Gaussian fluctuations for products of random matrices

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Addition and multiplication of Hermitian matrices

Consider real diagonal matrices

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_N \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & b_N \end{pmatrix}.$$

Choose $U, V \sim$ Haar measure on $U(N)$. Consider

$$C = UAU^* + VBV^* \qquad C = (UAU^*) \cdot (VBV^*)$$

Question: What do the eigenvalues of C look like as $N \to \infty$?

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- ... for a fixed number of matrices M? ($M = 2$ here)
- ... for $M \to \infty$?

Motivations

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This construction appears in several areas:

- $N \to \infty$, M fixed: free probability
- N fixed, $M \to \infty$: dynamical systems
- $N \to \infty$, $M \to \infty$: disordered systems and polymers, initializations of neural networks

Approach

For eigenvalue measure $d\mu(x)$ on $(x_1 \geq \cdots \geq x_N) \in \mathbb{R}^N$, study multivariate analogue of Laplace transform

$$\phi_\chi(s) := \mathbb{E}_\mu \left[\frac{\mathcal{B}(s; x)}{\mathcal{B}(\chi, x)} \right],$$

where $\mathcal{B}(s, x)$ is the **multivariable Bessel function.**

Limiting properties of $\phi_\chi(s)$	$\Longrightarrow$	Limiting properties of $d\mu$

I. The limit $N \to \infty$ with M fixed

II. The limit $N, M \to \infty$ jointly

III. Technique: Multivariate Bessel generating functions

Limit shapes as $N \to \infty$ with M fixed

Study **empirical spectral measures** of A, B, C:

$$\mu_A := \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{a_i} \quad \mu_B := \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{b_i} \quad \mu_C := \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{c_i}.$$

Theorem (Voiculescu, '80's)

As $N \to \infty$, if empirical spectral measures of A and B converge to deterministic μ_A and μ_B, same holds for C. Limit μ_C has:

$$\text{add: } R_{\mu_C}(z) = R_{\mu_A}(z) + R_{\mu_B}(z)$$
$$\text{mult: } S_{\mu_C}(z) = S_{\mu_A}(z) \cdot S_{\mu_B}(z),$$

where the R and S-transforms are defined by

$$R_\mu(z) = G_\mu^{-1}(z) - z^{-1} \qquad S_\mu(z) = \frac{z+1}{z} M_\mu^{-1}(z)$$

for $G_\mu(z) = \int (z-x)^{-1} \mu(dx)$ and $M_\mu(z) = \int \frac{xz}{1-xz} \mu(dx)$.

Fluctuations as $N \to \infty$ with M fixed

Study **height function** of C:

$$H_C(x) = \#\{c_i \le x\} = N\mu_C\Big((-\infty, x]\Big).$$

- Expectation $\mathbb{E}[H_C(x)]$ is determined by limit shape.
- Fluctuations $H_C(x) - \mathbb{E}[H_C(x)]$ are random functions on $\mathbb{R}$.

Theorem

As $N \to \infty$, the limit of fluctuations of the height function

$$\xi_C(x) := \lim_{N\to\infty}\Big(H_C(x) - \mathbb{E}[H_C(x)]\Big)$$

is an **explicit** Gaussian log-correlated field $\xi_C(x)$, meaning

$$\mathbb{E}[\xi_C(x)\xi_C(y)] \approx -\frac{1}{2\pi^2}\log|x-y| \qquad \text{for } x \approx y.$$

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Additive case:

- 2^{nd} order freeness: [Collins-Mingo-Śniady-Speicher '04]
- Stieltjes transform: [Pastur-Vasilchuk '07]

Multiplicative case:

- Gaussianity: [Guionnet-Novak '15] [Arizmendi-Mingo '18]
- Gaussianity + explicit covariance + log-correlation: [Gorin-S.]

I. The limit $N \to \infty$ with M fixed

II. The limit $N, M \to \infty$ jointly

III. Technique: Multivariate Bessel generating functions

The case $M \to \infty$: Matrix random walks

For i.i.d. matrices $U_i \sim$ Haar measure on $U(N)$, consider

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_N \end{pmatrix} \qquad A_i := U_i A U_i^*.$$

As $N, M \to \infty$, want to study eigenvalues of

$$\sum_{i=1}^{M} U_i A U_i^* \qquad \text{and} \qquad \prod_{i=1}^{M} (U_i A U_i^*).$$

$M \to \infty$, additive case

Define $X_{N,M} := \sum_{i=1}^{M} U_i A U_i^*$. As $N, M \to \infty$, have

$$\frac{1}{M} X_{N,M} \approx \frac{1}{N}\Big(\sum_{i=1}^{N} a_i\Big) \cdot \mathrm{Id}$$

$$\sqrt{\frac{N^2-1}{NM}}\Big(X_{N,M} - \mathbb{E}[X_{N,M}]\Big) \approx C_A \cdot \mathrm{GUE}_{N,\mathrm{Tr}=0}$$

for $C_A := \sqrt{\frac{1}{N}\sum_{i=1}^{N} a_i^2 - \frac{1}{N^2}\Big(\sum_{i=1}^{N} a_i\Big)^2}$.

Theorem (Johansson '98)

Fluctuations of height function of GUE_N converge as $N \to \infty$ to explicit log-correlated Gaussian field.

Implies: Fluctuations of height function are log-correlated Gaussian field for M fixed and $M \to \infty$.

$M \to \infty$, multiplicative case

For i.i.d. $U_i \sim$ Haar measure on $U(N)$ and $a_j > 0$, consider

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_N \end{pmatrix} \qquad A_i := U_i A U_i^*.$$

Define the Hermitized product

$$X_{N,M} := (A_1^{1/2} \cdots A_M^{1/2})^* (A_1^{1/2} \cdots A_M^{1/2}),$$

whose eigenvalues are squared singular values of $A_1^{1/2} \cdots A_M^{1/2}$.
Study **Lyapunov exponents** $\{\lambda_i\}$ defined by

$$\lambda_k := \frac{1}{M} \log \left(k^{\text{th}} \text{ eigenvalue of } X_{N,M} \right)$$

$M \to \infty$ limit shape, multiplicative case

Define $\mu := \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{a_i}$ with S-transform given by

$$S(z) := \frac{z+1}{z} M_\mu^{-1}(z) \qquad M_\mu(z) := \int \frac{xz}{1-xz}\mu(dx).$$

Theorem (Newman '86, Kargin '08, Tucci '10, Gorin-S.)

As $N, M \to \infty$ jointly, the empirical measure of Lyapunov exponents converges to the explicit measure

$$\lim_{N,M\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i} = \frac{-e^{-x}}{S'(S^{-1}(e^{-x}))} \mathbf{1}_{[-\log S(-1), -\log S(0)]} dx.$$

- For additive case, this limit was a point mass!

$M \to \infty$ fluctuations, multiplicative case

Lyapunov exp.: $\lambda_k = \frac{1}{M} \log \left(k^{\text{th}} \text{ eig of } (A_1^{1/2} \cdots A_M^{1/2})^* (A_1^{1/2} \cdots A_M^{1/2}) \right)$

Height function: $H_{N,M}(x) = \sum_{i=1}^{N} \mathbf{1}_{\lambda_i \le x}$

Theorem (Gorin-S.)

As $N, M \to \infty$ jointly, have convergence of rescaled fluctuations

$$M^{1/2} \Big(H_{N,M}(x) - \mathbb{E}[H_{N,M}(x)] \Big) \to \xi(x)$$

to explicit Gaussian field $\xi(x)$ with **white noise** component, i.e.

$$\mathbb{E}[\xi(x)\xi(y)] \approx \delta(x-y) \qquad \text{for } x \approx y.$$

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Compare to results for Ginibre, truncated unitary case:

- N fixed, $M \to \infty$: asymptotically indep. Gaussians [Akemann-Burda-Kieburg '14], [Forrester '15], [Reddy '16], [Kieburg-Kosters '17]
- $N, M \to \infty$ jointly: **local** trans. from sine ($N \gg M$) to delta ($N \ll M$) statistics [Akemann-Burda-Kieburg '18] [Liu-Wang-Wang '18]

Why does white noise appear?

Compare decompositions

$$X^{\text{add}}_{N,M} = \mathbb{E}[X^{\text{add}}_{N,M}] + \left(X^{\text{add}}_{N,M} - \mathbb{E}[X^{\text{add}}_{N,M}]\right)$$

$$\log X^{\text{mult}}_{N,M} = \mathbb{E}[\log X^{\text{mult}}_{N,M}] + \left(\log X^{\text{mult}}_{N,M} - \mathbb{E}[\log X^{\text{mult}}_{N,M}]\right).$$

Additive setting: $\frac{1}{M}\mathbb{E}[X^{\text{add}}_{N,M}] \approx (\text{const}) \cdot \text{Id}$

$$(k^{\text{th}} \text{ eigenval. of } X^{add}_{N,M}) \approx \underbrace{\boxed{(\text{const}_1) \cdot M}}_{\mathbb{E}[X^{\text{add}}_{N,M}]} + \underbrace{\boxed{(\text{const}_2) \cdot \sqrt{M} \cdot \gamma_k}}_{X^{\text{add}}_{N,M} - \mathbb{E}[X^{\text{add}}_{N,M}]}$$

for $\gamma_k \stackrel{d}{=} (k^{\text{th}} \text{ eigenval. of } \text{GUE}_{N,\text{Tr}=0})$.

Conclusion: Height function fluctuations come from matrix fluctuations!

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$$X^{\text{add}}_{N,M} = \mathbb{E}[X^{\text{add}}_{N,M}] + \left(X^{\text{add}}_{N,M} - \mathbb{E}[X^{\text{add}}_{N,M}]\right)$$

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Multiplicative setting: $\mathbb{E}[\log X^{\text{mult}}_{N,M}]$ has non-trivial spectrum

- k^{th} eigenval. of $\log X^{\text{mult}}_{N,M}$ **not easily determined** from $\mathbb{E}[\log X^{\text{mult}}_{N,M}]$ and $\left(\log X^{\text{mult}}_{N,M} - \mathbb{E}[\log X^{\text{mult}}_{N,M}]\right)$
- heuristically, fluctuations of $\left(\log X^{\text{mult}}_{N,M} - \mathbb{E}[\log X^{\text{mult}}_{N,M}]\right)$ are distributed along the spectrum of $\mathbb{E}[\log X^{\text{mult}}_{N,M}]$

Analogy with Dyson Brownian motion

Observation (Maurice Duits)

Double contour integral for correlation kernel of singular values of products of M Ginibre matrices looks similar to kernel for **Dyson Brownian motion** at time $t = M^{-1}$.

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For $N = 1$, if ξ_i are i.i.d. standard Gaussian, then

$$\frac{1}{tM} \log \prod_{i=1}^{tM} \xi_i \approx \mathbb{E}[\log \xi_i] + \sqrt{\mathrm{Var}(\log \xi_i)} \cdot \frac{1}{\sqrt{M}} t^{-1} B_t,$$
$$\overset{d}{\approx} \mathbb{E}[\log \xi_i] + \sqrt{\mathrm{Var}(\log \xi_i)} \cdot \frac{1}{\sqrt{M}} B_{t^{-1}},$$

where Brownian motion B_t satisfies $t^{-1} B_t \overset{d}{=} B_{1/t}$.

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[Akemann-Burda-Kieburg '18]: If $N/M \to \infty$ with $N/M \in (0, \infty)$, local law of Lyapunov exponents and DBM started at evenly spaced initial condition coincide.

Analogy with Dyson Brownian motion

Dyson Brownian Motion is a process $\{X_i(t)\}_{i=1}^N$ so that

$$X_k(t) = k^{\text{th}} \text{ eigenvalue of } \Big(\text{diag}(X_k(0)) + \frac{1}{2}(Y(t) + Y(t)^*)\Big),$$

where $Y(t)$ is $N \times N$ matrix of i.i.d. complex BM's. It satisfies

$$dX_k(t) = dB_k(t) + \sum_{j \neq k} \frac{dt}{X_k(t) - X_j(t)}.$$

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multiplication: $X_k(0) = k^{\text{th}}$ Lyapunov exp., [Duits-Johansson '18]:

small $t \implies$ white noise (BM near initial condition dominates)
finite $t \implies$ log-correlated (strong interactions dominate)

Relies on **non-trivial limit** of Lyapunov exponents!

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Suggestion of [Ahn-van Peski '21]:

Dyson BM	$\Longleftrightarrow$	Eigenvalues of BM on $GL_n(\mathbb{C})$
	$\Longleftrightarrow$	Products of random matrices

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III. Technique: Multivariate Bessel generating functions

Multivariate Bessel generating functions

Multivariate Bessel function is defined by

$$\mathcal{B}(s,x) := \frac{\det(e^{s_i x_j})_{i,j=1}^N}{\prod_{i<j}(s_i - s_j)\prod_{i<j}(x_i - x_j)}(N-1)!\cdots 1!.$$

- symmetric eigenfunctions of rat'l Dunkl operators at $\beta = 2$

For measure $d\mu(x)$ on N-tuples $(x_1 \geq \cdots \geq x_N) \in \mathbb{R}^N$ and $\chi \in \mathbb{R}^N$, the **Bessel generating function** is

$$\phi_\chi(s) := \mathbb{E}_\mu\left[\frac{\mathcal{B}(s;x)}{\mathcal{B}(\chi,x)}\right]$$

- normalized by $\phi_\chi(\chi) = 1$
- $\phi_\chi(s)$ is analogue of Schur generating function for discrete measures ($\chi = 0^N$ in [Bufetov-Gorin '13-'17])

Bessel generating functions for sums

Take $\phi_X^{\text{add}}(s) := \phi_{0^N}(s)$ for spectral measure of Herm. matrix X.

Lemma (HCIZ integral)

For $S = \text{diag}(x)$ and $X = \text{diag}(x)$, we have

$$\frac{\mathcal{B}(s,x)}{\mathcal{B}(0^N,x)} = \mathbb{E}\left[e^{\text{Tr} USU^* X}\right],$$

where expectation is taken over Haar measure on $U \in U(N)$.

Proposition

If X and Y are indep. unitarily-invariant Hermitian matrices, then

$$\phi_{X+Y}^{\text{add}}(s) = \phi_X^{\text{add}}(s)\phi_Y^{\text{add}}(s).$$

Bessel generating functions for products

For positive-definite matrix X, take $\phi_X^{\text{mult}}(s) := \phi_\rho(s)$ for $\rho = (N-1, \ldots, 0)$ and **log-spectral** measure of X.

Proposition

If X and Y are indep. unitarily invariant positive-definite Hermitian matrices, then

$$\phi_{XY}^{\text{mult}}(s) = \phi_X^{\text{mult}}(s) \cdot \phi_Y^{\text{mult}}(s).$$

(By analytic continuation of functional relation for $U(N)$-char.)

Note: Similar statement holds for real case, but replace multivariate Bessel with Heckman-Opdam.

MVB GF and Cholesky decomposition

Cholesky decomposition: $X = R^* R$ with R upper triangular.

Lemma (Kieburg-Kosters '15, Gorin-S.)

If X is pos. def., unitarily invariant, we have

$$\phi_X^{\text{mult}}(s) = \mathbb{E}\left[\prod_{k=1}^{N} R_{kk}^{2(s_k - \rho_k)}\right].$$

Analogue of fact from [Matsumoto-Novak '18]:

joint law of $X_{kk} \implies$ distribution of unitarily invariant X.

Corollary (Gorin-S.)

Let X be unitarily invariant pos. def. Hermitian with spectral measure limiting to $d\lambda$. For $t \in [0, 1]$, we have

$$-\log S_{d\lambda}(t-1) = \lim_{N \to \infty} \mathbb{E}[2 \log R_{\lfloor tN \rfloor, \lfloor tN \rfloor}].$$

Moments from Bessel generating functions

Consider differential operators

$$D_k := \prod_{i<j} (s_i - s_j)^{-1} \circ \sum_{i=1}^{N} \partial_i^k \circ \prod_{i<j} (s_i - s_j).$$

Proposition (Bufetov-Gorin '13-'17, Gorin-S.)

If $\phi_\chi(s)$ is Bessel generating function for measure $d\mu(x)$ on $(x_1 \geq \cdots \geq x_N)$, moments of μ are

$$\mathbb{E}[p_{k_1}(x) \cdots p_{k_r}(x)] = D_{k_1} \cdots D_{k_r} \phi_\chi(\chi)$$

for $p_k(x) = x_1^k + \cdots + x_N^k$.

Proof.

Analytic continuation from $D_k \phi_\chi(s) = p_k(x) \phi_\chi(s)$ via

$$\frac{\mathcal{B}(s, x)}{\mathcal{B}(\chi, x)} = \frac{\det(e^{s_i x_j})_{i,j=1}^N}{\prod_{i<j}(s_i - s_j)} \frac{\prod_{i<j}(\chi_i - \chi_j)}{\det(e^{\chi_i x_j})_{i,j=1}^N}.$$

LLN and CLT from Bessel generating functions

Theorem (Gorin-S.)

If $\phi_X^{\text{mult}}(s)$ for prob. measure $d\mu(x)$ on $(x_1 \geq \cdots \geq x_N)$ satisfies

$$\frac{1}{N}\partial_{r_i}[\log \phi_X^{\text{mult}}(rN)]\Big|_{r_k=\rho_k/N, k\neq j} \to \Psi'(r_i)$$
$$\partial_{r_i}\partial_{r_j}[\log \phi_X^{\text{mult}}(rN)]\Big|_{r_k=\rho_k/N, k\neq i,j} \to F^{(1,1)}(r_i, r_j)$$

then $\{p_k(x) - \mathbb{E}[p_k(x)]\}_{k\in\mathbb{N}}$ are Gaussian with covariance

$$\text{Cov}(p_k, p_l) \to \oint\oint\Big(\log(u/(u-1)) + \Psi'(u)\Big)^k$$
$$\Big(\log(w/(w-1)) + \Psi'(w)\Big)^l\Big(\frac{1}{(u-w)^2} + F^{(1,1)}(u,w)\Big)\frac{du}{2\pi\mathbf{i}}\frac{dw}{2\pi\mathbf{i}}.$$

- Corresponds to $\chi = \rho$. Have similar theorem for general χ.
- Similar theorem for MVB GF $\psi_X^{\text{mult}}(s) = \phi_X^{\text{mult}}(s)^M$ with $M \to \infty$, yields white noise.

LLN and CLT from Bessel generating functions

To prove CLT, need to verify conditions:

$$\frac{1}{N}\partial_{r_i}[\log \phi_X^{\text{mult}}(rN)]\Big|_{r_k=\rho_k/N, k\neq j} \to \Psi'(r_i)$$

$$\partial_{r_i}\partial_{r_j}[\log \phi_X^{\text{mult}}(rN)]\Big|_{r_k=\rho_k/N, k\neq i,j} \to F^{(1,1)}(r_i, r_j).$$

For $X = UAU^*$, boils down to asymptotics of $\frac{\mathcal{B}(\mu,a)}{\mathcal{B}(\rho,a)}$ for

$$\mu = \left(r_1 N, r_2 N, \ldots, r_k N, N-1, \ldots, \widehat{b_1 N}, \ldots, \widehat{b_k N}, \ldots, 0\right)$$

so that μ differs from ρ in k coordinates.

Summary

1. Global fluctuations for sums and products of M indep. $N \times N$ unitarily-invariant random matrices are described by explicit Gaussian fields as $N \to \infty$.
   - sums: log-correlated fields
   - products: transition from **log-correlated** (M finite) to **white noise** ($M \to \infty$).
2. Technique based on action of differential operators on **multivariate Bessel generating functions** of empirical measures of Lyapunov exponents.

Paper: V. Gorin and Y. S., Gaussian fluctuations for products of random matrices.

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