Limit theorems for Bessel and Dunkl processes of large dimension

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- Multivariate Bessel processes
- Associated ODEs
- Limit theorems for $N \to \infty$
- Connections to free probability
- Extensions to Dunkl processes

Motivation

- Multivariate Bessel and Dunkl processes of dimension N have a background in **interacting particle systems** of Calogero- Moser-Sutherland type, we will consider two types, the case A_{N-1} and B_N.

- Dunkl processes are **jump diffusions**, where the jumps occur when particles change position or sign. The radial part is a Bessel process.

- The joint distribution of the components of a multivariate Bessel process at time $t = 1$ corresponds to the joint distribution of the **eigenvalues of random matrices**, i.e. the ordered eigenvalues of β-**Hermite and** β-**Laguerre ensembles**.

Aim:
Derive the **semicircle, Marchenko-Pastur and related laws** for the empirical measure of Bessel and Dunkl processes with growing dimension as an analogon to the classical results for random matrices.

General outline of the technique

- Consider the **freezing limit** first.
- Derive limit results for the frozen process via **recurrence relations for the moments**.
- Interpret the limiting laws with the help of Stieltjes- and R-transforms from **free probability**.
-Extend the results to the original stochastic process setting with martingale techniques.

Observation:
The limiting laws stay the same.
All information on the limit is encoded in the frozen process.

Generators and Weyl chambers A_{N-1}

For A_{N-1}, we have a multiplicity $k \in]0, \infty[$, the processes live on the closed Weyl chamber

$$C_N^A := \{x \in \mathbb{R}^N : \quad x_1 \geq x_2 \geq \ldots \geq x_N\},$$

and the generator of the transition semigroup is

$$Lf := \frac{1}{2}\Delta f + k \sum_{i=1}^{N} \Big(\sum_{j \neq i} \frac{1}{x_i - x_j} \Big) \frac{\partial}{\partial x_i} f, \tag{1}$$

where we assume reflecting boundaries, i.e., the domain of L is

$$D(L) := \{f|_{C_N^A} : f \in C^{(2)}(\mathbb{R}^N), f \text{ invariant under all coordinate permutations}\}$$

Generators and Weyl chambers B_N

For B_N, we have the multiplicity $k = (k_1, k_2) \in]0, \infty[^2$, the processes live on

$$C_N^B := \{x \in \mathbb{R}^N : \quad x_1 \geq x_2 \geq \ldots \geq x_N \geq 0\},$$

and the generator of the transition semigroup is

$$Lf := \frac{1}{2}\Delta f + k_2 \sum_{i=1}^{N} \sum_{j \neq i} \Big(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \Big) \frac{\partial}{\partial x_i} f \quad + k_1 \sum_{i=1}^{N} \frac{1}{x_i} \frac{\partial}{\partial x_i} f,$$

where we again assume reflecting boundaries, i.e., L has the domain

$$D(L) := \{f|_{C_N^B} : f \in C^{(2)}(\mathbb{R}^N), \quad f \quad \text{invariant under all permutations}$$
$$\text{and sign changes of all coordinates}\}.$$

Stochastic differential equation

Theorem

(Chybiryakov, Gallardo and Yor (2008), Graczyk and Malecki (2014)) *Assume that* $k > 0$*. Then, for* $x \in C_N$ *in the closed Weyl chamber and* $t > 0$*, the Bessel process* $(X_{t,k})_{t\geq 0}$ *satisfies*

$$X_{0,k} = x, \qquad dX_{t,k} = dB_t + \frac{1}{2}(\nabla(\ln w_k))(X_{t,k})\, dt$$

with an N-dimensional Brownian motion $(B_t)_{t\geq 0}$ *and*

$$w_k^A(x) := \prod_{i<j}(x_i - x_j)^{2k}, \qquad w_k^B(x) := \prod_{i<j}(x_i^2 - x_j^2)^{2k_2} \cdot \prod_{i=1}^{N} x_i^{2k_1},$$

has a unique (strong) solution $(X_{t,k})_{t\geq 0}$*. If all components of* k *are at least* $1/2$*, then* $(X_{t,k})_{t>0}$ *lives on the interior on* C_N *almost surely.*

Freezing limit

An important role plays the limit $k \to \infty$ which in physics corresponds to the case of low temperature and a **decreasing influence of the Brownian motion**. For the Bessel processes $(X_{t,k})_{t\geq 0}$ of type A_{N-1}

$$dX^i_{t,k} = dB^i_t + k \sum_{j\neq i} \frac{1}{X^i_{t,k} - X^j_{t,k}} dt \qquad (i = 1, \ldots, N).$$

the **renormalized processes** $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k})_{t\geq 0}$ satisfy

$$d\tilde{X}^i_{t,k} = \frac{1}{\sqrt{k}} dB^i_t + \sum_{j\neq i} \frac{1}{\tilde{X}^i_{t,k} - \tilde{X}^j_{t,k}} dt \qquad (i = 1, \ldots, N).$$

and the limit leads to the following ODE.

ODE in the case A_{N-1}

$$\frac{dx}{dt}(t) = H(x(t)), \qquad x(0) = x_0 \tag{2}$$

with

$$H(x) := \Big(\sum_{j\neq 1} \frac{1}{x_1 - x_j}, \ldots, \sum_{j \neq N} \frac{1}{x_N - x_j}\Big).$$

has a unique solution for all $t \geq 0$ in the sense that $[0, \infty[\to C_N^A$, $t \mapsto x(t)$ is continuous such that $x(t)$ is in the interior of C_N^A and solves the ODE in (2) for $t > 0$.

We denote the solutions of the ODE by

$$\phi_N := (\phi_{N,1}, \ldots, \phi_{N,N})$$

.

Special solution

Lemma (Stieltjes)

For $y \in C_N^A$, the following statements are equivalent:

(1) *The function $2\sum_{i,j:i<j} \ln(x_i - x_j) - \|x\|^2/2$ is maximal at $y \in C_N^A$;*

(2) *For $i = 1, \ldots, N$: $\frac{1}{2}y_i = \sum_{j:j\neq i} \frac{1}{y_i - y_j}$;*

(3) *The vector*

$$z := (z_1, \ldots, z_N) := (y_1/\sqrt{2}, \ldots, y_N/\sqrt{2})$$

consists of the ordered zeroes of the Hermite polynomials H_N.

For the vector z as above and each $c \geq 0$, a solution of the ODE above with start in cz is given by

$$\phi(t) = \sqrt{2t + c^2} \cdot z.$$

The empirical measure

Aim: Determine the limit as $N \to \infty$ of the normalized empirical measure of the solutions of the ODE.
We take $(x_{N,1}, \ldots, x_{N,N}) \in C_N^A$ as starting points of the solution $\phi_N(t)$ and define the **empirical measure**

$$\mu_{N,t} := \frac{1}{N}(\delta_{\phi_{N,1}(t)/\sqrt{N}} + \ldots + \delta_{\phi_{N,N}(t)/\sqrt{N}}) \tag{3}$$

for $t \geq 0$. Denote the l-th moment ($l \in \mathbb{N}_0$) $\mu_{N,t}$ by

$$S_{N,l}(t) := \int_{\mathbb{R}} y^l \, d\mu_{N,t}(y) = \frac{1}{N^{l/2+1}}(\phi_{N,1}(t)^l + \ldots + \phi_{N,N}(t)^l).$$

Hence we have to calculate **symmetric polynomials**.

Calculation of empirical moments I

Clearly $S_{N,0}(t) = 1$. Moreover, by the ODE,

$$\frac{d}{dt}S_{N,1}(t) = \frac{1}{N^{3/2}} \sum_{i,j=1;i\neq j}^{N} \frac{1}{\phi_{N,i}(t) - \phi_{N,j}(t)} = 0,$$

i.e., $S_{N,1}(t) = S_{N,1}(0)$ for all $t \geq 0$.

$$\frac{d}{dt}S_{N,2}(t) = \frac{2}{N^2} \sum_{i,j=1;i\neq j}^{N} \frac{\phi_{N,i}(t)}{\phi_{N,i}(t) - \phi_{N,j}(t)} = \frac{2}{N^2} \cdot \frac{N(N-1)}{2} = \frac{N-1}{N} \tag{4}$$

and

$$\begin{aligned}\frac{d}{dt}S_{N,3}(t) =& \frac{3}{N^{5/2}} \sum_{i,j=1;i\neq j}^{N} \frac{\phi_{N,i}(t)^2}{\phi_{N,i}(t) - \phi_{N,j}(t)} \\ =& \frac{3}{2N^{5/2}} \sum_{i,j=1;i\neq j}^{N} (\phi_{N,i}(t) + \phi_{N,j}(t)) = \frac{3(N-1)}{N} S_{N,1}(0). \end{aligned} \tag{5}$$

Calculation of empirical moments II

for $l \geq 4$ we obtain

$$\frac{d}{dt} S_{N,l}(t) = \frac{l}{N^{l/2+1}} \sum_{i,j=1; i \neq j}^{N} \frac{\phi_{N,i}(t)^{l-1}}{\phi_{N,i}(t) - \phi_{N,j}(t)}$$

$$= \frac{l}{2} \Big(\frac{1-l}{N} S_{N,l-2}(t) + \sum_{k=0}^{l-2} S_{N,l-2-k}(t) S_{N,k}(t) \Big). \qquad (6)$$

Recurrence relation for the limiting empirical moments

Lemma

Let $(x_{N,k})_{1\leq k\leq N} \subset \mathbb{R}$ *be starting sequences such that for all* $l \in \mathbb{N}_0$, $c_l(0) := \lim_{N\to\infty} S_{N,l}(0) = \lim_{n\to\infty} \frac{1}{N^{l/2+1}}(x_{N,1}^l + \ldots + x_{N,N}^l) < \infty$ *exists. Then for all* $l \in \mathbb{N}_0$,

$$c_l(t) := \lim_{N\to\infty} S_{N,l}(t)$$

exists locally uniformly in $t \in [0,\infty[$. *For each* $l \in \mathbb{N}_0$, $c_l(t)$ *is a polynomial in* t *of degree at most* $\lfloor l/2 \rfloor$ *with a nonnegative "leading" coefficient of order* $\lfloor l/2 \rfloor$. *Moreover, the* $c_l(t)$ *satisfy*

$$c_l(t) = c_l(0) + \frac{l}{2}\int_0^t \Big(\sum_{k=0}^{l-2} c_{l-2-k}(s)c_k(s)\Big)\, ds. \tag{7}$$

Catalan numbers

The **Catalan numbers** may be defined via the recurrence relation

$$C_0 = C_1 = 1, \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad (n \geq 1).$$

Hence comparison with our derived recurrence relation implies:
The polynomial $c_{2l}(t)$ has the degree l with the Catalan number C_l as leading coefficient for $l \in \mathbb{N}_0$.

Example: Semicircle Law

Assume that $x_{N,k} = 0$ for all N, k. Then $c_0(0) = 1$ and $c_l(0) = 0$ for $l \geq 1$. Therefore $c_0(t) = 1$, $c_1(t) = 0$, $c_2(t) = t$ and $c_3(t) = 0$ for $t \geq 0$ and

$$c_{2l}(t) = C_l t^l \quad \text{and} \quad c_{2l+1}(t) = 0 \quad (t \geq 0, l \in \mathbb{N}_0).$$

For $R > 0$, a random variable X_R with the **semicircle law** $\mu_{sc,R}$ with density $f_R(x) := \frac{2}{\pi R^2}\sqrt{R^2 - x^2}$ for $|x| \leq R$ and $f_R(x) = 0$ otherwise has the moments

$$E(X_R^{2n}) = \left(\frac{R}{2}\right)^{2n} C_n \quad \text{and} \quad E(X_R^{2n+1}) = 0 \quad \text{for} \quad n \geq 0.$$

By the moment convergence theorem for $t > 0$ the empirical measures $\mu_{N,t}$ of the (renormalized) solutions of our ODEs with start in the origin tend weakly to $\mu_{sc,2\sqrt{t}}$ for $N \to \infty$.

General Moment Method

We choose $(x_{N,n})_{N\geq 1, 1\leq n\leq N} \subset \mathbb{R}$ with $x_{N,n-1} > x_{N,n}$ for $2 \leq n \leq N-1$ such that the empirical measures

$$\mu_{N,0} := \frac{1}{N}(\delta_{x_{N,1}/\sqrt{N}} + \ldots \delta_{x_{N,N}/\sqrt{N}})$$

tend weakly to μ for $N \to \infty$, i.e., by the moment convergence theorem, that

$$\lim_{N\to\infty} S_{N,l}(0) := \lim_{N\to\infty} \frac{1}{N^{l/2+1}}(x_{N,1}^l + \ldots + x_{N,N}^l) = c_l \quad (l \geq 0).$$

If

$$|c_l| \leq (cl)^l \qquad \text{for all} \quad l \geq 0 \quad \text{and some} \quad c > 0. \tag{8}$$

holds, then for each $t \in [0, \infty[$, the sequence $(c_l(t))_{l\geq 0}$ is the sequence of moments of some unique probability measure $\mu_t \in M^1(\mathbb{R})$ for which (8) also holds. Moreover, the $\mu_{N,t}$ tend weakly to μ_t for $N \to \infty$.

Limiting Distribution

Question:
How can we interpret the limiting distribution?

Hint:
For Gaussian unitary ensembles it is well-know that the limit is the free additive convolution of the semicircle law with the empirical measure of a second added random matrix.

Conjecture:
μ_t is the **free additive convolution** between the starting measure μ and the semicircle law $\mu_{sc,2\sqrt{t}}$.

Stieltjes Transform

Recall the definition of the **Stieltjes transform**

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x) \qquad \text{for} \quad z \in H := \{z \in \mathbb{C} : \Im z > 0\} \qquad (9)$$

of a probability measure $\mu \in M^1(\mathbb{R})$. Clearly, G_μ is analytic on H. We next derive by the recurrence relation the **PDE for the Stieltjes transform**

$$G(t,z) := G_{\mu_t}(z) \qquad (t \geq 0,\ z \in H)$$

of the measure μ_t. It satisfies **Burgers equation**

$$G_t(t,z) = -G(t,z)G_z(t,z).$$

Idea of the proof: Use

$$G(t,z) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu_t(x) = \sum_{l=0}^{\infty} \frac{c_l(t)}{z^{l+1}}$$

R-Transform

With the aid of the **R-transform** of measures $\mu \in M^1(\mathbb{R})$ which is defined as the formal power series $R_\mu(z) := \sum_{n=0}^{\infty} k_{n+1}(\mu) z^n$ with the free cumulants $k_n(\mu)$ of the measure μ for which all moments exist. The functions R_μ and G_μ are related by

$$z - \frac{1}{G_\mu(z)} = R_\mu(G_\mu(z)). \tag{10}$$

Hence we may transform the **PDE for the Stieltjes transform** into a **PDE for the R-transform**:

$$R_t(t, z) = z \tag{11}$$

with $R(0, z) = R_\mu(z)$ the R-transform of the starting measure μ. Therefore,

$$R(t, z) = zt + R(0, z).$$

and since the R-transform is additive with respect to free convolution

$$zt + R(0, z) = R_{\mu_{sc,2\sqrt{t}}}(z) + R_\mu(z) = R_{\mu_{sc,2\sqrt{t}} \boxplus \mu}(z).$$

Bessel processes of type A

To use the results for the ODEs we consider the renormalized processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k})_{t\geq 0}$ which satisfy the SDE

$$d\tilde{X}^i_{t,k} = \frac{1}{\sqrt{k}} dB^i_t + \sum_{j\neq i} \frac{1}{\tilde{X}^i_{t,k} - \tilde{X}^j_{t,k}} dt \qquad (i = 1, \ldots, N), \tag{12}$$

which agrees, for $k = \infty$, with the ODE. We also consider the renormalized empirical measures

$$\tilde{\mu}_{N,t} := \frac{1}{N}(\delta_{\tilde{X}^1_{t,k}/\sqrt{N}} + \ldots + \delta_{\tilde{X}^N_{t,k}/\sqrt{N}}). \tag{13}$$

Denote the l-th moment ($l \in \mathbb{N}_0$) of $\tilde{\mu}_{N,t}$ by

$$S_{N,l}(t) := \int_{\mathbb{R}} y^l \, d\tilde{\mu}_{N,t}(y) = \frac{1}{N^{l/2+1}} \sum_{i=1}^{N} (\tilde{X}^i_{t,k})^l.$$

The Semicircle law for Bessel processes of type A

Theorem

Consider the processes $(\tilde{X}_{t,k})_{t\geq 0}$ *with* $k \geq 1/2$ *and with starting sequence* $(x_i)_{i\geq 1}$ *as for the ODE. Then, for the sequences* $(c_l(t))_{l\geq 0}$ *corresponding to the ODE,*

$$S_{N,l}(t) \to c_l(t) \qquad \text{for} \quad N \to \infty \tag{14}$$

almost surely for all $l \geq 0$ *and locally uniformly for all* $t \in [0, \infty[$.

Ideas of the proof

- Consider the asymptotic behaviour of the expectation of the normalized moments, which is similar to the behaviour of the associated ODE.
- Deduce the asymptotic behaviour of the corresponding diffusion part of the associated SDE.
- Deduce the desired result for the normalized moments:
  First derive with Chebychev and Burkholder-Davis-Gundy inequality
  convergence in probability
  and finally with a Borel Cantelli argument **almost sure convergence**.

Empirical Moments of the Bessel process

Using Itô's formula we obtain for $l \geq 1$

$$\sum_{i=1}^{N}(\tilde{X}_{t,k}^{i})^{l} = \sum_{i=1}^{N} x_i^l + \frac{l}{\sqrt{k}}\sum_{i=1}^{N}\int_0^t (\tilde{X}_{s,k}^{i})^{l-1} dB_s^i \qquad (15)$$
$$+ \int_0^t \left(l \sum_{i=1}^{N}\sum_{j\neq i} \frac{(\tilde{X}_{s,k}^{i})^{l-1}}{\tilde{X}_{s,k}^{i} - \tilde{X}_{s,k}^{j}} + \sum_{i=1}^{N} \frac{l(l-1)}{2k}(\tilde{X}_{s,k}^{i})^{l-2} \right) ds$$

Note

$$\sum_{i=1}^{N}\int_0^t (\tilde{X}_{s,k}^{i})^{l-1} dB_s^i \stackrel{d}{=} \int_0^t \sqrt{\sum_{i=1}^{N}(\tilde{X}_{s,k}^{i})^{2l-2}}\, d\tilde{B}_s$$

for some one-dimensional Brownian motion $\tilde{B}$ by the Lévy characterization of the one-dimensional Brownian motion.

Example: Start in Zero

If the renormalized Bessel processes start in 0, i.e. if the starting measure $\mu = \delta_0$, then the limiting measure $\tilde{\mu}_t = \mu_{sc,2\sqrt{t}}$ for all $t > 0$.

Hence for the original Bessel processes $(X_{t,k}^N)_{t \geq 0}$ of dimension N with multiplicity $k \geq 1/2$, the associated empirical measures $\mu_{N,t}$ tend weakly to the **semicircle law** $\mu_{sc,2\sqrt{tk}}$ almost surely for $t > 0$.

The case B_N

The case B_N works with the same technique. We consider the multiplicities $k = (k_1, k_2) = (\nu \cdot \beta, \beta)$ with $\nu > 0$ fixed and $\beta \to \infty$ and the SDE

$$dX^i_{t,k} = dB^i_t + \beta \sum_{j \neq i} \Big(\frac{1}{X^i_{t,k} - X^j_{t,k}} + \frac{1}{X^i_{t,k} + X^j_{t,k}} \Big) dt + \frac{\nu \cdot \beta}{X^i_{t,k}} dt$$

for $i = 1, \ldots, N$ with an N-dimensional Brownian motion $(B^1_t, \ldots, B^N_t)_{t \geq 0}$. The associated dynamical system is then $\frac{dx}{dt}(t) = H(x(t))$ with

$$H : U_\epsilon \to \mathbb{R}^N, \quad x \mapsto \begin{pmatrix} \sum_{j \neq 1} \left(\frac{1}{x_1 - x_j} + \frac{1}{x_1 + x_j} \right) + \frac{\nu}{x_1} \\ \vdots \\ \sum_{j \neq N} \left(\frac{1}{x_N - x_j} + \frac{1}{x_N + x_j} \right) + \frac{\nu}{x_N} \end{pmatrix}$$

and an analogon to the Lemma in the case A_{N-1} holds where the zeros of the Hermite polynomials are replaced by the positive square root of the zeros of the Laguerre polynomials $L_N^{(\nu-1)}$ denoted by $(z_1^{(\nu-1)}, \ldots, z_N^{(\nu-1)})$.

Ideas for the case B

Wish to proceed similarly as in the case A_{N-1}:

- Consider the associated ODE first, i.e. derive a recurrence relation for the normalized empirical moments
- Identify the limiting distributions
- Transform the result to the SDE setting

Problem:
We cannot work with the process itself. Necessary polynom division for the recurrence relation does not work for odd moments.

Solution:
Pass to the squares.

Empirical moments for the ODE

Consider the associated solutions $\phi_N(t)$ and the normalized empirical measures

$$\mu_{N,t} := \frac{1}{N}(\delta_{\phi_{N,1}(t)^2/(2N)} + \ldots + \delta_{\phi_{N,N}(t)^2/(2N)}) \qquad (16)$$

Denote the l-th moment ($l \in \mathbb{N}_0$) of $\mu_{N,t}$ by

$$S_{N,l}(t) := \int_{\mathbb{R}} y^l \, \mu_{N,t}(y) = \frac{1}{2^l N^{l+1}}(\phi_{N,1}(t)^{2l} + \ldots + \phi_{N,N}(t)^{2l}).$$

Then

$$\frac{d}{dt} S_{N,l}(t) = l\Big(\frac{2N+\nu-l}{N} S_{N,l-1}(t) + \sum_{k=1}^{l-2} S_{N,l-1-k}(t) S_{N,k}(t)\Big). \qquad (17)$$

Limit in first case

First case: $\nu > 0$ fixed.

$$c_0(t) = 1 \quad \text{and} \quad c_l(t) = c_l(0) + l \int_0^t \sum_{k=0}^{l-1} c_{l-1-k}(s) c_k(s)\, ds \quad (l \geq 1). \tag{18}$$

We obtain the same result as for the even moments in the case A. Hence the limiting distribution for the case B is

$$|\mu_{sc,2\sqrt{t}} \boxplus \mu_{even}|$$

As special case with start in zero the limit is **quarter circle law** on the positive half line.
We use the notation: for a probability measure μ on $[0, \infty[$, let μ_{even} the unique even probability measure on $\mathbb{R}$ with $|\mu_{even}| = \mu$

Limit in the second case

Second case:

Let $(x_{N,k})_{1\leq k\leq N} \subset \mathbb{R}$ be starting sequences such that for all $l \in \mathbb{N}_0$,

$$c_l(0) := \lim_{N\to\infty} S_{N,l}(0) = \lim_{n\to\infty} \frac{1}{2^l N^{l+1}}(x_{N,1}^{2l} + \ldots + x_{N,N}^{2l}) < \infty$$

exists. Assume that $\nu = \nu(N)$ depends on N with

$$\lim_{N\to\infty} \frac{\nu(N)}{N} = \nu_0 \geq 0.$$

Then for $l \in \mathbb{N}_0$,

$$c_l(t) := \lim_{N\to\infty} S_{N,l}(t)$$

exists locally uniformly in $t \in [0,\infty[$ and satisfies the recurrence relation

$$c_0(t) = 1 \quad \text{and} \quad c_l(t) = c_l(0) + l\nu_0 \int_0^t c_{l-1}(s)ds + l \int_0^t \sum_{k=0}^{l-1} c_{l-1-k}(s)c_k(s)ds \tag{19}$$

Marchenko-Pastur law

Recall that for the parameters $c \geq 0$, $t > 0$, the **Marchenko-Pastur distribution** $\mu_{MP,c,t}$ is the probability measure on $[x_-, x_+] \subset [0, \infty[$ with $\mu_{MP,c,t} = \tilde{\mu}$ for $c \geq 1$ and $\mu_{MP,c,t} = (1-c)\delta_0 + c\tilde{\mu}$ for $0 \leq c < 1$, where $x_\pm := t(\sqrt{c} \pm 1)^2$ and $\tilde{\mu}$ has the Lebesgue density

$$\frac{1}{2\pi x t}\sqrt{(x_+ - x)(x - x_-)} \cdot \mathbf{1}_{[x_-, x_+]}(x).$$

The Marchenko-Pastur distributions have the R-transforms

$$R_{MP,c,t}(z) = \frac{ct}{1 - tz}. \tag{20}$$

As these R-transforms are linear in c, we in particular conclude that

$$\mu_{MP,a,t} \boxplus \mu_{MP,b,t} = \mu_{MP,a+b,t}. \tag{21}$$

Limiting distribution in the general case

Under the conditions as before the normalized empirical measures

$$\mu_{N,t} := \frac{1}{N}(\delta_{\frac{\phi_{N,1}(t)}{\sqrt{2N}}} + \ldots + \delta_{\frac{\phi_{N,N}(t)}{\sqrt{2N}}}) \qquad (t \geq 0),$$

tend weakly to

$$\sqrt{\mu_{MP,\nu_0,t} \boxplus (\mu_{sc,2\sqrt{t}} \boxplus \mu_{even})^2}.$$

Namely we may deduce as in the case A_{N-1} a PDE for the R-transform:

$$\begin{aligned} R_t(t,z) &= \nu_0 + 1 - 2zR(t,z) + z^2 R_z(t,z) \qquad (22) \\ R(0,z) &= R_\mu(z), \end{aligned}$$

with a solution $R_{\mu_{MP,\nu_0,t}}(z) + R_{(\mu_{sc,2\sqrt{t}} \boxplus \mu_{even})^2}(z)$ using the first case.

Bessel process of type B

Theorem

Consider the processes $(\tilde{X}_{t,k})_{t\geq 0}$ *with* $\beta \geq 1/2$, $\nu > 0$ *and with starting sequences* $(x_{N,k})_{k\geq 1} \subset [0,\infty[$ *as before such that*

$$c_l(0) := \lim_{N\to\infty} S_{N,l}(0) = \lim_{n\to\infty} \frac{1}{2^l N^{l+1}} (x_{N,1}^{2l} + \ldots + x_{N,N}^{2l}) < \infty$$

exists for $l \geq 0$*. Assume that* $\nu := \nu(N)$ *with* $\nu_0 := \lim_{N\to\infty} \nu(N)/N \geq 0$*. Then, for* $l \in \mathbb{N}_0$*,*

$$c_l(t) := \lim_{N\to\infty} S_{N,l}(t)$$

exists almost surely locally uniformly in $t \in [0,\infty[$ *with*

$$c_0(t) = 1 \quad \textit{and} \quad c_l(t) = c_l(0) + l\nu_0 \int_0^t c_{l-1}(s)ds + l \int_0^t \sum_{k=0}^{l-1} c_{l-1-k}(s)c_k(s)\, ds.$$

Extension to Dunkl processes

Dunkl processes are **jump-diffusions** with jumps, which **exchange the coordinates** or lead to a **random sign change**.
For A_{N-1} **only permutations** of particles occur which have **no influence** on our derived limit theorem.
For B_N we have additional random sign changes which lead to new limit theorems for the normalized empirical measure.
Consider $k = (k_1, k_2) = (\beta, \nu\beta)$ with $\beta > 0$ and $\nu \geq 0$. In this case we have for the renormalized process the generator

$$\tilde{\mathcal{L}}_{k_0,\beta} u(x) := \frac{1}{2\beta} \Delta u(x) + L_\nu u(x) \tag{23}$$

for $u \in C_c^2(\mathbb{R}^N)$

Frozen Dunkl process

Denote the frozen Dunkl process $X_{t,\nu}$ given by the generator

$$L_\nu u(x) := \sum_{i=1}^{N} \Big(\sum_{j:\, j \neq i} \frac{2x_i}{x_i^2 - x_j^2} + \frac{\nu}{x_i} \Big) u_{x_i}(x) + \frac{\nu}{2} \sum_{i=1}^{N} \frac{u(\sigma_i x) - u(x)}{x_i^2}$$

$$+ \frac{1}{2} \sum_{i,j:\, j \neq i} \Big(\frac{u(\sigma_{i,j} x) - u(x)}{(x_i - x_j)^2} + \frac{u(\sigma_{i,j}^- x) - u(x)}{(x_i + x_j)^2} \Big) \tag{24}$$

with $\sigma_i, \sigma_{i,j}, \sigma_{i,j}^-$ $(i \neq j)$ are reflections on $\mathbb{R}^N$ where σ_i changes the sign of the i-th coordinate, $\sigma_{i,j}$ exchanges the coordinates i, j, and $\sigma_{i,j}^-$ exchanges the coordinates i, j and changes the signs of these coordinates.
Note: The frozen Dunkl process is **random**, the jumps are still there.

Empirical measure of the frozen process I

Denote the components of $X_{t,\nu}$ by $X_{j,t,\nu}$ for $j = 1, \ldots, N$. Consider (random) normalized empirical measures

$$\mu_{N,t,\nu} := \frac{1}{N}(\delta_{X_{1,t,\nu}/\sqrt{N}} + \ldots + \delta_{X_{N,t,\nu}/\sqrt{N}}) \in M^1(\mathbb{R}). \quad (25)$$

and the corresponding moments

$$S_{N,l,\nu}(t) := \frac{1}{N^{l/2+1}}(X_{1,t,\nu}^l + \ldots + X_{N,t,\nu}^l) \qquad (l \geq 0). \quad (26)$$

Note: The even moments $S_{N,2l,\nu}(t)$ are **deterministic**, corresponding to frozen Bessel processes of type B. With the functions

$$u_l(x) := x_1^l + \ldots + x_N^l \qquad (l \geq 0)$$

we have

$$S_{N,l,\nu}(t) = \frac{1}{N^{1+l/2}} \cdot u_l(X_{t,\nu}) \qquad (l \geq 0). \quad (27)$$

The u_l are **invariant under permutations of coordinates**.

Empirical measure of the frozen process II

For all $u := u_l$

$$L_\nu u(x) = \sum_{i=1}^{N}\Big(\sum_{j:\, j\neq i} \frac{2x_i}{x_i^2 - x_j^2} + \frac{\nu}{x_i}\Big) u_{x_i}(x) + \frac{\nu}{2}\sum_{i=1}^{N} \frac{u(\sigma_i x) - u(x)}{x_i^2}$$

$$+ \frac{1}{2}\sum_{i,j:\, j\neq i} \frac{u(\sigma_{i,j}^{-} x) - u(x)}{(x_i + x_j)^2}. \tag{28}$$

Moreover, Dynkin's formula for Markov processes implies

$$\Big(u_l(X_{t,\nu}) - u_l(X_{0,\nu}) - \int_0^t (L_\nu u_l)(X_{s,\nu})\, ds\Big)_{t\geq 0} \tag{29}$$

are martingales. Hence, for all $l \geq 0$,

$$\frac{d}{dt} E(u_l(X_{t,\nu})) = E((L_\nu u_l)(X_{t,\nu})), \tag{30}$$

which now is the analogon to our ODE in the Bessel case and we may proceed as there with **polynom division**.

Recurrence relation for the expectations of moments

For the **odd** moments, we obtain

$$\frac{d}{dt}E(S_{N,2l+1,\nu}(t)) = \frac{2l(2N+\nu-(l+1))}{N}E(S_{N,2l-1,\nu}(t)) \\ + 4\sum_{h=1}^{l-1}(l-h)S_{N,2h,\nu}(t)E(S_{N,2l-1-2h,\nu}(t)). \tag{31}$$

and in the **even** case we obtain

$$\frac{d}{dt}S_{N,2l,\nu}(t) = \frac{2l(\nu+l+1)}{N}S_{N,2l-2,\nu}(t)+2l\sum_{h=0}^{l-1}S_{N,2l-2-2h,\nu}(t)S_{N,2h,\nu}(t). \tag{32}$$

Recall that the **even moments are deterministic** and for them we have the same **recurrence relation for the moments of Bessel processes of type B** up to an factor 2.

This factor 2 is caused by by the slightly different scalings of the empirical measures.

Lemma

Let $(x_{N,k})_{1\leq k\leq N} \subset \mathbb{R}$ *be the starting sequences of the frozen Dunkl processes* $(X_{t,\nu})_{t\geq 0}$ *for* $N \in \mathbb{N}$ *with* $X_{0,\nu} = (x_{N,1}, \ldots, x_{N,N})$ *such that*

$$c_l(0) := \lim_{N\to\infty} S_{N,l,\nu}(0) = \lim_{n\to\infty} \frac{1}{N^{l/2+1}}(x_{N,1}^l + \ldots + x_{N,N}^l) < \infty$$

exists for all $l \in \mathbb{N}_0$. *Assume that* $\lim_{N\to\infty} \frac{\nu(N)}{N} = \nu_0 \geq 0$. *Then for* $l \in \mathbb{N}_0$, $c_l(t) := \lim_{N\to\infty} E(S_{N,l,\nu(N)}(t))$ *exists locally uniformly in* $t \in [0,\infty[$ *and satisfies the recurrence relations*

$c_0(t) = 1$, $c_1(t) = c_1(0)$, *and for* $l \geq 1$,

$$c_{2l}(t) = c_{2l}(0) + 2l\int_0^t \Big(\nu_0 c_{2l-2}(s) + \sum_{h=0}^{l-1} c_{2h}(s)c_{2l-2h-2}(s)\Big)\, ds$$

$$c_{2l+1}(t) = c_{2l+1}(0) + \int_0^t \Big(2l\nu_0 c_{2l-1}(s) + 4\sum_{h=0}^{l-1}(l-h)c_{2h}(s)c_{2l-2h-1}(s)\Big)\, ds.$$

Description of the limiting law

We define the **reflected probability measures** $\mu_t^* \in M^1(\mathbb{R})$ with $\mu_t^*(A) = \mu_t(-A)$ for Borel sets $A \subset \mathbb{R}$ and

$$\mu_{t,even} := \frac{1}{2}(\mu_t + \mu_t^*), \quad \mu_{t,odd} := \frac{1}{2}(\mu_t - \mu_t^*).$$

Note $\mu_{t,odd}$ usually is a signed measure with that $\mu_t = \mu_{t,even} + \mu_{t,odd}$. We now introduce the Stieltjes transforms

$$G^{even}(t,z) := G_{\mu_{t,even}}(z), \quad G^{odd}(t,z) := G_{\mu_{t,odd}}(z)$$

with $G = G^{even} + G^{odd}$ and obtain by the recurrence relation the **quasilinear system of PDEs**

$$\begin{aligned} G_t^{even}(t,z) &= \nu_0\Big(\frac{G^{even}(t,z)}{z^2} - \frac{G_z^{even}(t,z)}{z}\Big) - 2G^{even}(t,z)G_z^{even}(t,z) \\ G_t^{odd}(t,z) &= \Big(-\frac{\nu_0}{z} - 2G^{even}(t,z)\Big)G_z^{odd}(t,z) \end{aligned} \tag{33}$$

for $t \geq 0$.

Corollary

For $t \geq 0$, the even parts $\mu_{t,even}$ of the limiting probability measures μ_t are the unique even probability measures on $\mathbb{R}$ whose pushforwards under the mapping $x \mapsto x^2/2$ are given by $\mu_{MP,\nu_0,t} \boxplus (\mu_{sc,2\sqrt{t}} \boxplus \mu_{even})^2$. Hence,

$$\mu_{t,even} = \left(\sqrt{\mu_{MP,\nu_0,2t} \boxplus (\mu_{sc,2\sqrt{2t}} \boxplus \mu_{even})^2}\right)_{even} \qquad (t \geq 0). \qquad (34)$$

Interpretation

If the initial measure $\mu_0 = \mu$ is even, then the associated linear PDE for G^{odd} has the solution $G^{odd} = 0$, i.e., the limiting μ_t is given by the measure in Eq. (34).
If we start the frozen Dunkl process in a **symmetric measure** the limiting measure is the **two-sided version of the law for the corresponding Bessel processes**.
Hence the jumps with the sign changes **keep the symmetry** of the starting measure.
Question:
What happens, if we start in an asymmetric measure?

Indeed we get a **different class of limiting measure**, which cannot be interpreted as free convolutions.

Lemma

Let $\mu \in M^1(\mathbb{R})$ for which the moment condition (8) holds. Let $D \subset H$ be some non-empty open domain such that there is some (analytical) function K with

$$K[G_{\mu_{even}}(z) \cdot (G_{\mu_{even}}(z) + \nu_0 G_{\delta_0}(z))] = z \qquad (z \in D).$$

Then, with the measures $\mu_{t,even}$ from the previous corollary,

$$G(t,z) := G_{\mu_{odd}}[K\left(G_{\mu_{t,even}}(z) \cdot (G_{\mu_{t,even}}(z) + \nu_0 G_{\delta_0}(z))\right)] + G_{\mu_{t,even}}(z) \quad (35)$$

is the solution of the system of PDEs with the initial condition $G(0,z) = G_\mu(z)$ for $z \in D$.
In particular, for $\nu_0 = 0$ this solution simplifies to

$$G(t,z) = G_\mu(G^{-1}_{\mu_{even}}(G_{\mu_{sc,2\sqrt{2t}} \boxplus \mu_{even}}(z))). \quad (36)$$

Extension to Dunkl processes

Theorem

Consider the Dunkl processes $(\tilde{X}_{t,\nu,\beta})_{t\geq 0}$ *with* $\beta \geq 1/2$, $\nu > 0$ *and with starting sequences* $(x_{N,i})_{i\geq 1} \subset \mathbb{R}$ *as before such that for* $l \geq 0$,

$$c_l(0) := \lim_{N\to\infty} S_{N,l,\nu,\beta}(0) = \lim_{n\to\infty} \frac{1}{N^{l/2+1}}(x_{N,1}^l + \ldots + x_{N,N}^l) < \infty$$

exists. Let $\nu_0 := \lim_{N\to\infty} \nu(N)/N \geq 0$. *Then, for* $l \in \mathbb{N}_0$,

$$c_l(t) := \lim_{N\to\infty} S_{N,l,\nu,\beta}(t)$$

exists a.s. locally uniformly in $t \in [0,\infty[$. *Furthermore, the* $c_l(t)$ *satisfy the recurrence relation from the frozen process.*

Further considerations on the limiting laws

Questions:
How do the limiting laws in the asymmetric case look like?
How is the **long term behaviour**, is the asymmetry kept or vanishing?

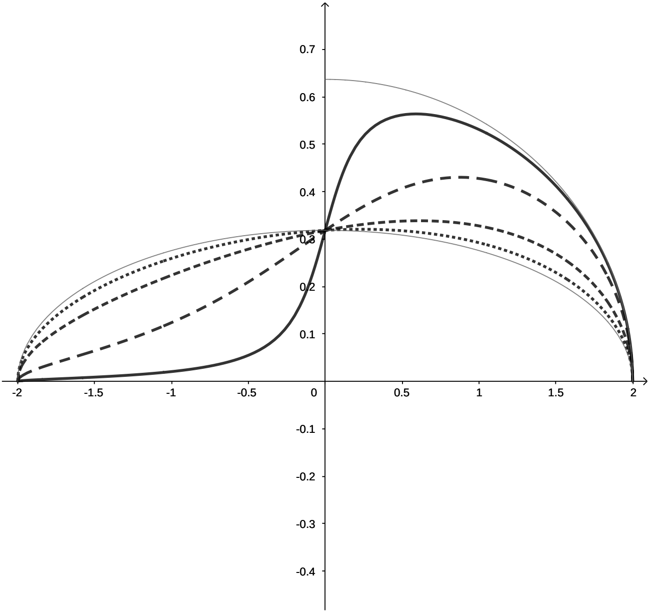
Example: Consider $\nu_0 = 0$ and start in the quartercircle distribution on $[0,2]$, the we may calculate the density of the limiting law

$$
\begin{aligned}
f_t(x) &= \frac{-1}{\pi} \lim_{\epsilon \downarrow 0} \Im G(t, x + i\epsilon) \qquad (37)\\
&= \frac{1}{(2t+1)\pi} \left(\frac{1}{2} + \frac{t+1}{\pi} \arctan \frac{x}{2t} \right) \sqrt{4(2t+1) - x^2} \\
&\quad - \frac{1}{\pi^2} \frac{tx}{2(2t+1)} \ln \left(\frac{2(t+1) + \sqrt{4(2t+1) - x^2}}{2(t+1) - \sqrt{4(2t+1) - x^2}} \right).
\end{aligned}
$$

Now look at rescaled densities

$$\tilde{f}_t(x) = \sqrt{2t+1} f(x\sqrt{2t+1}),$$

i.e. probability measures on $[-2, 2]$.



$\tilde{f}_t(x)$ for $t = 0,\ 0.1,\ 1,\ 10,\ 100$ and $t = \infty$.

Conclusion

- We derived that crucial information of the limiting laws for the empirical measures of Bessel and Dunkl processes is already encoded in their frozen versions.
- We derived semicircle, quarter circle and Marchenko-Pastur type laws for ODEs associated with frozen Bessel and Dunkl processes of type A_{N-1} and B_N as $N \to \infty$.
- We deduced that the same limit laws hold for the processes themselves.
- We saw that the case of a Dunkl process of type B with start in an asymmetric configuration leads to new limiting laws, which cannot be interpreted in terms of free probability.