Modern Analysis Related to Root Systems with Applications
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Dispersive equations on noncompact symmetric spaces

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Notation

Symmetric space

- G noncompact semisimple Lie group (connected, finite center)
- K maximal compact subgroup of G

$\rightsquigarrow G/K$ Riemannian symmetric space of noncompact type

e.g., $\mathbb{H}^n(\mathbb{R}) = \mathrm{SO}(n,1)^\circ / \mathrm{SO}(n)$, $\mathrm{SPos}(n) = \mathrm{SL}(n,\mathbb{R}) / \mathrm{SO}(n)$

Weyl chamber

$\mathfrak{a}$ maximal connected, totally geodesic, flat sub-manifold of G/K

$$\ell = \dim \mathfrak{a} = \operatorname{rank} G/K$$

- Cartan decomposition: $G = K(\exp \overline{\mathfrak{a}^+})K$ (polar coordinates)
- Weyl chamber $\mathfrak{a} \cong \mathbb{R}^\ell$ generated by the roots

Spherical Fourier analysis

Fourier transform on $\mathbb{R}^d$

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}^d} \mathrm{d}x \, e^{-i\langle\lambda,x\rangle} \, f(x)$$

$$f(x) = \text{const.} \int_{\mathbb{R}^d} \mathrm{d}\lambda \, e^{i\langle\lambda,x\rangle} \, \mathcal{F}f(\lambda)$$

Harish-Chandra transform for bi-K-invariant data

$$\mathcal{H}f(\lambda) = \int_{G/K} \mathrm{d}x \, \varphi_{-\lambda}(x) \, f(x)$$

$$f(x) = \text{const.} \int_{\mathfrak{a}} \mathrm{d}\lambda \, |\mathbf{c}(\lambda)|^{-2} \, \varphi_\lambda(x) \, \mathcal{H}f(\lambda)$$

Fact: the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$ is not a differential symbol

Plancherel density

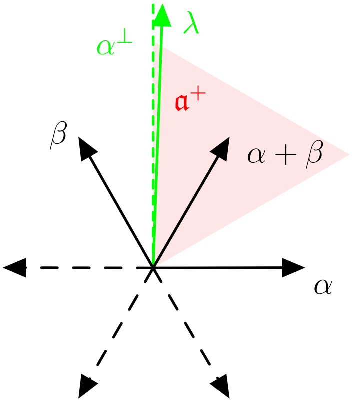


Figure: Root system A_2

When λ is closed to the wall $\alpha^\perp$

$\rightsquigarrow$ $|\langle \pm\alpha, \lambda \rangle|$ is very small

$|\langle \sigma, \lambda \rangle| \asymp |\lambda|, \forall\, \sigma \in \Sigma \smallsetminus \{\pm\alpha\}$

$\rightsquigarrow$ $\partial_{\pm\alpha} |\mathbf{c}(\lambda)|^{-2}$ has no enough decay

$\rightsquigarrow$ $|\mathbf{c}(\lambda)|^{-2}$ is not a symbol

i.e. $(\frac{\partial}{\partial\lambda})^k |\mathbf{c}(\lambda)|^{-2} = \mathrm{O}(|\lambda|^{d-\ell}) \quad |\lambda| \geq 1$

Plancherel density (continued)

General case: $|\mathbf{c}(\lambda)|^{-2}$ is not a symbol

$$|\mathbf{c}(\lambda)|^{-2} = \prod_{\alpha\in\Sigma_r^+} \left|c_\alpha\left(\tfrac{\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle}\right)\right|^{-2}$$

where

$$c_\alpha(\nu) \;=\; \text{const.}\; \frac{\Gamma(i\nu)}{\Gamma(i\nu+\frac{m_\alpha}{2})}\frac{\Gamma(\frac{i}{2}\nu+\frac{m_\alpha}{4})}{\Gamma(\frac{i}{2}\nu+\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2})}$$

and $|c_\alpha|^{-2}$ is a symbol on $\mathbb{R}$ of order $m_\alpha + m_{2\alpha}$ for every $\alpha \in \Sigma_r^+$

Special examples: $|\mathbf{c}(\lambda)|^{-2}$ is a differential symbol

- If $G/K = \mathbb{H}^d(\mathbb{R})$: $|\mathbf{c}(\lambda)|^{-2} = \text{const.} \left|\frac{\Gamma(i\lambda+\rho)}{\Gamma(i\lambda)}\right|^2$
- If G is complex: $|\mathbf{c}(\lambda)|^{-2} = \sum_{\alpha\in\Sigma_r^+}\langle\alpha,\lambda\rangle^2$

Motivation

- Δ Laplace-Beltrami operator on G/K

Schrödinger equation

$$i\partial_t u(t,x) + \Delta_x u(t,x) = 0 \qquad u(0,x) = u_0(x)$$

Homogeneous solution:

$$u(t,x) = e^{it\Delta_x} u_0(x) = u_0 * s_t(x)$$

Looking for the pointwise estimate of the Schrödinger kernel:

$$s_t(x) = \text{const.} \int_{\mathfrak{a}} \mathrm{d}\lambda \, \underbrace{|\mathbf{c}(\lambda)|^{-2}}_{\text{amplitude}} \, \underbrace{\varphi_\lambda(x)\, e^{-it(|\lambda|^2+|\rho|^2)}}_{\text{"phase part"}}$$

Recall: On $\mathbb{R}^d$

$$S_t(x) = (4\pi i t)^{-\frac{d}{2}} e^{i\frac{|x|^2}{t}}$$

A new tool

Barycentric decomposition of the Weyl chamber [Anker-Z. 2020]

There exist smooth cut-off functions $\chi_{w.S_j} \in \mathcal{C}_c^\infty(\mathfrak{a} \smallsetminus \{0\})$ such that for every $w \in W$ and $1 \le j \le \ell$, $\chi_{w.S_j}$ are homogeneous symbols in $\mathfrak{a}$ and satisfy

$$\sum_{w\in W} \sum_{1\le j\le \ell} \chi_{w.S_j} = 1 \qquad \text{on} \qquad \mathfrak{a} \smallsetminus \{0\}.$$

Moreover, a root $\alpha \in \Sigma$ satisfies either $\langle \alpha, w.\Lambda_j \rangle = 0$ or

$$|\langle \alpha, \lambda \rangle| \asymp |\lambda| \qquad \forall\, \lambda \in \operatorname{supp} \chi_{w.S_j}$$

where $w.\Lambda_j$ are the "well-chosen" directions.

Barycentric decomposition of A_2

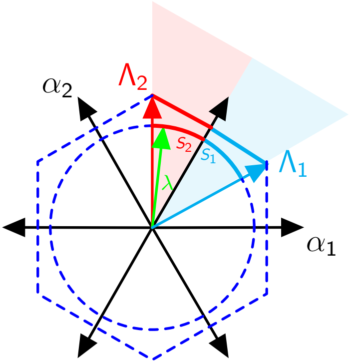


Figure: Root system A_2

$$\langle \Lambda_j, \alpha_k \rangle = \delta_{jk}, \quad \alpha_k \in \Sigma_s^+$$

If $\lambda \in \chi_{S_2}$, consider Λ_2

- $\langle \pm\alpha_1, \Lambda_2 \rangle = 0$

  $\rightsquigarrow \partial_{\Lambda_2} f(\langle \pm\alpha_1, \lambda \rangle) = 0$

- $|\langle \sigma, \lambda \rangle| \asymp |\lambda|$

  $\forall\, \sigma \in \Sigma \smallsetminus \{\pm\alpha_1\}, \forall\, \lambda \in \chi_{S_2}$

If $\lambda \in \chi_{S_1}$, consider Λ_1...

Barycentric decomposition of A_3

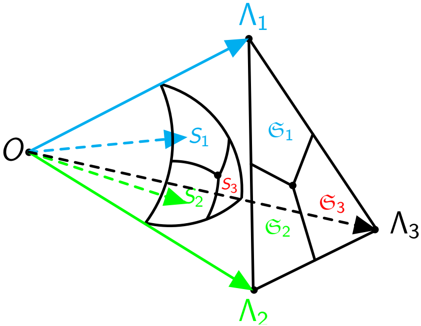


Figure: Root system A_3

Application

Dichotomy

A root $\alpha \in \Sigma$ satisfies either $\langle \alpha, w.\Lambda_j \rangle = 0$ or

$$|\langle \alpha, \lambda \rangle| \asymp |\lambda| \qquad \forall\, \lambda \in \operatorname{supp} \chi_{w.S_j}$$

Application to the Plancherel density

$$|\mathbf{c}(\lambda)|^{-2} = \prod_{\alpha \in \Sigma_r^+} \left|\mathbf{c}_\alpha\left(\tfrac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\right)\right|^{-2}$$

- $\langle \alpha, w.\Lambda_j \rangle = 0 \quad \rightsquigarrow \quad \partial^k_{w.\Lambda_j} |\mathbf{c}_\alpha(\tfrac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle})|^{-2} = 0$
- $\langle \alpha, w.\Lambda_j \rangle \neq 0 \quad \rightsquigarrow$

$$\left|\partial^k_{w.\Lambda_j} |\mathbf{c}_\alpha(\tfrac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle})|^{-2}\right| \lesssim |\langle \alpha, \lambda \rangle|^{m_\alpha + m_{2\alpha} - k} \asymp |\lambda|^{m_\alpha + m_{2\alpha} - k}$$

$$\Longrightarrow \qquad \partial^k_{w.\Lambda_j} |\mathbf{c}(\lambda)|^{-2} = O(|\lambda|^{d-\ell-k}) \qquad \forall\, \lambda \in \operatorname{supp} \chi_{w.S_j}$$

Pointwise kernel estimates

$$\widetilde{s}_t^{\infty}(x) = \int_{\mathfrak{a}\smallsetminus\{0\}} \mathrm{d}\lambda\, |\mathbf{c}(\lambda)|^{-2}\, \varphi_\lambda(x)\, e^{-it|\lambda|^2}$$

$$= \sum_{w\in W} \sum_{1\leq j\leq \ell} \int_{\mathfrak{a}\smallsetminus\{0\}} \mathrm{d}\lambda\, \underbrace{\chi_{w.S_j}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}}_{\text{"symbol"}}\, \underbrace{\varphi_\lambda(x)\, e^{-it|\lambda|^2}}_{\text{"phase part"}}$$

Stationary phase method, Hadamard parametrix, etc. $\rightsquigarrow$

Theorem (Anker-Meda-Pierfelice-Vallarino-Z. 2021)

There exist $C>0$ and $N>0$ such that the following estimates hold for all $t \in \mathbb{R}^*$ and $x \in G/K$:

$$|s_t(x)| \leq C\,(1+|x^+|)^N\, e^{-\langle \rho, x^+\rangle} \begin{cases} |t|^{-\frac{d}{2}} & \text{if } 0<|t|<1, \\ |t|^{-\frac{D}{2}} & \text{if } |t|\geq 1. \end{cases}$$

Dispersive estimates for Schrödinger operator

In the Euclidean setting $\mathbb{R}^d$

$$\|e^{it\Delta_{\mathbb{R}^d}}\|_{L^{q'}(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \lesssim |t|^{-(\frac{1}{2}-\frac{1}{q})\,d} \qquad \forall\, t \in \mathbb{R}^*$$

On real hyperbolic spaces $\mathbb{H}^d(\mathbb{R})$

$$\|e^{it\Delta_{\mathbb{H}^d}}\|_{L^{q'}(\mathbb{H}^d)\to L^q(\mathbb{H}^d)} \lesssim \begin{cases} |t|^{-(\frac{1}{2}-\frac{1}{q})\,d} & \text{if } 0<|t|<1 \\ |t|^{-\frac{3}{2}} & \text{if } |t|\geq 1 \end{cases}$$

(Anker-Pierfelice, Banica, Ionescu-Staffilani, etc... 2007-2009)

On G/K (AMPVZ. 2021)

$$\|e^{it\Delta}\|_{L^{q'}(G/K)\to L^q(G/K)} \lesssim \begin{cases} |t|^{-(\frac{1}{2}-\frac{1}{q})\,d} & \text{if } 0<|t|<1 \\ |t|^{-\frac{D}{2}} & \text{if } |t|\geq 1 \end{cases}$$

Strichartz inequality

Linear Cauchy problem

$$i\partial_t u(t,x) + \Delta_x u(t,x) = F(t,x), \qquad u(0,x) = u_0(x)$$

Duhamel's formula $\rightsquigarrow$

$$u(t,x) = e^{it\Delta_x}\, u_0(x) - i \int_0^t \mathrm{d}s\, e^{i(t-s)\Delta_x}\, F(s,x)$$

Space-time Strichartz norm:

$$\|u\|_{L^p(\mathbb{R};L^q(G/K))} := \left[\int_{\mathbb{R}} \mathrm{d}t \left(\int_{G/K} \mathrm{d}x\, |u(t,x)|^q\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}$$

(Global-in-time) Strichartz inequality

$$\|u\|_{L^p(\mathbb{R};L^q(G/K))} \lesssim \|u_0\|_{L^2(G/K)} + \|F\|_{L^{\tilde{p}'}(\mathbb{R};L^{\tilde{q}'}(G/K))}$$

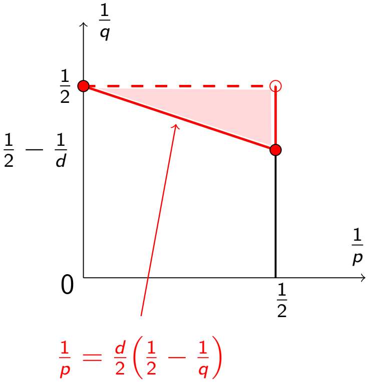
for all admissible pairs (p,q) and $(\tilde{p},\tilde{q})$

Admissible pairs for G/K

Definition

A pair (p, q) is admissible for G/K if $(\frac{1}{p}, \frac{1}{q})$ belongs to the triangle

$$\{(\tfrac{1}{p}, \tfrac{1}{q}) \in (0, \tfrac{1}{2}] \times (0, \tfrac{1}{2}) \mid \tfrac{2}{p} + \tfrac{d}{q} \geq \tfrac{d}{2}\} \cup \{(0, \tfrac{1}{2})\}$$



Semi-linear Schrödinger equation on G/K

Semi-linear Cauchy problem

$$\begin{cases} i\partial_t u(t,x) + \Delta_x u(t,x) = F(u(t,x)) \\ u(0,x) = u_0(x) \end{cases} \tag{NLS}$$

with power-like nonlinearities of order $\gamma > 1$:

$$|F(u)| \lesssim |u|^\gamma, \; |F(u) - F(v)| \lesssim (|u|^{\gamma-1} + |v|^{\gamma-1})\,|u - v|$$

Global well-posedness

- For all $1 < \gamma \leq 1 + \frac{4}{d}$, (NLS) is globally well posed for small L^2 initial data
- For all $1 < \gamma \leq 1 + \frac{4}{d-2}$, (NLS) is globally well posed for small H^1 initial data

Semi-linear Schrödinger equation on G/K

Scattering

- $1 < \gamma \leq 1 + \frac{4}{d}$, there exists $u_\pm \in L^2$ such that

$$\|u(t,.) - e^{it\Delta}u_\pm\|_{L^2(G/K)} \longrightarrow 0 \qquad \text{as } t \to \pm\infty$$

- $1 < \gamma \leq 1 + \frac{4}{d-2}$, there exists $u_\pm \in H^1$ such that

$$\|u(t,.) - e^{it\Delta}u_\pm\|_{H^1(G/K)} \longrightarrow 0 \qquad \text{as } t \to \pm\infty$$

On $\mathbb{R}^d$

Scattering in H^1 fails for $1 < \gamma \leq 1 + \frac{2}{d}$

Thank you! Merci! 谢谢！