On β-ensembles for large β and high dimensions

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Joint work with Sergio Andraus and Michael Voit

- A β-Hermite (Gaussian) ensemble is a set of random variables with the joint density

$$c_\beta e^{\frac{-||x||^2}{2}} \prod_{1 \leq i < j \leq N} (x_j - x_i)^\beta$$

on the weyl chamber $C_N^A = \{x \in \mathbb{R}^N | x_1 \leq ... \leq x_N\}$.
- A β-Laguerre (Wishart) ensemble is a set of random variables with the joint density

$$c_{\beta,\nu} \prod_{1 \leq i < j \leq N} (x_j - x_i)^\beta \prod_{j=1}^{N} x_j^{\beta\nu} e^{-\frac{x_j}{2}}$$

on the weyl chamber $C_N^B = \{x \in \mathbb{R}^N | 0 \leq x_1 \leq ... \leq x_N\}$ with some parameter $\nu > 0$.

- A β-Jacobi (Manova) ensemble is a set of random variables with the joint density

$$c_{\beta,a,b} \prod_{1\le i<j\le N} (x_j - x_i)^{\beta} \prod_{j=1}^{N} (1-x_j)^{\beta a}(1+x_j)^{\beta b}$$

on the alcove $A = \{x \in \mathbb{R}^N | -1 \le x_1 \le ... \le x_N \le 1\}$ with some parameters $a, b > 0$.

Theorem (Dumitriu, Edelmann(2005),Voit (2018))

Let Y_β be a $\mathbb{R}^N$ dimensional random variable with the density

$$\tilde{c_\beta} e^{\frac{-||x||^2}{2}} \prod_{i<j} (x_j - x_i)^\beta$$

on $C_N^A = \{x \in \mathbb{R}^N | x_1 \leq ... \leq x_N\}$ and $z_N^H = (z_{1,N}^H, ..., z_{N,N}^H)$ vector consisting of the ordered zeros of the N-th Hermite Polynomial. Then

$$\sqrt{\beta}\left(\frac{Y_\beta}{\sqrt{\beta}} - z_N^H\right) = Y_\beta - \sqrt{\beta} z^H \to \mathcal{N}(0, \Sigma_N^H)$$

with $(\Sigma_N^H)^{-1} = (s_{i,j}^H)_{i,j=1,...,N}$ given by

$$s_{i,j}^H = \begin{cases} 1 + \sum_{l=1, l\neq j}^N (z_{j,N}^H - z_{l,N}^H)^{-2} & \text{for } i = j \\ -(z_{i,N}^H - z_{j,N}^H)^{-2} & \text{for } i \neq j \end{cases}.$$

Theorem (Andraus, Voit (2019))

Let $z_N^H = (z_{1,N}^H, ..., z_{N,N}^H)$ *be vector consisting of the ordered zeros of the N-th Hermite Polynomial. For the Matrix* $(\Sigma_H^N)^{-1} = (s_{i,j}^H)_{i,j=1,...,N}$ *with*

$$s_{i,j}^H = \begin{cases} 1 + \sum_{l=1, l\neq j}^N (z_{j,N}^H - z_{l,N}^H)^{-2} & \text{for } i = j \\ -(z_{i,N}^H - z_{j,N}^H)^{-2} & \text{for } i \neq j \end{cases}.$$

the eigenvalues are $\lambda_k^H = k$*. Each* λ_k^H *has an eigenvector of the form*

$$v_k^H = \left(Q_{k-1,N}^H(z_{1,N}^H), ..., Q_{k-1,N}^H(z_{N,N}^H) \right)$$

where $\left\{ Q_{k-1,N}^H \right\}_{k=0}^{N-1}$ *is the system of orthonormal Polynomials w.r.t the measure*

$$\mu_N^H = \frac{1}{N} \left(\delta_{z_{1,N}^H} + ... + \delta_{z_{N,N}^H} \right)$$

Theorem (H., Voit (2019))

Let $Y_{\beta,a,b}$ *be a* $\mathbb{R}^N$ *dimensional random variable with the density*

$$c_{\beta,a,b} \prod_{1\le i<j\le N} (x_j - x_i)^{\beta} \prod_{j=1}^{N} (1-x_j)^{\beta a}(1+x_j)^{\beta b}$$

on $A = \{x \in \mathbb{R}^N | -1 \le x_1 \le ... \le x_N \le 1\}$ *and* $z_N^J = (z_{1,N}^J, ..., z_{N,N}^J)$ *vector consisting of the ordered zeros of the N-th Jacobi Polynomial* $P_N^{(2a-1,2b-1)}$. *Then*

$$\sqrt{\beta}(Y_{\beta,a,b} - z_N^J) \to \mathcal{N}(0, \Sigma_N^J)$$

for $\beta \to \infty$ *in distribution with* $(\Sigma_N^J)^{-1} = (s_{i,j}^J)_{i,j=1,...,N}$ *given by*

$$s_{i,j}^J = \begin{cases} \sum_{l=1,l\ne j}^{N} \frac{1}{(z_{j,N}^J - z_{l,N}^J)^2} + a\frac{1}{(1-z_{j,N}^J)^2} + b\frac{1}{(1+z_{j,N}^J)^2} & \text{for } i = j \\ \frac{-1}{(z_{i,N}^J - z_{j,N}^J)^2} & \text{for } i \ne j \end{cases}$$

Problem: For the Matrix $(\Sigma_N^J)^{-1}$ the eigenvalues and eigenvector seem not be easy to get. Therefore we studied the ensemble in a trigonometric form. This means the pushforward measure $\tilde{Y}_{\beta,a,b} = T(Y_{\beta,a,b})$ under the Transformation

$$T : A \to \tilde{A}, \quad T(x_1, ..., x_N) = \frac{1}{2}(\arccos(x_1), ..., \arccos(x_N))$$

from A to $\tilde{A} = \{t \in \mathbb{R}^N | \frac{\pi}{2} \geq t_1 \geq ... \geq t_N \geq 0\}$. What we get is a transformed CLT

$$\sqrt{\beta}(\tilde{Y}_{\beta,a,b} - \tilde{z}_N^J) \to \mathcal{N}(0, \tilde{\Sigma}_N^J)$$

with $\tilde{z}_N^J = \frac{1}{2}\left(\arccos(z_{1,N}^J), ..., \arccos(z_{N,N}^J)\right)$ and a transformed Matrix $\tilde{\Sigma}_N^J$.

Theorem (H., Voit (2019))

Let $z_N^J = (z_{1,N}^J, ..., z_{N,N}^J)$ *be vector consisting of the ordered zeros of the N-th Jacobi Polynomial* $P_N^{(2a-1,2b-1)}$. *Then the Matrix* $(\tilde{\Sigma}_N^J)^{-1}$ *has the eigenvalues* $\lambda_k^J = 2k(2N+2a+b-1-k)$ *and corresponding eigenvectors are given by*

$$v_k^J = \left(\sqrt{1-(z_{1,N}^J)^2}\,Q_{k-1}^{(a,b)}(z_{1,N}^J), ..., \sqrt{1-(z_{N,N}^J)^2}\,Q_{k-1}^{(a,b)}(z_{N,N}^J)\right)$$

where $\left\{Q_{k-1}^{(a,b)}\right\}_{k=0}^{N-1}$ *is the system of orthonormal Polynomials w.r.t the measure*

$$\mu_{N,a,b}^J = \left(1-(z_{1,N}^J)^2\right)\delta_{z_{1,N}^J} + ... + \left(1-(z_{N,N}^J)^2\right)\delta_{z_{N,N}^J}$$

The monic polynomials $\{\hat{Q}_{k,N}\}_{k=0}^{N-1}$ which are orthogonal w.r.t to the measure

$$\mu_N^H = \frac{1}{N}\left(\delta_{z_{1,N}^H} + \dots + \delta_{z_{N,N}^H}\right)$$

satisfy the three term recurrence

$$\hat{Q}_{0,N} = 1,\ \hat{Q}_{1,N}(x) = x,\ \hat{Q}_{k+1,N}(x) = x\hat{Q}_{k,N}(x) - \left(\frac{N-k}{2}\right)\hat{Q}_{k-1,N}(x) \tag{1}$$

for $k = 1, \dots, N-2$.
While the monic Hermite Polynomials $\{\hat{H}_k\}_{k=0}^{N-1}$ satisfiy

$$\hat{H}_0 = 1,\ \hat{H}_1(x) = x,\ \hat{H}_{k+1}(x) = x\hat{H}_k(x) - \frac{k}{2}\hat{H}_{k-1}(x)$$

for $k = 1, \dots, N-2$.
This type of duality due to de Boor and Saff (1986) appears in all three cases.

Let $(\hat{P}_n)_{n=0}^{\infty}$ be a sequence of monic orthogonal polynomials where the corresponding sequence of orthonormal polynomials $(\tilde{P}_n)_{n=0}^{\infty}$ satisfies the three term recurrence relation

$$x\tilde{P}_n(x) = b_{n+1}\tilde{P}_{n+1}(x) + a_n\tilde{P}_n(x) + b_n\tilde{P}_{n-1}(x) \quad (n \geq 1)$$

and denote by J_k and J^k the upper left principal submatrix resp. lower right principal submatrix of

$$\begin{pmatrix} a_0 & b_1 & & & \\ b_1 & a_1 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & a_{N-2} & b_{N-1} \\ & & & b_{N-1} & a_{N-1} \end{pmatrix},$$

then $\hat{P}_k(x) = \det(xId_k - J^k)$ and $\hat{Q}_k(x) = \det(xId_k - J_k)$.

The Gauss quadrature implies that the finite set of polynomials $(\hat{P}_k)_{k=0}^{N-1}$ obeys the discrete orthogonality relation

$$\sum_{i=1}^{N} w_i \hat{P}_k(z_{i,N}) \hat{P}_l(z_{i,N}) = \xi_k \delta_{kl},$$

with the Christoffel numbers w_i and the N ordered zeroes $z_{1,N} < \ldots < z_{N,N}$ of $\hat{P}_N$.

Theorem (de Boor, Saff (1986), Vinet, Zhedanov, 2004)

The dual poylnomals also obey an discrete orthogonality relation

$$\sum_{i=1}^{N} w_i^* \hat{Q}_{k,N}(z_{i,N}) \hat{Q}_{l,N}(z_{i,N}) = \xi_k^* \delta_{kl} \quad (k, l = 0, \ldots, N-1)$$

for some dual weights w_i^*.
These dual weights are particular easy in the three cases:
$\mu_N^H = \frac{1}{N}\left(\delta_{z_1^H} + \ldots + \delta_{z_N^H}\right)$: $w_i^* = \frac{1}{N}$, $\mu_N^L = \left(z_1^L \delta_{z_1^L} + \ldots + z_N^L \delta_{z_N^L}\right)$: $w_i^* = \frac{z_i^L}{\kappa_N}$, $\mu_{N,a,b}^J = (1 - z_1^2)\delta_{z_1} + \ldots + (1 - z_N^2)\delta_{z_N}$: $w_i^* = \frac{1 - z_i^2}{\kappa_N}$

From the diagonalisation of $\Sigma_H^{-1}, \Sigma_L^{-1}, \Sigma_J^{-1}$ we derived formuals for $\Sigma_H, \Sigma_L, \Sigma_J$ in terms of dual polynomials and in terms of Hermite/Laguerre/Jacobi polynomials. In the Hermite and Laguerre case they are much simpler compared to those from the tridiagonal matrix model.

Theorem (Andraus,H.,Voit (2021))

$$[\Sigma_N^H]_{i,j} = \sqrt{w_i^H w_j^H}(-1)^{i+j} \sum_{k=0}^{N-1} \frac{\tilde{H}_k(z_{i,N}^H)\tilde{H}_k(z_{j,N}^H)}{N-k}$$

$$[\Sigma_N^L]_{i,j} = \sqrt{w_i^L w_j^L}(-1)^{i+j} \sum_{k=0}^{N-1} \frac{\tilde{L}_k(z_{i,N}^L)\tilde{L}_k(z_{j,N}^L)}{2N-2k}$$

$$[\tilde{\Sigma}_N^J]_{i,j} = \sqrt{w_i^J w_j^J}(-1)^{i+j} \sum_{k=0}^{N-1} \frac{\tilde{P}_k^{2a-1,2b-1}(z_{i,N}^J)\tilde{P}_k^{2a-1,2b-1}(z_{j,N}^J)}{\lambda_{N-k}^J}$$

Theorem (Andraus,H.,Voit(2021), Gorin,Kleptsyn)

If we denote $\Sigma_N^H =: (\sigma_{i,j})_{i,j=1,\dots,N}$ *and with* Ai *beeing the Airy function with* Ai *and* a_r *its* r*-th largest zero we have*

$$\lim_{N\to\infty} N^{\frac{1}{3}}\sigma_{N-r+1,N-r+1} = \int_0^\infty \frac{\mathrm{Ai}(x+a_r)^2}{(\mathrm{Ai}'(a_r))^2 x}dx = 0,834...$$

And if we consider $(X_1^\beta,\dots,X_N^\beta)^T = X_\beta$ *to be a* β*-Hermite ensemble and assuming that* G *is a random variable with the* $G \sim \mathcal{N}(0,\rho_r)$, *where*

$$\rho_r^2 := \int_0^\infty \frac{\mathrm{Ai}(x+a_r)^2}{(\mathrm{Ai}'(a_r))^2 x}dx$$

the statement can be rewritten in terms of distributional convergence as

$$\lim_{N\to\infty}\lim_{\beta\to\infty} N^{\frac{2}{3}}\sqrt{2\beta}\left(\frac{X_{N-r+1}^\beta}{\sqrt{2N\beta}} - z_{N-r+1,N-r+1}^H\right) = G$$

Thank you for your attention.