expectation wrt X_D,
- $P\varphi$ = harmonic extension of φ on Σ
- $\mathcal{A}^0_{\Sigma,g}(\varphi) = e^{-\frac{1}{2}\langle (\mathbf{D}_\Sigma - \mathbf{D})\varphi, \varphi\rangle}$ half-density term ($\mathbf{D}_\Sigma$=Dirichlet to Neumann map on Σ, $\mathbf{D} = \sqrt{\Delta_{\mathbb{S}^1}}$).

Segal Axioms are satisfied for Liouville CFT

Theorem (G-Kupiainen-Rhodes-Vargas '21)

1) Let (Σ, g) be Riemannian surface with b parametrized boundary circles, marked points $x = (x_1, \dots, x_m)$ with weight $\alpha = (\alpha_1, \dots, \alpha_m)$. Then if $\sum_i \alpha_i + Q\chi(\Sigma) > 0$

$$\mathcal{A}_{\Sigma,g,x,\alpha} \in L^2(H^{-\varepsilon}(\mathbb{S}^1)^b, d\mu^b) = \mathcal{H}^{\otimes b}.$$

2) The amplitudes satisfy conformal invariance required in Segal axioms.
3) The amplitudes satisfy gluing properties required in Segal axioms.

The propagator and the Hamiltonian

For the flat annulus $\mathbb{A}_t = (\{z \in \mathbb{C} \,|\, e^{-t} \leq |z| \leq 1\}, g = \frac{|dz|^2}{|z|^2})$, define the amplitude as above

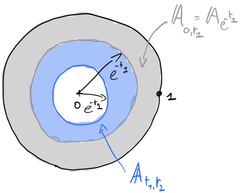
$$A_{\mathbb{A}_t}(\varphi, \varphi') := \mathbb{E}\Big[e^{-\frac{1}{4\pi}\int_{\mathbb{A}_t} e^{\gamma(X_D + P(\varphi,\varphi'))}\mathrm{dv}_g}\Big] e^{-\frac{1}{2}\langle(\mathbf{D}_{\mathbb{A}_t} - \mathbf{D})(\varphi,\varphi'),(\varphi,\varphi')\rangle}$$

where $\mathbf{D}_{\mathbb{A}_t}$ =Dirichlet-to-Neumann of $\mathbb{A}_t$ and $\mathbf{D} = |\partial_\theta|$ (note: $\mathbf{D}_{\mathbb{A}_t} - \mathbf{D}$ is smoothing).

Define the associated operator $S(t) : \mathcal{H} \to \mathcal{H}$:

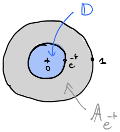
$$\forall \varphi \in H^{-\varepsilon}(\mathbb{S}^1), \quad \boxed{(S(t)F)(\varphi) := \int_{H^{-\varepsilon}(\mathbb{S}^1)} A_{\mathbb{A}_t}(\varphi, \varphi')F(\varphi')d\mu(\varphi')}$$

idea 1: gluing two annuli produces bigger annuli $\Longrightarrow$ $S(t)$ should be a semi-group.



with $\mathbb{A}_{t_1,t_2} = \{|z| \in [e^{-t_2}, e^{-t_1}]\}$

idea 2: gluing annulus $\mathbb{A}_t$ with a disk $\mathbb{D}$ produces a bigger disk $\Longrightarrow$ $S(t)A_{\mathbb{D},0,\alpha} = e^{\lambda t}A_{\mathbb{D},0,\alpha}$.



Proposition (G-Kupiainen-Rhodes-Vargas '20)

The operator $e^{-(\frac{1+6Q^2}{12})t}S(t) = e^{-t\mathbf{H}}$ *is a Markov process, a contraction semi-group on* $\mathcal{H} = L^2(\mathbb{R} \times \Omega; dc \otimes \mathbb{P})$ *with self-adjoint generator*

$$\boxed{\mathbf{H} = \frac{1}{2}(-\partial_c^2 + Q^2 + 2\mathbf{P} + e^{\gamma c}V) =: \mathbf{H}_0 + \frac{1}{2}e^{\gamma c}V}$$

with $\mathbf{P}$ *the infinite harmonic oscillator and* $V \in L^{\frac{2}{\gamma^2}-}(\Omega)$ *a positive potential/measure:*

$$\mathbf{P} := \sum_{n=1}^{\infty} n[(\partial_{x_n})^*\partial_{x_n} + (\partial_{y_n})^*\partial_{y_n}], \quad V(\tilde{\varphi}) := \frac{1}{2\pi}\int_{\mathbb{S}} e^{\gamma\tilde{\varphi}(\theta)}d\theta$$

where $\tilde{\varphi} = \varphi - \frac{1}{2\pi}\int_{\mathbb{S}^1}\varphi(\theta)d\theta = \varphi - c$.

Tool: Feynmann-Kac representation of $e^{-t\mathbf{H}} \Longrightarrow$ Vincent's talk.

Spectral resolution for the free field Hamiltonian $\mathbf{H}_0$

Fact 1: $\mathbf{H}_0 = -\partial_c^2 + Q^2 + \mathbf{P}$ has continuous spectrum $[Q^2, \infty)$, eigenfunctions are

$$e^{ipc}\psi_{\mathbf{kl}}, \qquad \mathbf{H}_0(e^{ipc}\psi_{\mathbf{kl}}) = (p^2 + Q^2 + |\mathbf{k}| + |\mathbf{l}|)e^{ipc}\psi_{\mathbf{kl}}$$

with

$$\psi_{\mathbf{kl}} = \prod_n h_{k_n}(x_n)h_{l_n}(y_n), \quad \mathbf{P}\psi_{\mathbf{kl}} = (|\mathbf{k}| + |\mathbf{l}|)\psi_{\mathbf{kl}}$$

indexed by $\mathbf{k} = (k_1, \dots, k_n, 0, \dots), \mathbf{l} = (l_1, \dots, l_{n'}, 0, \dots) \in \mathbb{N}^{\mathbb{N}}$ finite sequences, $h_k(x)$ Hermite polynomial and $|\mathbf{k}| = \sum_n nk_n \in \mathbb{N}$.

Fact 2: Plancherel formula: for $u_1, u_2 \in \mathcal{H} = L^2(\mathbb{R} \times \Omega)$

$$\boxed{\langle u_1, u_2\rangle_{\mathcal{H}} = \sum_{\mathbf{k},\mathbf{l}\in\mathcal{N}} \int_{\mathbb{R}} \langle u_1, e^{ipc}\psi_{\mathbf{kl}}\rangle_{\mathcal{H}} \langle e^{ipc}\psi_{\mathbf{kl}}, u_2\rangle_{\mathcal{H}}\, dp}$$

Fact 3: $p \mapsto e^{ipc}\psi_{\mathbf{kl}}$ extends analytically to $\mathbb{C}$, in particular for $ip = \alpha \in \mathbb{R}^-$

Diagonalization of $\mathbf{H}$ using scattering theory:

Theorem (G-Kupiainen-Rhodes-Vargas '20)

Let $\gamma \in (0,2)$, $Q = 2/\gamma + \gamma/2$. Then

- *$\exists$ a complete family of eigenstates $\Phi_{Q+ip,\mathbf{k},\mathbf{l}} \in e^{-\varepsilon c} L^2(\mathbb{R}_c \times \Omega)$ labeled by $p \in \mathbb{R}_+$ and $\mathbf{k}, \mathbf{l} \in \mathbb{N}^{\mathbb{N}}$ s.t.*

$$\mathbf{H}\Phi_{Q+ip,\mathbf{k},\mathbf{l}} = \Big(\frac{Q^2}{2} + \frac{p^2}{2} + |\mathbf{k}| + |\mathbf{l}|\Big)\Phi_{Q+ip,\mathbf{k},\mathbf{l}}.$$

- *$\Phi_{Q+ip,\mathbf{k},\mathbf{l}}$ is a complete family diagonalizing $\mathbf{H}$: $\forall u_1, u_2 \in L^2(\mathbb{R} \times \Omega)$*

$$\boxed{\langle u_1, u_2\rangle_{L^2} = \frac{1}{2\pi}\sum_{\mathbf{k},\mathbf{l}\in\mathcal{N}} \int_0^\infty \langle u_1, \Phi_{Q+ip,\mathbf{k},\mathbf{l}}\rangle_{L^2} \langle \Phi_{Q+ip,\mathbf{k},\mathbf{l}}, u_2\rangle_{L^2}\, dp}$$

Link with the amplitude of the disk

Proposition (G-Kupiainen-Rhodes-Vargas'20)

1) The (probabilistic) amplitude of the unit disk $(\mathbb{D}, |dz|^2)$ *with* 1 *marked point at* $x = 0$*, weight* $\alpha < Q$

$$\Phi_\alpha(\varphi) := A_{\mathbb{D},\alpha}(\varphi) = \mathbb{E}\Big[e^{\alpha(X_D+P\varphi)(0)}e^{-\frac{1}{4\pi}\int_{\mathbb{D}} e^{\gamma(X_D+P\varphi)}\mathrm{dv}_{\mathbb{D}}}\Big] \in e^{(\alpha-Q-\varepsilon)c}L^2(\mathbb{R}\times\Omega)$$

is an eigenfunction of H*:*

$$\boxed{H\Phi_\alpha = \alpha(Q-\frac{\alpha}{2})\Phi_\alpha = 2\Delta_\alpha\Phi_\alpha}$$

2) The map $\alpha \mapsto \Phi_\alpha$ *extends analytically to* $\mathrm{Re}(\alpha) \le Q$ *and* $\Phi_{Q+ip,0,0} = \Phi_{Q+ip}$
3) Same for $\alpha \mapsto \Phi_{\alpha,\mathbf{k},\mathbf{l}}$

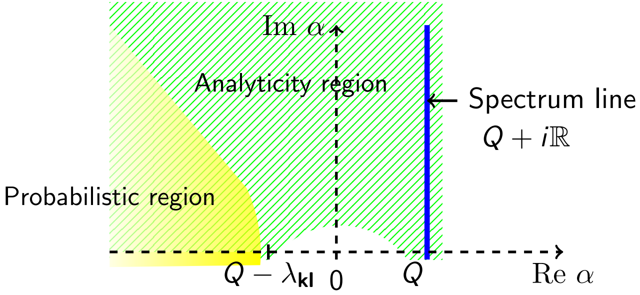


Figure: Analytic continuation of eigenstates $\Psi_{\alpha,\mathbf{k},\mathbf{l}}$ and probabilistic region, $\lambda_{\mathbf{kl}} := |\mathbf{k}| + |\mathbf{l}|$.

In probabilistic region, intertwining (scattering):

$$\Phi_{\alpha,\mathbf{k},\mathbf{l}} = \lim_{t\to\infty} e^{t(\Delta_\alpha+|\mathbf{k}|+|\mathbf{l}|)} e^{-t\mathbf{H}} (\underbrace{e^{(\alpha-Q)c}\psi_{\mathbf{kl}}}_{\mathbf{H}_0 \text{ eigenst}}).$$

A different basis related to Virasoro algebra

- For two Young diagrams $\nu = (\nu_1 \geq \cdots \geq \nu_k)$ and $\tilde{\nu} = (\tilde{\nu}_1 \geq \cdots \geq \tilde{\nu}_{\tilde{k}})$, $(\nu_j \in \mathbb{N})$ there is a canonical eigenfunction of $\mathbf{H}$

$$\Psi_{Q+ip,\nu,\tilde{\nu}} \in \mathrm{Span}\{\Phi_{Q+ip,\mathbf{k},\mathbf{l}} \,|\, |\mathbf{k}| + |\mathbf{l}| = |\nu| + |\tilde{\nu}|\}$$

  obtained from Virasoro algebra, more adapted to the problem
- $(\Psi_{Q+ip,\nu,\tilde{\nu}})_{p,\nu,\tilde{\nu}}$ is a basis but not orthonormal

$$\boxed{\langle u_1, u_2\rangle = \sum_{|\nu'|=|\nu|} \sum_{|\tilde{\nu}'|=|\tilde{\nu}|} \int_0^\infty \langle u_1, \Psi_{Q+ip,\nu,\tilde{\nu}}\rangle \langle \Psi_{Q+ip,\nu',\tilde{\nu}'}, u_2\rangle \mathcal{Q}^{-1}_{Q+ip}(\nu,\nu') \mathcal{Q}^{-1}_{Q+ip}(\tilde{\nu},\tilde{\nu}')\, dp}$$

$\mathcal{Q}_{Q+ip}(\nu,\tilde{\nu})$ are Gram matrices of change of basis.

Conformal Bootstrap for general surfaces

Theorem (G-Kupiainen-Rhodes-Vargas 21': modular bootstrap)

For a closed Riemannian surface (Σ, g) with n marked points $x = (x_1, \dots, x_n) \in \Sigma^n$ and weights $\alpha = (\alpha_1, \dots, \alpha_n) \in (0, Q)^n$, then

$$\langle V_{\alpha_1}(x_1) \dots V_{\alpha_n}(x_n) \rangle_{\Sigma, g} = C_g \int_{\mathbb{R}_+^{3h-3+n}} \rho(p, \alpha) |\mathcal{F}_{p,\alpha}(q)|^2 dp$$

- $\rho(p, \alpha)$ *is a product of 3-point correlations functions on* $\mathbb{S}^2$
- $q \mapsto \mathcal{F}_{p,\alpha}(q) =$ *conformal blocks are holomophic in* $q = (q_1, \dots, q_{3h-3+n})$*, the plumbing (complex) coordinates on the moduli space* $\mathcal{M}_{h,n}$ *of Riemann surfaces,* $h = \mathrm{genus}(\Sigma)$.
- $C_g > 0$ *an explicit constant depending on* g.

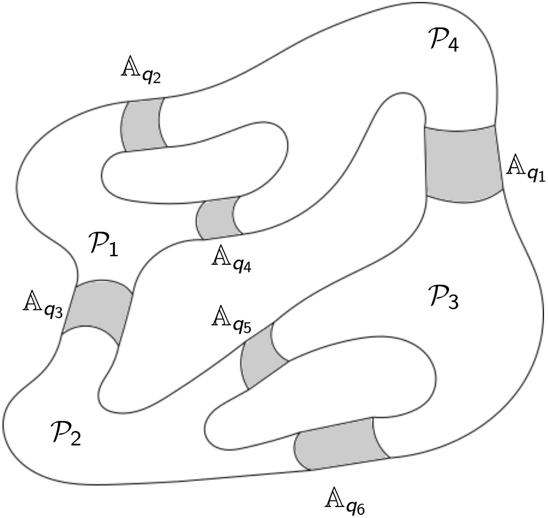


Figure: The plumbed surfaces $\Sigma_{\mathbf{q}}$ with four pairs of pants $\mathcal{P}_1, \ldots, \mathcal{P}_4$ and six annuli $\mathbb{A}_{q_1}, \ldots, \mathbb{A}_{q_6}$

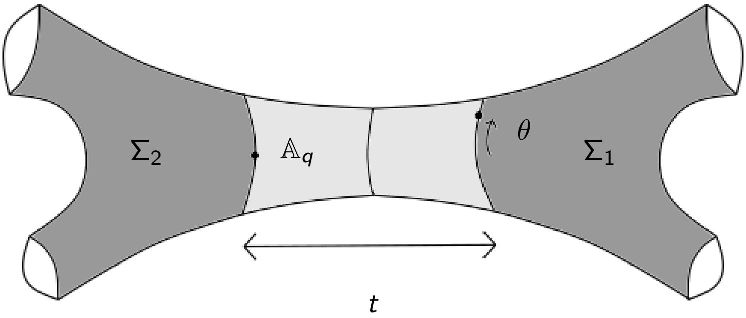
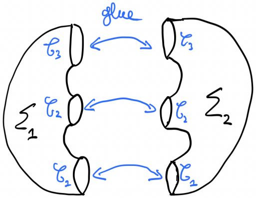


Figure: The plumbing with parameter $q = e^{-t+i\theta}$ of two pairs of pants, viewed as gluing an annulus $\mathbb{A}_q = \{z \in \mathbb{D} \,|\, |q| \leq |z| \leq 1\}$ with a twist of angle θ between the two pairs of pants. The length for the flat metric $|dz|^2/|z|^2$ of the annulus is t.

In terms of amplitudes: composition with $e^{-tH+i\Pi}$ where Π is generator of rotations $z \mapsto e^{i\theta}z$.

Idea of proof : genus 2

(Forget the Young diagrams to simplify): assume Φ_{Q+ip} basis of eigenfunctions of $\mathbf{H}$



1) Use gluing rule (Segal axiom)

$$\langle 1\rangle_{\Sigma,g} = \int_{H^{-s}(\mathbb{S}^1)^3} A_{\Sigma_1,g}(\varphi_1,\varphi_2,\varphi_3)A_{\Sigma_2,g}(\varphi_1,\varphi_2,\varphi_3)d\mu^3(\varphi_1,\varphi_2,\varphi_3)$$
$$=\langle A_{\Sigma_1,g}, A_{\Sigma_2,g}\rangle_{\mathcal{H}^{\otimes 3}}$$

2) Use spectral decomposition (Plancherel formula)

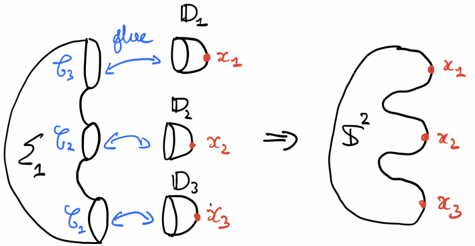
$$\langle A_{\Sigma_1,g}, A_{\Sigma_2,g}\rangle_{\mathcal{H}^{\otimes 3}} = \int_{\mathbb{R}^3_+} \langle A_{\Sigma_1,g}, \otimes_{j=1}^3 \Phi_{Q+ip_j}\rangle_{\mathcal{H}^{\otimes 3}} \langle \otimes_{j=1}^3 \Phi_{Q+ip_j}, A_{\Sigma_1,g}\rangle_{\mathcal{H}^{\otimes 3}} dp_1 dp_2 dp_3$$

3) Analytic continuation in $\alpha_j = Q + ip_j$ to come back to $\alpha < Q$ real of

$$W(\alpha_1,\alpha_2,\alpha_3) := \int_{(\mathbb{R}^+)^3} \langle A_{\Sigma_1,g}, \otimes_{j=1}^3 \Phi_{\alpha_j}\rangle_{\mathcal{H}^{\otimes 3}}$$

4) Gluing rule: $\Phi_{\alpha_j} = A_{\mathbb{D}_{0,\alpha_j}}$ thus

$$W(\alpha_1,\alpha_2,\alpha_3) = \langle V_{\alpha_1}(x_1) V_{\alpha_2}(x_2) V_{\alpha_3}(x_3)\rangle_{\mathbb{S}^2,g} = C^{\mathrm{DOZZ}}(\alpha_1,\alpha_2,\alpha_3)$$



Conclusion: using analytic continuation again in α_j

$$\langle 1\rangle_{\Sigma,g} = \int_{(\mathbb{R}^+)^3} C^{\mathrm{DOZZ}}(Q+ip_1, Q+ip_2, Q+ip_3)C^{\mathrm{DOZZ}}(Q-ip_1, Q-ip_2, Q-ip_3)dp_1dp_2dp_3$$

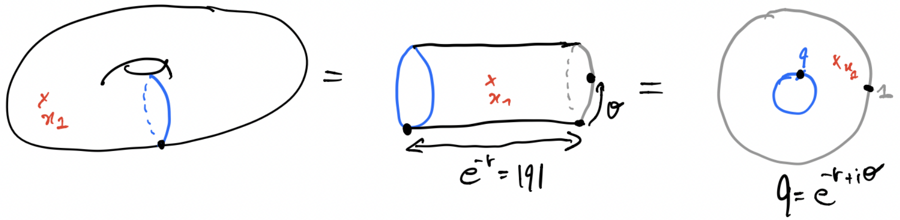
But if take into account the Young diagram part of spectral resolution: use Ward identities and obtain conformal block

$$\langle 1\rangle_{\Sigma,g} = \int_{(\mathbb{R}^+)^3} C^{\mathrm{DOZZ}}(Q+ip_1, Q+ip_2, Q+ip_3)C^{\mathrm{DOZZ}}(Q-ip_1, Q-ip_2, Q-ip_3)|\mathcal{F}_p|^2dp$$

Change of moduli of surface: glue annuli of moduli $q=(q_1,q_2,q_3)\in\mathbb{D}^3$ between Σ_1 and Σ_2, this only enters the conformal block

$$\langle 1\rangle_{\Sigma_q,g_q} = \int_{(\mathbb{R}^+)^3} C^{\mathrm{DOZZ}}(Q+ip_1, Q+ip_2, Q+ip_3)C^{\mathrm{DOZZ}}(Q-ip_1, Q-ip_2, Q-ip_3)|\mathcal{F}_p(q)|^2dp$$

Another example: torus 1 point



1-point function on torus $\mathbb{T}^2_\tau = \mathbb{C}/(2\pi\mathbb{Z} + 2\pi\tau\mathbb{Z})$, with $q = e^{2i\pi\tau}$

$$\langle V_{\alpha_1}(x_1)\rangle_{\mathbb{T}^2_\tau} = \frac{|q|^{-\frac{1+6Q^2}{12}}}{2\pi} \int_0^\infty C(Q+ip, \alpha_1, Q-ip)|q|^{-2\Delta_{Q+ip}}|\mathcal{F}_{p,\alpha_1}(q)|^2 dp$$

Remarks:

- first mathematical proof of the explicit expressions proposed by physicists (Knizhnik, Belavin, Sonoda, Polchinski, Teschner ...).
- the bootstrap formula depends on the chosen decomposition into pairs of pants, annuli with 1 marked point/insertion and disks with 1 or 2 marked points/insertions
- proves crossing symmetries: formulas for correlations functions given by bootstrap approach do not depend on the decomposition into geometric blocks (although conformal blocks do)
- implies convergence a.e. $P \in \mathbb{R}$ of conformal block series (this was an open problem)

$$\mathcal{F}_{P,\alpha}(q) = \sum_{k \in \mathbb{N}_0^{3h-3+n}} w_k(\alpha, p) q_1^{k_1} \dots q_{3h-3+n}^{k_{3h-3+n}}$$

for $q = (q_1, \dots, q_{3h-3+n}) \in \mathbb{D}^{3h-3+n}$ Marden-Kra plumbing coordinates; here $w_k(\alpha, p)$ are representation theoretic constants depending only on Virasoro commutation relations.