

Twisted Ruijsenaars model

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## Plan :

- ① Ruijsenaars model
- ② Main result. Example
- ③ 1st approach:  
From BA functions
- ④ 2nd approach:  
Hamiltonian reduction
- ⑤ 3rd approach:  
via DAHA
- ⑥ 4th approach: via  
elliptic Cherednik  
operators

C. - Etingof '13

C. - Fairon '17

Braverman - Etingof -  
Finkelberg '16

C.  
(under preparation)

① Ruijsenaars model (Trigonometric case)

$$H_k = \sum_{\substack{I \subset \{1, \dots, n\} \\ |I|=k}} \left( \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \right) T_I$$

Ruijsenaars '87

also known as  
Macdonald operators

$$T_i : x_i \mapsto qx_i$$

$$T_I = \prod_{i \in I} T_i$$

$$H_1, H_2, \dots, H_n$$

$$[H_i, H_j] = 0$$

quantum  
integrable  
system

$$H_1 = \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_i$$

## ② Main result

For  $\ell \geq 2$ ,  $\exists$  a completely integrable model with commuting hamiltonians  $\tilde{H}_1, \dots, \tilde{H}_n$  such that

$$\tilde{H}_k \approx (H_k)^\ell$$

$\tilde{H}_k$  are quite complicated

First implicit construction given in

C.: Appendix in C.-Etingof '13

Example ( C.-Fairon '17)

$$\ell=2 : \quad \tilde{H}_k \approx (H_k)^2$$

$$\tilde{H}_1 = \sum_i a_i T_i^2 + \sum_{i < j} b_{ij} T_i T_j$$

$$a_i = \prod_{j \neq i} \frac{(tx_i - x_j)(qt x_i - x_j)}{(x_i - x_j)(qx_i - x_j)}$$

$$b_{ij} = \prod_{k \neq i, j} \frac{(tx_i - x_k)(tx_j - x_k)}{(x_i - x_k)(x_j - x_k)}$$

$$\times q^{\frac{k}{2}}(t-1)(t-q) \frac{(x_i + x_j)(x_i x_j)^{\frac{k}{2}}}{(x_i - qx_j)(x_j - qx_i)}$$

$$\leftrightarrow -2qt(x_i^2 + x_j^2) + ((q+t)(qt+1) + (q-t)^2)x_i x_j$$

$\tilde{H}_1$  twisted,  $\ell=2 \leftrightarrow (H_1)^2$  non-twisted

### ③ 1st approach: BA functions

Change notation:  $x_i \rightsquigarrow q^{x_i}$

key fact: For  $t = q^{-m}$ ,  $m \in \mathbb{Z}_+$

$\exists$  common eigenfunction of  $H_1, \dots, H_n$  of the form

$$\psi(\lambda, x) = q^{\langle \lambda, x \rangle} P(q^\lambda, q^x)$$

$$\langle \lambda, x \rangle = \sum_{i=1}^n \lambda_i x_i$$

$$q^\lambda = (q^{\lambda_1}, \dots, q^{\lambda_n})$$

$$q^x = (q^{x_1}, \dots, q^{x_n})$$

$\psi$  is called BA function

C. '02

True for all root systems

non-symmetric

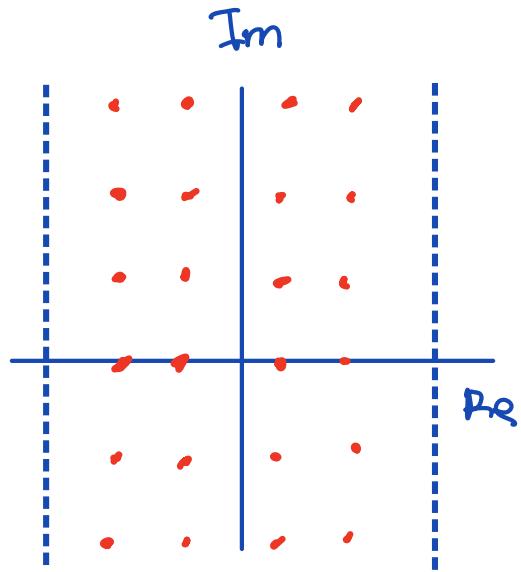
Cherednik - Macdonald - Mehta  
identity for BA function

$$\psi(\lambda, x) \quad \lambda, x \in \mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$$

THM (CE13): For large  $\xi \in \mathbb{R}^n$ ,

$$\begin{aligned} & \int_{\xi + i\mathbb{R}^n} \frac{\psi(\lambda, x) \psi(\mu, x)}{\Delta(x) \Delta(-x)} q^{-\frac{\langle x, x \rangle}{2}} dx \\ &= q^{\frac{\langle \lambda, \lambda \rangle + \langle \mu, \mu \rangle}{2}} \psi(\lambda, \mu). \end{aligned}$$

$n=1$  case



$$\Delta(x) = \prod_{i < j} \prod_{k=1}^m (q^{-k} q^{x_i} - q^{x_j})$$

$\xi$  is large if  $|\xi_i - \xi_{i+1}| > m$

$\xi + i\mathbb{R}$

## Twisted BA function $\tilde{\psi}$

Given  $\ell \in \mathbb{N}$ , define  $\tilde{\psi}(\lambda, x)$  by

$$\int_{\xi + i\mathbb{R}^n} \frac{\psi(\lambda, x) \psi(\mu, x)}{\Delta(x) \Delta(-x)} q^{-\frac{\ell \langle x, x \rangle}{2}} dx \\ = q^{\frac{\langle \lambda, \lambda \rangle + \langle \mu, \mu \rangle}{2\ell}} \tilde{\psi}(\lambda, \mu)$$

Then (C'13):

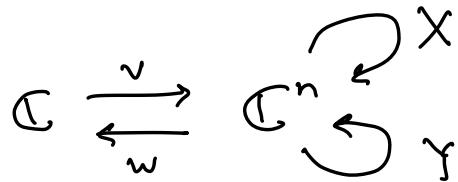
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$$\bullet \tilde{\psi}(\lambda, x) \in q^{\frac{\langle \lambda, x \rangle}{\ell}} \underbrace{\tilde{P}(q^{\frac{\lambda}{\ell}}, q^{\frac{x}{\ell}})}_{\text{polynomial}}$$

- $$\bullet \tilde{\psi}$$
- is a joint eigenfunction
- 
- of
- $\tilde{H}_i \approx (H_i)^{\ell} \quad i=1\dots n$

④ 2nd approach: Hamiltonian reduction

(a) Ruijsenaars-Schneider model



$$\mathcal{M} = \{(X, Y, v, w)\} \cong \mathbb{C}^{2n^2+2n}$$

$$\cong T^* \mathbb{C}^{n^2+n}$$

$(\mathcal{M}, \omega, G, \mu)$       Hamiltonian  
G-space

$\text{GL}_n$       moment map

[KKS'78]: Rational CM system  
by Hamiltonian reduction on cl.

$$X, Y \in \text{Mat}_{n,n}(\mathbb{C})$$

$$v \in \text{Mat}_{1,n}$$

$$w \in \text{Mat}_{n,1}$$

$$\mu = XY - YX + vw$$

$$C_n = \mu^{-1}(I_n) // G$$

$$\dim C_n = 2n$$

Fock-Rosly '99: To get RS system,  
one needs to choose different  
Poisson structure on  $\mathcal{M}$ .

Alternatively, use quasi-Poisson  
picture:

$$(\mathcal{M}, \Pi, G, \Phi)$$

quasi-Poisson  
bivector  $\Pi$   
 $\uparrow$   
 $G \subset \mathrm{GL}_n$

$$\Phi : \mathcal{M} \rightarrow G \quad \text{multiplicative moment map}$$

$$\Phi = xyx^{-1}y^{-1}(I_n + vw)$$

$$C_{n,t} = \Phi^{-1}(tI_n) // G$$

Poisson variety of  $\dim = 2n$

This is a particular example of a multiplicative quiver variety

Crawley-Boevey, Shaw '05  
Van den Bergh '08

$$h_i = \text{tr } Y^i \quad i=1 \dots n$$

These are  $G$ -invariant so well-defined on  $C_{n,t}$

$$\{h_i, h_j\} = 0$$

To see connection to RS system,  
assume  $X$  is diagonalisable.

Then

$$X = \text{diag}(x_1, \dots, x_n)$$

$$Y = (Y_{ij}), \quad v=\dots, w=\dots$$

$$Y_{ij} = \frac{t-1}{t - x_i/x_j} \prod_{k \neq j} \frac{tx_j - x_k}{x_j - x_k} \delta_{ij}$$

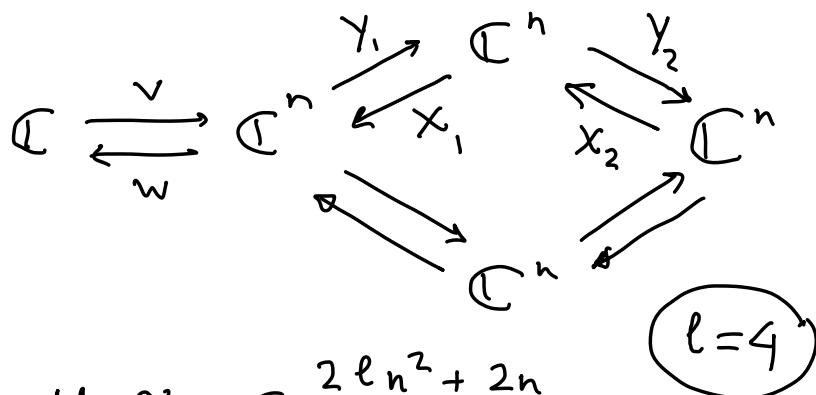
$$h_i = \text{tr } Y = \sum_i \prod_{k \neq i} \frac{tx_i - x_k}{x_i - x_k} \delta_{ii}$$

$$\{x_i, x_j\} = \{\epsilon_i, \epsilon_j\} = 0$$

$$\{x_i, \delta_j\} = \delta_{ij} x_i \delta_j$$

Ruijsenaars-Schneider '86  
 $\gamma$  = Lax matrix for RS

(b) Cyclic quiver generalisation  
 (C.-Faron '17)



$$\mathcal{M} \cong \mathbb{C}^{2\ell n^2 + 2n}$$

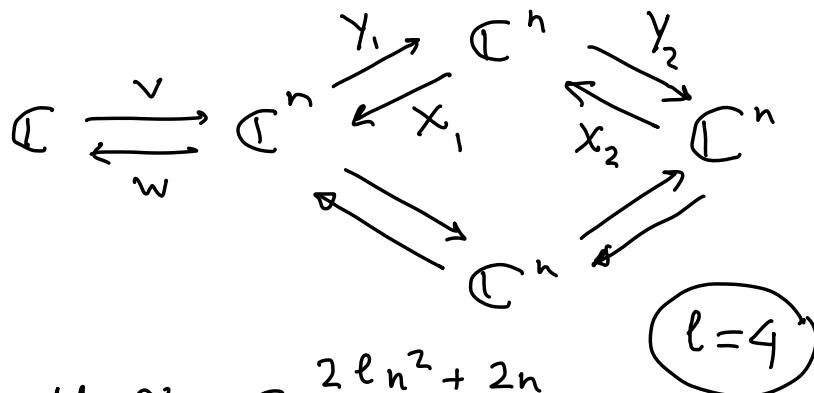
$G = (GL_n)^\ell$  acts on  $\mathcal{M}$

$(\mathcal{M}, \Pi, \Phi)$  quasi-Ham.  
 G-space

$\Phi: \mathcal{M} \rightarrow G$

mult. moment map

(b) Cyclic quiver generalisation  
 (C.-Faron '17)



$G = (GL_n)^{\ell}$  acts on  $M$

$(M, \Pi, \Phi)$  quasi-Ham.  
 $G$ -space

$\Phi: M \rightarrow G$   
 mult. moment map

$$\Phi = (\Phi_1, \dots, \Phi_{\ell})$$

$$\Phi_1 = (y_1 x_1)^{-1} x_1 y_1 (I_n + vw)$$

$$\Phi_2 = (y_1 x_1)^{-1} x_2 y_2$$

...

$$\Phi_{\ell} = (y_{\ell-1} x_{\ell-1})^{-1} x_{\ell} y_{\ell}$$

$$C_{n,t}^{(\ell)} = \Phi^{-1}(t_1 I_n, \dots, t_{\ell} I_n) // G$$

$t_1, \dots, t_{\ell} \neq 0$  arbitrary

$$t := t_1 \dots t_{\ell}$$

$\dim C_{n,t}^{(\ell)} = 2n$  smooth  
 symplectic

$$\tilde{h}_k = \text{tr} (Y_l \dots Y_1)^k \quad k=1 \dots n$$

$$\{ \tilde{h}_i, \tilde{h}_j \} = 0 \Rightarrow \text{IS}$$

Next, modulo G-action,

$$X_1 = \dots = X_{l-1} = I_n, \quad X_l = X$$

$$Y_r = (t_1 \dots t_r) Y \quad r=1 \dots l-1$$

$$Y_l = t X^{-1} Y$$

where  $X, Y$  are the same  
as for RS model.

$$X = \text{diag}(x_1, \dots, x_n)$$

$$Y = (Y_{ij})$$

$$Y_{ij} = \frac{t-1}{t - x_i/x_j} \prod_{k \neq j} \frac{tx_j - x_k}{x_j - x_k} \delta_j$$

$$\tilde{h}_k = \text{tr} (Y_{\ell} \dots Y_1)^k \quad k=1 \dots n$$

$$\{\tilde{h}_i, \tilde{h}_j\} = 0 \Rightarrow \text{IS}$$

Next, modulo G-action,

$$X_1 = \dots = X_{\ell-1} = \mathbb{1}_n, \quad X_\ell = X$$

$$Y_r = (t_1 \dots t_r) Y \quad r=1 \dots \ell-1$$

$$Y_\ell = t X^{-1} Y$$

where  $X, Y$  are the same as for RS model.

$$X = \text{diag}(x_1, \dots, x_n)$$

$$Y = (Y_{ij})$$

$$Y_{ij} = \frac{t-1}{t - x_i/x_j} \prod_{k \neq j} \frac{tx_j - x_k}{x_j - x_k} \delta_j$$

$$\tilde{h}_k = \text{tr} (Y_{\ell} \dots Y_1)^k \simeq \text{tr} (X^{-1} Y^{\ell})^k$$

up to constant factor

Thus, on the same space

$C_n, t$  we have RS system

with  $h_k = \text{tr } Y^k$ , and

a twisted RS system

with  $\tilde{h}_k = \text{tr} (X^{-1} Y^{\ell})^k$ .

Remark : On  $C_{n,t}$

$$\tilde{H}_k = \text{tr}(X^{-1}Y^l)^k \longleftrightarrow \tilde{G}_k := \text{tr}(X^{-l}Y)^k$$

under Ruijsenaars duality

complicated

simple

Same is true at quantum level.

E.g.  $\tilde{G}_i = \sum_i x_i^{-l} \prod_{j \neq i} \frac{t\alpha_i - \alpha_j}{\alpha_i - \alpha_j} T_i$

(cf. Di Francesco - Kedem '17 )

⑤ 3rd approach : via DAHA

(Braverman - Etingof - Finkelberg )  
2016

This allows to connect 1st and 2nd  
approaches

$H_{q,t}$  DAHA (  $GL_n$ -type )

It contains :

- $\mathbb{C} [x_1^{\pm 1}, \dots, x_n^{\pm 1}]$
- $\mathbb{C} [y_1^{\pm 1}, \dots, y_n^{\pm 1}]$
- Hecke algebra of  $S_n$

$$\left\{ \begin{array}{l} T_i \quad i=1 \dots n-1 \\ (T_i - t^{y_2})(T_i + t^{-y_2}) = 0 \\ \text{+ braid relations} \end{array} \right.$$

(plus relations between  $X_i, Y_i, T_i$ )

## ⑤ 3rd approach : via DATA

(Braverman - Etingof - Finkelberg )  
2016

This allows to connect 1st and 2nd approaches

$H_{q,t}$  DATA (  $GL_n$ -type )

It contains :

- $\mathbb{C} [x_1^{\pm 1}, \dots, x_n^{\pm 1}]$
- $\mathbb{C} [y_1^{\pm 1}, \dots, y_n^{\pm 1}]$
- Hecke algebra of  $S_n$

$$\begin{cases} T_i & i=1 \dots n-1 \\ (T_i - t^{y_2})(T_i + t^{-y_2}) = 0 \\ \text{+ braid relations} \end{cases}$$

(plus relations between  $x_i, y_i, T_i$ )

$H_{q,t}$  has a representation by difference-reflection operators in vars  $X_1, \dots, X_n$ :

$$X_i \mapsto X_i$$

$$T_i \mapsto t^{y_2} S_i + \frac{t^{y_2} - t^{-y_2}}{X_i / X_{i+1} - 1} (S_i - 1)$$

$$\text{where } S_i = \sigma_{i,i+1}$$

$$Y_i \mapsto T_i \dots T_n \pi T_1^{-1} \dots T_{i-1}^{-1}$$

where

$$\pi : (X_1, \dots, X_n) \mapsto (X_2, \dots, X_n, q X_1)$$

In this representation, symmetric comb. of  $y_i$  after restriction onto  $\mathbb{C} [x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}$  give Ruijsenaars hamiltonians.  
(aka Macdonald operators)

BEF'16 define cyclotomic DATA

$$\mathbb{H}_{q,t}^l \subset \mathbb{H}_{q,t} \text{ (skipped)}$$

Part of their results  
(relevant to this story):

1. Define  $D_i^{(l)}$  ( $i = 1, \dots, n$ ) by

$$D_i^{(l)} = T_{i-1}^{-1} \dots T_1^{-1} \underbrace{x_1^{-1} y_1^l T_1^{-1} \dots T_{i-1}^{-1}}_{y_1^l}$$

Then these elements commute.

2. Symmetric combinations of  $D_1^{(l)}, \dots, D_n^{(l)}$  after restriction

$$\text{onto } \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}$$

give quantum hamiltonians  
 $\tilde{H}_1, \dots, \tilde{H}_n$  from 1st approach.

3. Classical limits of  $\tilde{H}_i$   
coincide with  $\tilde{h}_i$  from  
cyclic quiver story.

⑥ 4th approach: via elliptic  $\mathbb{Y}'$ 's

( C. : in preparation )

$$H_i = \sum_i \prod_{j \neq i} \sigma_\mu(x_i - x_j) e^{\beta \frac{\partial}{\partial x_i}}$$

$$\sigma_\mu(z) = \frac{\Theta(z - \mu) \Theta'(0)}{\Theta(z) \Theta(-\mu)}$$

$\Theta = \Theta_1(z|\tau)$  odd  $\Theta$ -function

$$\text{Ruijsenaars '87 : } [H_i, H_j] = 0$$

$H_K$  can be constructed via  
elliptic Cherednik operators

( Komori-Hikami '98 )

Trig. limit:  $\tau \rightarrow +i\infty$

$$t = e^\mu \quad q = e^\beta$$

$$R_{ij} = \zeta_\mu (\alpha_i - \alpha_j) - \zeta_{\tilde{\beta}_i - \tilde{\beta}_j} (\alpha_i - \alpha_j) S_{ij}$$

Shibukawa - Ueno '92

$$R_{ij} R_{ik} R_{jk} = R_{jk} R_{ik} R_{ij}$$

$$\zeta = (\zeta_1, \dots, \zeta_n)$$

$$Y_1 = R_{12} R_{13} \dots R_{1n} e^{\beta \partial / \partial x_1}$$

[K-H'98]: Let  $\rho = \frac{1}{2} \sum_{i < j} e_i - e_j$ .

Then: for  $\zeta = -\mu \rho$  we have

$$Y_1 : \mathbb{C}(x)^{S_n} \rightarrow \mathbb{C}(x)^{S_n} \text{ and}$$

$$Y_1 |_{\mathbb{C}(x)^{S_n}} = H_1.$$

$\zeta$  "spectral" variables  
("dynamical")

Similar construction

works for other root systems,  
see Komori-Hikami'98

Higher  $H_K$  constructed similarly.

$\gamma_1 = \gamma_1(\xi)$  as above.

Given  $l$ , define

$$\tilde{\gamma}_1 = \prod_{j=0 \dots l-1} \gamma_1\left(\xi + \frac{j\xi}{l}\right) e^{-2\pi i x_1}$$

THM: For  $\xi = -\mu p$ ,  $\tilde{\gamma}_1$

preserves  $(\mathbb{C}(x))^{S_n}$  and

$$\tilde{H}_1 := \tilde{\gamma}_1 \restriction_{\mathbb{C}(x)^{S_n}}$$

is integrable, i.e.

$$\exists \tilde{H}_k \quad k = \dots n,$$

$$[\tilde{H}_i, \tilde{H}_j] = 0.$$

Trig. limit:  $\tilde{\gamma}_1 \xrightarrow[n \rightarrow \infty]{} (\gamma_1)^l e^{-2\pi i x_1}$

THANK YOU !







