The diagonal centraliser of sl_3 and its E_6-symmetry

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Modern Analysis Related to Root Systems with Applications

CIRM, October 2021

Diagonal centralisers

Let $\mathfrak{g}$ a Lie algebra, $\quad U(\mathfrak{g})^{\otimes n} = U(\mathfrak{g}) \otimes \cdots \otimes U(\mathfrak{g})$.

The diagonal embedding :

$$\delta^{(n)} \;:\; \begin{array}{ccc} U(\mathfrak{g}) & \to & U(\mathfrak{g})^{\otimes n} \\ g & \mapsto & g \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes g \end{array} \qquad (g \in \mathfrak{g})$$

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The diagonal centraliser is :

$$Z_n(\mathfrak{g}) = \{x \in U(\mathfrak{g})^{\otimes n} \mid [x, \delta^{(n)}(g)] = 0\,, \;\; \forall g \in \mathfrak{g}\}\ .$$

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Example. $\underline{n = 1 :}$ $Z_1(\mathfrak{g})$ is the center of $U(\mathfrak{g})$.

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- Rems. :

- ($n > 3$) "Higher rank" Racah algebra : actively studied (multivar. OPs)
- q-deformation for $U_q(sl_2) \rightsquigarrow$ Askey–Wilson algebra.

Goal of the talk

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Goal of the talk :

$$Z_2(sl_3)$$

- A first example outside sl_2.
- Can we describe this algebra ? (Do we get an algebra similar to the Racah algebra ?)
- Application :

  Tensor product of two irreps of sl_3 : $sl_3 \curvearrowright V \otimes W \curvearrowleft$ centraliser ?

$$Z_2(sl_3) \twoheadrightarrow \mathrm{End}_{U(sl_3)}(V \otimes W)$$

"Universal centraliser" "Represented centraliser"

Algebraic description

- **Generators of** $Z_2(sl_3)$: *(fundamental invariants for* $SL(3) \curvearrowright U(sl_3)^{\otimes 2}$*)*

$$\boxed{\underbrace{k_1\,,\ k_2\,,\ k_3\,,\ \ell_1\,,\ \ell_2\,,\ \ell_3}_{\text{central elements = "parameters"}} \quad \text{and} \quad X\,,\ Y}$$

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- **Relations of** $Z_2(sl_3)$:

$$\begin{aligned} [X,Y] &= Z && (a_2, a_5, a_6, a_8, a_9, a_{12} \in \mathbb{C}[k_\cdot, \ell_\cdot]) \\ [X,Z] &= -6Y^2 + a_2X^2 + a_5X + a_8 \\ [Y,Z] &= -2X^3 - a_2\{X,Y\} - a_5Y + a_6X + a_9 \\ \Omega &= a_{12} && (\Omega \text{ is a certain central element, } \Omega = Z^2 + \dots) \end{aligned}$$

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$(a_2, a_5, a_6, a_8, a_9, a_{12} \in \mathbb{C}[k_\cdot, \ell_\cdot])$

(Ω is a certain central element, $\Omega = Z^2 + \ldots$)

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- To be compared with the Racah algebra (Heun-Hahn algebra?)

The missing label

Take 3 irreps of $sl_3 \leftrightarrow$ 3 highest weights : $\underbrace{(m_1, m_2), (m'_1, m'_2), (m''_1, m''_2)}_{\underline{m}}$

$$V_{(m_1,m_2)} \otimes V_{(m'_1,m'_2)} \cong \quad \ldots\ldots \quad \oplus \quad M_{\underline{\mathbf{m}}} \otimes V_{(m''_1,m''_2)} \quad \oplus \quad \ldots\ldots$$

↑ "multiplicity space" of dimension $d_{\underline{m}}$

• The problem : We can NOT unambiguously specify a vector in $V_{(m_1,m_2)} \otimes V_{(m'_1,m'_2)}$ using only operators from the diagonal sl_3 ($d_{\underline{\mathbf{m}}} > 1$).

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Example :  $\otimes$  $\cong \ldots \oplus 2$  $\oplus \ldots \rightsquigarrow$ 2-dim irrep of $Z_2(sl_3)$

(We have calculated the explicit matrices representing $Z_2(sl_3)$ on $M_{\underline{\mathbf{m}}}$, $\forall \underline{\mathbf{m}}$.)

(Bethe Ansatz to find the eigenvalues)

Symmetries

We have :

- The algebra $Z_2(sl_3)$;
- its representations on the multiplicity spaces $M_{\underline{\mathbf{m}}}$, $\forall \underline{\mathbf{m}}$.

Question

What are the transformations on the parameters $\underline{\mathbf{m}}$ which preserve :

- The algebra $Z_2(sl_3)$?
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"specialised centraliser"

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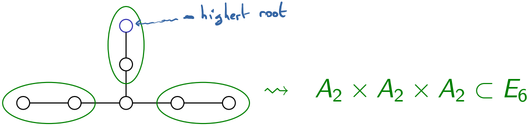
- Embed $A_2 \times A_2 \times A_2$ into E_6 :



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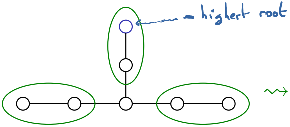

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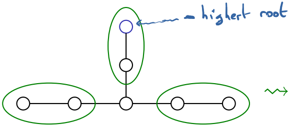

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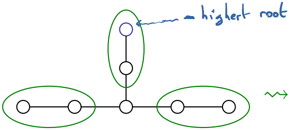

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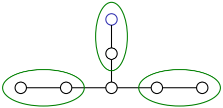
Theorem (Symmetry of the algebra : $W(E_6)$)

$s \in W(E_6)$: $\quad \underline{\boldsymbol{m}}' = s.\underline{\boldsymbol{m}} \quad \Rightarrow \quad Z_2(sl_3)_{\underline{\boldsymbol{m}}} \cong Z_2(sl_3)_{\underline{\boldsymbol{m}}'}$.



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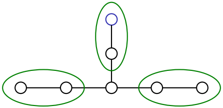

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- Rem. : [a particular case of Etingof-Loktev-Oblomkov-Rybnikov '08]

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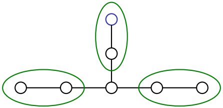

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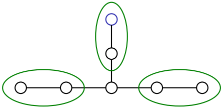

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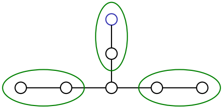

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- **Application to the missing label.**

Theorem (Symmetry of the missing label)

We extract from $W(E_6)$ the symmetry group of the missing label.

$\rightsquigarrow$ A group of order 144, explicitly described, containing the 12 natural symmetries + other "hidden" symmetries.

Thank You