The role of the simplex in Dunkl theory

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Outline

Dunkl operators
- Basic definitions
- Dunkl transform
- Intertwining operator

Results of Xu
- Dihedral groups
- Expression for the intertwining operator

Our approach
- General formulas
- Application to dihedral groups

Definition of Dunkl operators:

Data:

- G finite reflection group in $\mathbb{R}^m$
- R its root system
- $\kappa : R \to \mathbb{R}^+$ a G-invariant multiplicity function
- $\langle \cdot, \cdot \rangle$ the Euclidean inner product on $\mathbb{R}^m$
- $\sigma_\alpha x := x - 2\langle x, \alpha\rangle \alpha / ||\alpha||^2$ is a reflection

Then the Dunkl operator for $\xi \in \mathbb{R}^m$ is

$$\boxed{T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} \kappa_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \qquad x \in \mathbb{R}^m}$$

We write $T_j = T_{e_j}$ when ξ are the basis vectors

C. F. Dunkl, Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.* **311** (1989), 167-183.

C. F. Dunkl, Y. Xu, *Orthogonal polynomials of several variables.* Cambridge university press, 2014.

Example: one dimensional case

Only one reflection group: $\mathbb{Z}_2$ (root system A_1), acting on $\mathbb{R}$

$$\boxed{T_x = \frac{d}{dx} + k\frac{1-R}{x}}$$

with $Rf(x) = f(-x)$

For this case essentially everything is known, including Dunkl transform and intertwining operator (see later)

Example: the symmetric group

- $G = S_m$, acting on $\mathbb{R}^m$ by permutation of standard vectors
- $R = A_{m-1} = \{\pm(e_i - e_j), i < j\}$
- κ becomes a constant (only one orbit of G on R)

Dunkl operators for $i = 1, \ldots, m$

$$\boxed{T_i = \partial_i + \kappa \sum_{j \neq i} \frac{1 - \sigma_{ij}}{x_i - x_j}}$$

with

$$\sigma_{ij} f(\ldots, x_i, \ldots, x_j, \ldots) = f(\ldots, x_j, \ldots, x_i, \ldots)$$

Basic properties:

$$\boxed{T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} \kappa_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \qquad x \in \mathbb{R}^m}$$

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- commutative: $T_i T_j = T_j T_i$
- T_ξ maps polynomials to polynomials, lowers degree by 1
- Dunkl Laplacian $\Delta_\kappa = \sum_{j=1}^m T_j^2$
- With weight $\omega_\kappa(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2\kappa(\alpha)}$ the inner product

  $$\langle f, g \rangle_\kappa = \int_{\mathbb{R}^m} f(x) \overline{g(x)} \omega_\kappa(x) dx$$

  makes T_j skew-adjoint:

  $$\langle T_j f, g \rangle_\kappa = -\langle f, T_j g \rangle_\kappa$$

Two important operators:

- the Dunkl transform (generalizes the Fourier transform)
- the intertwining operator (maps the ordinary situation to the Dunkl situation)

We will discuss both.

Summary: some general results are known, but there is limited information for specific reflection groups

M. de Jeu, The Dunkl transform. *Invent. Math.* **113** (1993), 147–162.

C. F. Dunkl, M. de Jeu, E. Opdam, Singular polynomials for finite reflection groups. *Trans. Amer. Math. Soc.* **346** (1994), 237-256.

M. Rösler, Positivity of Dunkl's intertwining operator. *Duke Math. J.* **98** (1999), 445-464.

Recall that the Fourier transform

$$\mathcal{F}(f)(y) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle} f(x)dx$$

satisfies

$$\mathcal{F}(\partial_{x_j} f(x)) = iy_j \mathcal{F}(f), \qquad j = 1, \ldots, m$$

$\Longrightarrow$ we want an integral transform with these properties for the Dunkl operators!

Dunkl transform: Put for $y \in \mathbb{R}^m$

$$\boxed{\mathcal{F}_\kappa f(y) := c_\kappa^{-1} \int_{\mathbb{R}^m} E_\kappa(-ix, y) f(x) \omega_\kappa(x) dx}$$

with E_κ the joint eigenfunction of all T_j,

$$T_j E_\kappa(x, y) = y_j E_\kappa(x, y), \qquad j = 1, \ldots, m$$

This transform satisfies

$$\mathcal{F}_\kappa(T_j f(x)) = i y_j \mathcal{F}(f), \qquad j = 1, \ldots, m$$

Proof: use skew-adjointness of T_j

Abstract results on Dunkl transform:

- the integral kernel E_κ exists and is real-analytic
- boundedness: $|E_\kappa(-ix, y)| \leq 1$
- variants of Paley-Wiener theorems

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M. de Jeu, Paley-Wiener theorems for the Dunkl transform. *Trans. Amer. Math. Soc.* **358** (2006), 4225-4250.

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Concrete results on Dunkl transform:

Explicit expressions for the kernel E_κ are known for:

- $\mathbb{Z}_2$: sum of 2 Bessel functions
- some low dimensional cases, often with restrictions e.g. on κ
- dihedral groups (see later, this is our work)

Intertwining operator V_κ:

Theorem

There exists an unique linear and homogenous isomorphism on $\mathcal{P}$ which satisfies $V_\kappa 1 = 1$ and which intertwines the partial differential operators and the Dunkl operators,

$$T_j V_\kappa = V_\kappa \partial_j, \qquad j = 1, 2, \dots, m.$$

$\mathcal{P}$ is the space of polynomials on $\mathbb{R}^m$

C. F. Dunkl, Operators commuting with Coxeter group actions on polynomials, in Invariant Theory and Tableaux, Editor D. Stanton, Springer, Berlin - Heidelberg - New York, 1990, 107-117.

C. F. Dunkl, M. de Jeu, E. Opdam, Singular polynomials for finite reflection groups. *Trans. Amer. Math. Soc.* **346** (1994), 237-256.

Abstract results on the intertwining operator:

- V_κ can be represented by an integral transform

$$V_\kappa(p)(x) = \int_{\mathbb{R}^m} p(y) d\mu_x^{(\kappa)}(y)$$

- V_κ is positive for $\kappa \geq 0$: it maps positive functions to positive functions

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Concrete results on the intertwining operator:

Explicit expressions for integral representation of V_κ:

- $\mathbb{Z}_2$: see formula below
- some low dimensional cases, often with restrictions e.g. on κ
- dihedral groups (see later, this is our work)

Example: one dimensional case

For practical use:

$$\boxed{V_k(p)(x) = \frac{\Gamma(k+1/2)}{\Gamma(1/2)\Gamma(k)} \int_{-1}^{1} p(xt)(1-t)^{k-1}(1+t)^k dt}$$

Example: one dimensional case

For practical use:

$$\boxed{V_k(p)(x) = \frac{\Gamma(k+1/2)}{\Gamma(1/2)\Gamma(k)} \int_{-1}^{1} p(xt)(1-t)^{k-1}(1+t)^k dt}$$

The Bochner representation

$$V_k(p)(x) = \int_{\mathbb{R}} p(y) d\mu_x^{(k)}(y)$$

with

$$d\mu_x^{(k)}(y) \sim \frac{1}{|x|^{2k}}(|x| + \mathrm{sign}(x)y)(x^2 - y^2)^{k-1} 1_{(-|x|,+|x|)}(y) dy$$

reflects the positivity result of Rösler

Conclusion:

Little is known about the explicit form of the Dunkl transform and intertwining operator for specific classes of reflection groups.

Why is this problematic?

- hard analysis for Dunkl operators needs these explicit forms
- important for improving the appeal of the theory

What is a 'good' formula for E_κ or V_κ?

Several points of view possible:

- explain combinatorics / linear algebra behind V_κ
- explain relation with orthogonal polynomials
- just any integral expression is progress
- **allow for observational proofs of abstract results**

Ideally, the method is somewhat structural

Some examples for dihedral groups (many more in literature)

C. F. Dunkl, Polynomials associated with dihedral groups. *SIGMA Symmetry Integrability Geom. Methods Appl.* **3** (2007), Paper 052, 19 pp.

Y. Xu, Intertwining operators associated to dihedral groups. *Constr. Approx.* 52 (2020), 395-422

Our results:

The (integral over the) simplex seems to play a crucial role in explicit realizations of Dunkl kernel and intertwining operator.

Evidence:

- conclusive in dihedral case
- for A_n: special case
- for B_n: special case in progress

Outline

Dihedral groups:

- infinite family of finite reflection groups acting in $\mathbb{R}^2$
- the dihedral group I_k is the group of symmetries of the regular $k-$gon
- when k is odd, there is one orbit on the root system

$$\kappa = \alpha \in \mathbb{C}$$

- when k is even, there are two orbits

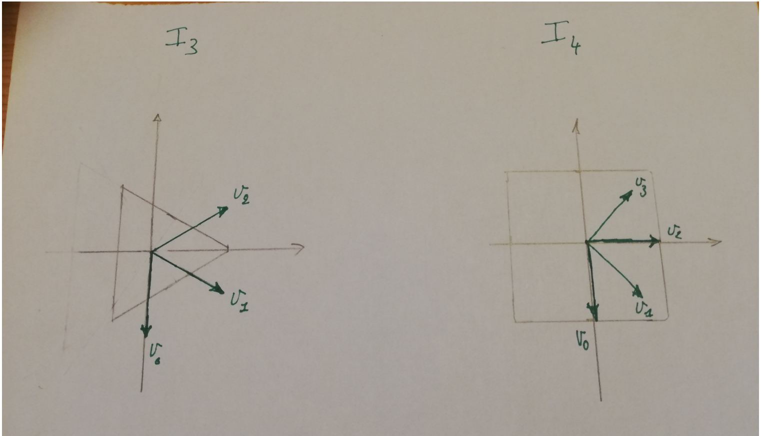
$$\kappa = (\alpha, \beta) \in \mathbb{C}^2$$

Dihedral groups

Pick as positive roots $j = 0, 1, \cdots, k-1$

$$v_j = \left(\sin\left(\frac{\pi j}{k}\right), -\cos\left(\frac{j\pi}{k}\right)\right) = e^{i\left(\frac{\pi j}{k} - \frac{\pi}{2}\right)}$$

E.g.: I_3, symmetries of regular triangle and I_4, symmetries of square



Dunkl operators for dihedral group I_k with k odd

They act on functions $f(x_1, x_2)$ by:

$$T_1 = \partial_{x_1} + \alpha \sum_{j=0}^{k-1} \sin(j\pi/k) \frac{1-\sigma_j}{\sin(j\pi/k)x_1 - \cos(j\pi/k)x_2}$$

$$T_2 = \partial_{x_2} - \alpha \sum_{j=0}^{k-1} \cos(j\pi/k) \frac{1-\sigma_j}{\sin(j\pi/k)x_1 - \cos(j\pi/k)x_2}$$

with σ_j the reflection over the line perpendicular to v_j

Xu's theorem: $\alpha = \beta = \lambda$

We first give a proof of Theorem 1.1, which we reformulate it below, using the polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$.

Theorem 3.1. *Let f be a differentiable function on $\mathbb{R}$. For $0 \le p \le 2k-1$, define*

$$F_p(x_1, x_2) := f\left(\cos\left(\frac{p\pi}{k}\right)x_1 + \sin\left(\frac{p\pi}{k}\right)x_2\right).$$

Then, for $k = 2, 3, 4, \ldots$, the intertwining operator V_λ for the dihedral group I_k with one parameter λ satisfies

$$V_\lambda F_p(x_1, x_2) = a_\lambda^{(k)} \int_{T^{k-1}} f\left(r\sum_{j=0}^{k-1}\cos\left(\theta - \frac{p\pi}{k} - \frac{2j\pi}{k}\right)u_j\right)u_0 \prod_{i=0}^{k-1} u_i^{\lambda-1}\mathrm{d}u. \tag{3.1}$$

Here T^{k-1} is the simplex defined by

$$T^{k-1} := \left\{u \in \mathbb{R}^{k-1} : u_1 \geq 0, \ldots, u_{k-1} \geq 0, u_1 + \ldots + u_{k-1} \leq 1\right\}.$$

and $u_0 = 1 - u_1 - u_2 - \ldots - u_{k-1}$

Comments:

- proof by direct verification is elementary but complicated (3 pages of trigonometry)
- Xu: 'There is little methodology for identifying this integral transform. The discovery of our formula is motivated by an integral formula in [Xu, Proc. Amer. Math. Soc. 143 (2015)] and is the result of trial and error, starting from I_4.'
- formula is not as general as we would want (conditions on the functions, and on κ)
- **appearance of integral over simplex is intriguing**

Y. Xu, Intertwining operators associated to dihedral groups. *Constr. Approx.* 52 (2020), 395-422

Y. Xu, An integral identity with applications in orthogonal polynomials. *Proc. Amer. Math. Soc.* 143 (2015), 5253-5263.

Outline

Our approach:

A well-known formula

$$\boxed{E_\kappa(x,y) = V_\kappa\left(e^{\langle\cdot,y\rangle}\right)(x)}$$

seems to imply that it is best to start with the intertwining operator.

This is in our experience not true:

- expression for E_κ easier to obtain than for V_κ

Instead:

- we derive a general formula for V_κ based on knowledge of E_κ
- E_κ for dihedral groups follows from the Poisson kernel
- E_κ for dihedral groups as 2nd Humbert function
- 2nd Humbert function is connected with simplex

Technical details explained for Dunkl Bessel function, not Dunkl kernel

D. Constales, H. De Bie, P. Lian, Explicit formulas for the Dunkl dihedral kernel and the (κ, a)-generalized Fourier kernel. *J. Math. Anal. Appl.* **460** (2018), 900-926.

H. De Bie, P. Lian, The Dunkl kernel and intertwining operator for dihedral groups. *J. Funct. Anal.* **280**, 108932 (2021)

General formulas:

Theorem

Let p be a polynomial and $K(iy, z) := e^{-\Delta_y/2} E_\kappa(iy, z)$. Then for any $z \in \mathbb{R}^m$, the intertwining operator V_κ satisfies

$$V_\kappa(p)(z) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K(iy, z) \mathcal{F}\left(p(\cdot)e^{-|\cdot|^2/2}\right)(y) dy.$$

Here

- $\mathcal{F}$ is the classical Fourier transform
- $\Delta_y = \sum_{j=1}^m \partial_j^2$
- using an identity of Macdonald

$$e^{-\Delta_y/2} E_\kappa(iy, z) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle} e^{-|x|^2/2} E_\kappa(-x, z) e^{|y|^2/2} dx$$

Formal manipulations:

Recall $K(iy,z) := e^{-\Delta_y/2}E_\kappa(iy,z)$

$$\begin{aligned}
V_\kappa(\partial_j p)(z) &= \int_{\mathbb{R}^m} K(iy,z)\mathcal{F}\left((\partial_j p)(\cdot)e^{-|\cdot|^2/2}\right)(y)dy \\
&= \int_{\mathbb{R}^m} K(iy,z)\mathcal{F}\left((\partial_j + x_j)(p(x)e^{-|x|^2/2})\right)(y)dy \\
&= \int_{\mathbb{R}^m} K(iy,z)i(y_j + \partial_{y_j})\mathcal{F}\left(p(x)e^{-|x|^2/2}\right)(y)dy \\
&= \int_{\mathbb{R}^m} \left(i(y_j - \partial_{y_j})K(iy,z)\right)\mathcal{F}\left(p(x)e^{-|x|^2/2}\right)(y)dy \\
&= \int_{\mathbb{R}^m} \left(e^{-\Delta_y/2}\left(iy_jE_\kappa(iy,z)\right)\right)\mathcal{F}\left(p(x)e^{-|x|^2/2}\right)(y)dy \\
&= T_jV_\kappa(p)(z)
\end{aligned}$$

Remarks:

- when $\kappa = 0$ then $V_\kappa = \mathrm{id}$
- case $G = \mathbb{Z}_2$ is easy to verify
- the general formula is **probably new**
- similar formula exists for V_κ^{-1} which satisfies

$$\partial_j V_\kappa^{-1} = V_\kappa^{-1} T_j$$

Generalized Bessel function GBF and invariant polynomials:

Symmetric version of the Dunkl kernel

$$\boxed{\mathcal{J}_\kappa(x,y) = \frac{1}{|G|}\sum_{g\in G} E_\kappa(x, g\cdot y)}$$

Corollary

Let p be a G-invariant polynomial. Then for any $z \in \mathbb{R}^m$, the intertwining operator V_κ satisfies

$$V_\kappa(p)(z) = \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}\left(e^{-\Delta_y/2}\mathcal{J}_\kappa(iy,z)\right)\mathcal{F}\left(p(\cdot)e^{-|\cdot|^2/2}\right)(y)dy.$$

G-invariant means $p(gz) = p(z)$ for all $g \in G$

Now let us specialize to dihedral groups.

Steps:

- compute GBF as Humbert function
- Humbert function represented by integral over simplex
- use previous result to get intertwining operator

We have done the same thing for the Dunkl kernel and the full intertwining operator
More complicated so I won't show those details

2nd Humbert function:

$$\boxed{\Phi_2^{(m)}(\beta_1, \ldots, \beta_m; \gamma; x_1, \ldots, x_m) := \sum_{j_1,\ldots,j_m \geq 0} \frac{(\beta_1)_{j_1} \cdots (\beta_m)_{j_m}}{(\gamma)_{j_1+\cdots+j_m}} \frac{x_1^{j_1}}{j_1!} \cdots \frac{x_m^{j_m}}{j_m!}}$$

Integral representation:

$$\Phi_2^{(m)} \sim \int_{T^m} e^{\sum_{j=1}^m x_j t_j} \left(1 - \sum_{j=1}^m t_j\right)^{\gamma - \sum_{j=1}^m \beta_j - 1} \prod_{j=1}^m t_j^{\beta_j - 1} dt_1 \ldots dt_m$$

with T^m the open unit simplex in $\mathbb{R}^m$ given by

$$T^m = \{(t_1, \ldots, t_m) : t_j > 0, j = 1, \ldots, m, \sum_{j=1}^m t_j < 1\}.$$

Generalized Bessel function GBF

Theorem (Demni, J. Lie Theory, 2012)

The GBF $\mathcal{J}_\kappa(z,w)$ associated to $I_{2k}, k \geq 2$ and $\kappa = (\alpha, \beta)$ is

$$\begin{aligned}\mathcal{J}_\kappa(z,w) \;\sim\; & \int_{-1}^{1}\int_{-1}^{1}\Big(f_{2k,\alpha+\beta}(|zw|, \xi_{u,v}(k\phi_1, k\phi_2), 1) \\ & + f_{2k,\alpha+\beta}(|zw|, -\xi_{u,v}(k\phi_1, k\phi_2), 1)\Big) d\nu^\alpha(u) d\nu^\beta(v)\end{aligned}$$

with $\xi_{u,v}(\phi_1, \phi_2) = v\cos(\phi_1)\cos(\phi_2) + u\sin(\phi_1)\sin(\phi_2)$ *and*

$$f_{2k,\lambda}(b, \xi, t) = \left(\frac{2}{b}\right)^{k\lambda} \sum_{j=0}^{\infty} \frac{j+\lambda}{\lambda} \mathcal{I}_{k(j+\lambda)}(bt) C_j^{(\lambda)}(\xi)$$

with $C_j^{(\lambda)}$ *Gegenbauer polynomial and* $\mathcal{I}_k$ *mod. Bessel function*

Here $z = |z|e^{i\phi_1}$, $w = |w|e^{i\phi_2}$ and $b = |z||w|$

We find a new expression for

$$f_{2k,\lambda}(b,\xi,t) = \left(\frac{2}{b}\right)^{k\lambda} \sum_{j=0}^{\infty} \frac{j+\lambda}{\lambda} \mathcal{I}_{k(j+\lambda)}(bt) C_j^{(\lambda)}(\xi)$$

We find a new expression for (later put $t = 1$)

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Take Laplace transform with respect to $t \to s$ with $S = \sqrt{s^2 - b^2}$

$$\mathcal{L}(f_{2k,\lambda})(s) = \left(\frac{2}{S(s+S)}\right)^{k\lambda} \boxed{\sum_{j=0}^{\infty} \frac{j+\lambda}{\lambda} \left(\frac{b^k}{(s+S)^k}\right)^j C_j^{(\lambda)}(\xi)}$$

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Now we can use the **Poisson formula**:

$$\boxed{\frac{1-z^2}{(1-2\xi z + z^2)^{\lambda+1}} = \sum_{j=0}^{\infty} \frac{j+\lambda}{\lambda} C_j^{\lambda}(\xi) z^j}$$

to get a closed form in the Laplace domain!

Lemma

For $k \geq 2$, the Laplace transform of $f_{2k,\lambda}(b, \xi, t)$ with respect to t is

$$\begin{aligned}
\mathcal{L}[f_{2k,\lambda}(b, \xi, \cdot)](s) &\sim \frac{1}{S} \frac{(S+s)^k - (s-S)^k}{((S+s)^k - 2b^k\xi + (s-S)^k)^{\lambda+1}} \\
&\sim \frac{d}{ds} \left(\frac{1}{((S+s)^k - 2b^k\xi + (s-S)^k)^{\lambda}} \right)
\end{aligned}$$

where $S = \sqrt{s^2 - b^2}$.

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For $k \geq 2$, the Laplace transform of $f_{2k,\lambda}(b,\xi,t)$ with respect to t is

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\begin{aligned}
\mathcal{L}[f_{2k,\lambda}(b,\xi,\cdot)](s) &\sim \frac{1}{S}\frac{(S+s)^k-(s-S)^k}{((S+s)^k-2b^k\xi+(s-S)^k)^{\lambda+1}} \\
&\sim \frac{d}{ds}\left(\frac{1}{((S+s)^k-2b^k\xi+(s-S)^k)^{\lambda}}\right)
\end{aligned}
$$

where $S=\sqrt{s^2-b^2}$.

Looks like a complicated function...

- ... but it turns out that the denominator $(S+s)^k-2b^k\xi+(s-S)^k$ is a polynomial in s that can be factored completely!

We have ($S = \sqrt{s^2 - b^2}$)

$$\mathcal{L}(f_{2k,\lambda})(s) \sim \frac{d}{ds}\left(\frac{1}{(S+s)^k - 2b^k\xi + (s-S)^k}\right)^\lambda$$

with

$$\frac{1}{2^k}\left((S+s)^k - 2b^k\xi + (s-S)^k\right) \quad = \quad \prod_{l=0}^{k-1}\left(s - b\cos\left(\frac{q-2\pi l}{k}\right)\right),$$

where $q = \arccos\xi$

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where $q = \arccos\xi$

How to invert this Laplace transform?

- λ integer: use partial fraction decomposition to find expression in elementary functions
- λ non integer: recognize Humbert $\Phi_2^{(k)}$
- and put $t = 1$

We started from

$$f_{2k,\lambda}(b,\xi,t) = \left(\frac{2}{b}\right)^{k\lambda} \sum_{j=0}^{\infty} \frac{j+\lambda}{\lambda} \mathcal{I}_{k(j+\lambda)}(bt) C_j^{(\lambda)}(\xi)$$

to reach

Theorem

For $k \geq 2$, we have

$$\begin{aligned} f_{2k,\lambda}(b,\xi,1) &= \Phi_2^{(k)}(\lambda,\ldots,\lambda;k\lambda;b_0,\ldots,b_{k-1}) \\ &= e^{b_0}\Phi_2^{(k-1)}(\lambda,\ldots,\lambda;k\lambda;b_1-b_0,\ldots,b_{k-1}-b_0) \end{aligned}$$

where $b_j = b\cos((q-2j\pi)/k)$, $j=0,\ldots,k-1$ in which $q = \arccos(\xi)$.

We started from

$$f_{2k,\lambda}(b,\xi,t) = \left(\frac{2}{b}\right)^{k\lambda} \sum_{j=0}^{\infty} \frac{j+\lambda}{\lambda} \mathcal{I}_{k(j+\lambda)}(bt) C_j^{(\lambda)}(\xi)$$

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Theorem

For $k \geq 2$, we have

$$\begin{aligned} f_{2k,\lambda}(b,\xi,1) &= \Phi_2^{(k)}(\lambda,\ldots,\lambda;k\lambda;b_0,\ldots,b_{k-1}) \\ &= e^{b_0}\Phi_2^{(k-1)}(\lambda,\ldots,\lambda;k\lambda;b_1-b_0,\ldots,b_{k-1}-b_0) \end{aligned}$$

where $b_j = b\cos((q-2j\pi)/k)$, $j = 0,\ldots,k-1$ in which $q = \arccos(\xi)$.

- Is this not conservation of misery?
- No: Humbert $\Phi_2^{(k)}$ can be represented by integral of exponential over simplex!

Theorem

The GBF associated to the dihedral group $I_{2k}, k \geq 2$, is

$$\begin{aligned}\mathcal{J}_\kappa(z,w) \quad \sim \quad & \int_{-1}^{1}\int_{-1}^{1}\int_{T^{k-1}} \left(e^{\sum_{j=0}^{k-1} a_j^+ t_j} + e^{\sum_{j=0}^{k-1} a_j^- t_j}\right) \\ & \times \prod_{j=0}^{k-1} t_j^{\alpha+\beta-1} dt_1 \ldots dt_{k-1} d\nu^\alpha(u) d\nu^\beta(v)\end{aligned}$$

where $t_0 = 1 - \sum_{j=1}^{k-1} t_j$ *and* $a_j^- = b\cos\left(\frac{q-2j\pi}{k}\right)$,
$a_j^+ = b\cos\left(\frac{\pi-q-2j\pi}{k}\right)$, $j = 0, \ldots, k-1$ *and* $q = \arccos\xi$.

$$\xi(u,v) = u\cos(k\phi_1)\cos(k\phi_2) + v\sin(k\phi_1)\sin(k\phi_2)$$

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Immediate advantage/consequence:

- $\mathcal{J}_\kappa(x,y) > 0$
- complexified GBF is bounded by 1

$\Rightarrow$ this is indeed a 'useful' formula

Dunkl kernel for dihedral group I_k: with the auxiliary variable t

Starting point for $x, y \in \mathbb{R}^2$ and $b = |x||y|$

$$\boxed{E_\kappa(-ix, y, t) \sim \sum_{j=0}^{\infty} (-i)^j b^{-\langle \kappa \rangle} J_{j+\langle \kappa \rangle}(bt) P_j \left(I_k; \frac{x}{|x|}, \frac{y}{|y|} \right)}$$

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Approach and complications

- We can still Laplace transform in t
- However, P_j is no longer a Gegenbauer polynomial
- Instead of Poisson kernel, we have to use Dunkl Poisson kernel
- Fortunately Dunkl Poisson kernel is already established

S. Ben Saïd, On the integrability of a representation of $sl(2, \mathbb{R})$. *J. Funct. Anal.* **250** (2007), 249-264.

C. F. Dunkl, Poisson and Cauchy kernels for orthogonal polynomials with dihedral symmetry, *J. Math. Anal. Appl.* **143** (1989), 459-470.

Theorem

For each dihedral group I_{2k} and positive multiplicity function κ, the Dunkl kernel is given by

$$E_\kappa(z,w) = \int_{-1}^{1}\int_{-1}^{1}\left[(1+u)(1+v) - \frac{2}{\alpha+\beta}(\alpha u(1+v) + \beta v(1+u))\right] \times h_{\alpha+\beta}(q(u,v))d\nu^\alpha(u)d\nu^\beta(v).$$

where

$$h_\alpha(q(u,v)) = \Phi_2^{(k+1)}(\alpha,\ldots,\alpha,1;k\alpha+1;a_0,\ldots,a_{k-1},a_k)$$

with $a_l = b\cos\left(\frac{q(u,v)+2\pi l}{k}\right)$, $l = 0,\ldots k-1$ *and* $a_k = \mathrm{Re}\,(z\overline{w})$.

- recall that $\Phi_2^{(k+1)}$ is integral over simplex
- from this expression, the Dunkl kernel satisfies $E_\kappa(z,w) \geq 0$

The full intertwining operator

From the expression for E_κ:

- expression for full intertwining operator
- various other expressions for V_κ, also on restricted classes of functions
- conceptual proof of Xu's result

Outro 1:

I claim that we have found ‘good’ formulas for the Dunkl kernel and intertwining operator for dihedral groups:

- the abstract results (positivity, boundedness) follow now by observation
- our approach follows a structural method

Nevertheless, there may be better formulas from a computational point of view:

- it is possible to rewrite our results in various other ways

Outro 2: Intertwining operator for symmetric groups, theorem by Xu

Let V_κ be the intertwining operator associated to the symmetric group S_d. Our main result in this section is the following integral representation of V_κ. Let T^{d-1} denote the simplex

$$T^{d-1} := \{u \in \mathbb{R}^{d-1} : t_1 \geq 0, \dots, t_{d-1} \geq 0,\, t_1 + \cdots + t_{d-1} \leq 1\}.$$

Written in homogeneous coordinates of $\mathbb{R}^d$, it is equivalent to the simplex

$$\mathcal{T}^d = \{(t_0, \dots, t_{d-1}) \in \mathbb{R}^d : t_i \geq 0, \quad t_0 + t_1 + \cdots + t_{d-1} = 1\}.$$

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$. For $1 \leq \ell \leq d$, define $F(x_1, \dots, x_d) = f(x_\ell)$. Let*

$$V_\kappa F(x) = c_\kappa \int_{\mathcal{T}^d} f(x_1 t_0 + x_2 t_1 + \cdots + x_d t_{d-1}) t_{\ell-1} (t_0 \dots t_{d-1})^{\kappa-1} \mathrm{d}t, \tag{2.1}$$

where the constant c_κ is given by

$$c_\kappa = c_{\kappa,d} = \Gamma(d\kappa + 1)/(\kappa \Gamma(\kappa)^d).$$

Then the operator V_κ satisfies

$$D_i V_\kappa F(x) = V_\kappa(\partial_i F)(x), \qquad 1 \leq i \leq d.$$

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Then the operator V_κ satisfies

$$D_i V_\kappa F(x) = V_\kappa(\partial_i F)(x), \qquad 1 \leq i \leq d.$$

- proof elementary (1 page)
- restricted class of functions
- again appearance of simplex!

Y. Xu, Intertwining operator associated to symmetric groups and summability on the unit sphere. *J. Approx. Theory* 272 (2021), 105649, 16 pp.

Our approach:

- consider trigonometric Dunkl case and take limit
- find GBF and Dunkl kernel via 2nd Humbert function
- explains links with simplex!
- reobtain intertwining operator from Dunkl kernel

Theorem

Assume $\kappa \geq 0$, $\nu \in \mathbb{C}$ and $x \in \mathbb{V} \subset \mathbb{R}^n$. The GBF for A_{n-1} is

$$\begin{aligned} J_\kappa(\lambda, x) &= \Phi_2^{(n)}[\kappa, \ldots, \kappa, n\kappa; \nu x_1, \ldots, \nu x_n] \\ &= e^{\nu x_n} \Phi_2^{(n-1)}[\kappa, \ldots, \kappa, n\kappa; \nu(x_1 - x_n), \ldots, \nu(x_{n-1} - x_n)] \end{aligned}$$

where $\lambda = \left(-\frac{\nu}{n}, \ldots, -\frac{\nu}{n}, \frac{(n-1)\nu}{n}\right)$ *and the hyperplane $\mathbb{V}$ of $\mathbb{R}^n$ is*

$$\mathbb{V} = \{x \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 0\}.$$

N. Shimeno, Y. Tamaoka. *The hypergeometric function for the root system of type A with a certain degenerate parameter.* Tsukuba J. Math. 42(2): 155-172 (2018).

H. De Bie, P. Lian, Dunkl intertwining operator for symmetric groups. *Proc. Amer. Math. Soc.* **149**, 4871-4880 (2021).