Oscillating multipliers on rank one locally symmetric spaces

Effie Papageorgiou

University of Crete,
Department of Mathematics and Applied Mathematics

Modern Analysis Related to Root Systems with Applications,
Marseille, Luminy, CIRM

October 20, 2021

Supported by the Hellenic Foundation for Research and Innovation, Project HFRI-FM17-1735

The problem in $\mathbb{R}^n$

- Multiplier transformation

$$(\widehat{T_{\alpha,\beta} f})(\xi) = m_{\alpha,\beta}(\xi)\hat{f}(\xi), \quad f \in C_0^\infty(\mathbb{R}^n),$$

where

$$m_{\alpha,\beta}(\xi) = \|\xi\|^{-\beta} e^{i\|\xi\|^\alpha}\theta(\xi), \quad \alpha > 0,\ \beta > 0,$$

and θ a smooth function, vanishing near zero.

- How much cancellation to lead to estimates of the form

$$\|T_{\alpha,\beta} f\|_p \lesssim \|f\|_p?$$

Noncompact symmetric spaces X=G/K

- G noncompact semisimple Lie group (connected, finite center)
- K maximal compact subgroup
- Cartan decomposition $\sim$ polar decomposition:

$$G = K(\exp \overline{\mathfrak{a}^{+}})K \ \rightsquigarrow \ X = K(\exp \overline{\mathfrak{a}^{+}})$$

- The rank of G/K is the dimension of $\mathfrak{a}$

Rank one noncompact symmetric spaces $X = G/K$

- Rank one: $\mathfrak{a} \simeq \mathbb{R}$, $\mathfrak{a}^{+} \simeq \mathbb{R}_{+}$.

Rank one noncompact symmetric spaces $X = G/K$

- Rank one: $\mathfrak{a} \simeq \mathbb{R}$, $\mathfrak{a}^+ \simeq \mathbb{R}_+$.
- Bottom of the L^2 spectrum of $-\Delta$ equal to ρ^2.
- Classification in rank 1:

X	$\mathbb{H}^n(\mathbb{R})$	$\mathbb{H}^n(\mathbb{C})$	$\mathbb{H}^n(\mathbb{H})$	$\mathbb{H}^2(\mathbb{O})$
G	$SO(n,1)$	$SU(n,1)$	$Sp(n,1)$	$F_{4(-20)}$
K	$SO(n)$	$S[U(n) \times U(1)]$	$Sp(n)$	$SO(9)$
ρ	$\frac{n-1}{2}$	n	$2n+1$	11

Spherical Fourier transform

For good enough K-bi-invariant (radial) functions on G, the spherical Fourier transform $\mathcal{H}$ is defined by

$$\mathcal{H}f(\lambda) = \int_G f(x)\,\varphi_{-\lambda}(x)\,dx, \quad \lambda \in \mathbb{R}.$$

Spherical Fourier transform

For good enough K-bi-invariant (radial) functions on G, the spherical Fourier transform $\mathcal{H}$ is defined by

$$\mathcal{H}f(\lambda) = \int_G f(x)\,\varphi_{-\lambda}(x)\,dx, \quad \lambda \in \mathbb{R}.$$

Inversion formula

$$f(x) = \text{const.} \int_{\mathbb{R}} \mathcal{H}f(\lambda)\varphi_\lambda(x)\,\frac{d\lambda}{|\mathbf{c}(\lambda)|^2}, \quad x \in G,$$

where $|\mathbf{c}(\lambda)|^{-2}$ is the Plancherel density, and $\varphi_\lambda(x)$ the elementary spherical functions.

- Γ=discrete, torsion free subgroup of G
- Locally symmetric space $M = \Gamma \backslash X = \Gamma \backslash G/K$: Riemannian manifold.

Example

Every compact Riemann surface of genus ≥ 2: quotient of the upper half plane $\rightsquigarrow$ hyperbolic plane when endowed with the Poincaré metric.

Poincaré series

- Poincaré series

$$P_s(xK, yK) = \sum_{\gamma \in \Gamma} e^{-sd(xK, \gamma yK)}, \quad \forall s > 0,\ x, y \in G.$$

- Critical exponent of the group Γ

$$\begin{aligned} \delta(\Gamma) &= \inf \{ s > 0 : P_s(xK, yK) < +\infty \} \\ &= \limsup_{R \to +\infty} \frac{\log N_R(xK, yK)}{R}, \end{aligned}$$

where $N_R(xK, yK) = |\gamma \in \Gamma : \ d(xK, \gamma yK) \leq R|$ the orbital counting function.

- $\delta(\Gamma)$ independent of the choice of xK and yK.

Poincaré series

- Critical exponent of Γ thus measures the exponential growth rate of Γ orbits in X.
- Poincaré series converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$.
- Always $0 \leq \delta(\Gamma) \leq 2\rho$.

Example

- If $G = KAN$ and $\Gamma = \langle a \rangle$, $a \in A$, then $\delta(\Gamma) = 0$.
- If $\Gamma \subset G = SL(2, \mathbb{R})$ contains unipotent elements, then $\delta(\Gamma) > c(G) > 0$ where $c(G)$ is some universal constant.
- If Γ a lattice, i.e. $[G : \Gamma] < \infty$, then $\delta(\Gamma) = 2\rho$.

Oscillating multiplier operator:

$$T_{\alpha,\beta}(f)(x) = \int_G \kappa_{\alpha,\beta}(y^{-1}x) f(y)\, dy, \quad f \in C_0^\infty(X),$$

where

$$\kappa_{\alpha,\beta} = \mathcal{H}^{-1} m_{\alpha,\beta},$$

and

$$m_{\alpha,\beta}(\lambda) = (\lambda^2+\rho^2)^{-\beta/2} e^{i(\lambda^2+\rho^2)^{\alpha/2}}, \quad \alpha > 0,\ \beta > 0.$$

Oscillating multipliers on symmetric spaces

Theorem (Giulini, Meda 1990)

Assume that X is an n-dimensional rank one symmetric space.
(i) If $\alpha < 1$, then $T_{\alpha,\beta}$ is bounded on $L^p(X)$, $p \in (1, \infty)$, provided that $\beta > \alpha n|1/p - 1/2|$.
(ii) If $\alpha = 1$, then $T_{\alpha,\beta}$ is bounded on $L^p(X)$, $p \in (1, \infty)$, provided that $\beta > (n-1)|1/p - 1/2|$.
(iii) If $\alpha > 1$, then $T_{\alpha,\beta}$ is bounded only on $L^2(X)$.

Theorem (Ionescu 2000)

(ii) is true even for $\beta = (n-1)|1/p - 1/2|$.

The results resemble closely the euclidean ones, $0 < \alpha \leq 1$:

$$\kappa_{\alpha,\beta} = \kappa^0_{\alpha,\beta} + \kappa^\infty_{\alpha,\beta} \longleftrightarrow T_{\alpha,\beta} = T^0_{\alpha,\beta} + T^\infty_{\alpha,\beta}$$

- Part at infinity $T^\infty_{\alpha,\beta}$ behaves nicely (bounded on $L^p(X)$, for all $1 < p < \infty$),
- Local part $T^0_{\alpha,\beta}$ is essentially euclidean.

$$T_{\alpha,\beta}f(x) = \int_G \kappa_{\alpha,\beta}(y^{-1}x)f(y)dy = \int_G \kappa_{\alpha,\beta}(y^{-1})f(xy)dy,$$

for all smooth compactly supported functions on X.

$$T_{\alpha,\beta}f(x) = \int_G \kappa_{\alpha,\beta}(y^{-1}x)f(y)dy = \int_G \kappa_{\alpha,\beta}(y^{-1})f(xy)dy,$$

for all smooth compactly supported functions on X.

- If f is Γ-left-invariant, then so is $T_{\alpha,\beta}f$.
- $\widehat{T}_{\alpha,\beta}$: the restriction of $T_{\alpha,\beta}$ on Γ-left-inv. functions on X.
- Is $\widehat{T}_{\alpha,\beta}$ bounded on $L^p(M)$?

The case $0 < \alpha < 1$

- For the local part $\widehat{T}_{\alpha,\beta} = *\kappa^0_{\alpha,\beta}$ of the operator we use the following result.
- Lohoué-Marias2009: Let $S_\kappa f(x) = \int_G f(y)\kappa(y^{-1}x)dy$ be a convolution operator with a K-bi-invariant kernel κ. Assume κ^0 is supported around the origin. If $S^0_\kappa = *\kappa^0$ is L^p bounded on G, $p \in (1,\infty)$, then it is L^p bounded on M.
- Transfer the local result of Giulini-Meda for $T^0_{\alpha,\beta}$ on X to M, to get that $\widehat{T}^0_{\alpha,\beta}$ is $L^p(M)$ bounded, $p \in (1,\infty)$.
- This implies the necessary condition $\beta > \alpha n|1/p - 1/2|$.
- For the part at infinity, we use a Kunze and Stein type phenomenon.

Kunze and Stein phenomenon on X

- On X: if $p \geq 1$ and κ is a K-bi-invariant (radial) kernel, then

$$|| * |\kappa| \, ||_{L^p(X) \to L^p(X)} = C \int_G |\kappa(g)| \, \varphi_{-i\rho_p}(g) dg$$
$$\leq C \int_{\mathbb{R}_+} |\kappa(r)| \, \varphi_{-i\rho_p}(r) e^{2\rho r} \, dr,$$

where $r = \mathrm{dist}(gK, eK)$ on X and $\rho_p = \left| \frac{2}{p} - 1 \right|$.

Kunze and Stein phenomenon on M

- Variants on locally symmetric spaces $M = \Gamma\backslash X$, if Γ is "good" enough: Lohoué-Marias2012, Zhang2019:

$$\| * |\kappa| \|_{L^p(M) \to L^p(M)} \lesssim \int_G |\kappa(g)| \, \varphi_{good}(g)^{s(p)} dg,$$

  where $s(p) = 2\min\{1/p, 1/p'\}$, $p \in (1, \infty)$.
- vector good in LM'12 is equal to $-i\eta_\Gamma$, $|\eta_\Gamma|^2 = \rho^2 - \lambda_0$,
- vector good in Z'19 is equal to 0.

Then we can follow some ideas introduced by Anker1990.

Aim: Show that if Γ is "good", then the operator $\widehat{T}^{\infty}_{\alpha,\beta}$ is bounded on L^p, $p \in (1, \infty)$.

The case $0 < \alpha < 1$: Part at infinity

$$\|\widehat{T}^{\infty}_{\alpha,\beta}\|_{p\to p} \lesssim \int_G |\kappa^{\infty}_{\alpha,\beta}(g)|\varphi_{good}(g)dg \lesssim \int_{|g|\geq 1} |\kappa_{\alpha,\beta}(g)|\varphi_{good}(g)dg$$
$$= \sum_{j\geq 1}\int_{j\leq |g|\leq j+1} |\kappa_{\alpha,\beta}(g)|\varphi_{good}(g)dg := \sum_{j\geq 1} I_j.$$

- Cauchy-Schwarz:

$$I_j \leq \left(\int_{j\leq|g|\leq j+1} \varphi_{good}(g)^2 dg\right)^{1/2} \cdot \left(\int_{j\leq|g|\leq j+1} |\kappa_{\alpha,\beta}(g)|^2 dg\right)^{1/2}.$$

The case $0 < \alpha < 1$: Part at infinity

$$\|\widehat{T}^{\infty}_{\alpha,\beta}\|_{p\to p} \lesssim \int_G |\kappa^{\infty}_{\alpha,\beta}(g)|\varphi_{good}(g)dg \lesssim \int_{|g|\geq 1} |\kappa_{\alpha,\beta}(g)|\varphi_{good}(g)dg$$
$$= \sum_{j\geq 1} \int_{j\leq |g|\leq j+1} |\kappa_{\alpha,\beta}(g)|\varphi_{good}(g)dg := \sum_{j\geq 1} I_j.$$

- Cauchy-Schwarz:

$$I_j \leq \left(\int_{j\leq |g|\leq j+1} \varphi_{good}(g)^2 dg \right)^{1/2} \cdot \left(\int_{j\leq |g|\leq j+1} |\kappa_{\alpha,\beta}(g)|^2 dg \right)^{1/2}.$$

First integral: independent of the kernel, depends only on Γ and grows exponentially on j.

THE CASE $0 < \alpha < 1$: PART AT INFINITY

$$\|\widehat{T}^{\infty}_{\alpha,\beta}\|_{p\to p} \lesssim \int_G |\kappa^{\infty}_{\alpha,\beta}(g)|\varphi_{good}(g)dg \lesssim \int_{|g|\geq 1} |\kappa_{\alpha,\beta}(g)|\varphi_{good}(g)dg$$
$$= \sum_{j\geq 1}\int_{j\leq |g|\leq j+1} |\kappa_{\alpha,\beta}(g)|\varphi_{good}(g)dg := \sum_{j\geq 1} I_j.$$

- Cauchy-Schwarz:

$$I_j \leq \left(\int_{j\leq |g|\leq j+1} \varphi_{good}(g)^2 dg\right)^{1/2} \cdot \left(\int_{j\leq |g|\leq j+1} |\kappa_{\alpha,\beta}(g)|^2 dg\right)^{1/2}.$$

First integral: independent of the kernel, depends only on Γ and grows exponentially on j.
Second integral: L^2-norm of $\kappa_{\alpha,\beta}$ on the annulus. Plancherel to get to $m_{\alpha,\beta}(\lambda)$, use analyticity and how derivatives decay:

$$|\partial^J m_{\alpha,\beta}(\lambda)| \leq c(1+|\lambda|)^{-\beta-|J|(1-\alpha)}.$$

Case $\alpha = 1$

Theorem (P.)

If $0 < \delta(\Gamma) < 2\rho$, then $T_{1,\beta}$ is bounded on $L^p(M)$, $p \in (1, \infty)$, provided that $\beta > (n-1)|1/p - 1/2|$.

Schonbek:

$$T_{1,\beta}(f)(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty \sigma^{\beta-1}(f * q_\sigma)(x)\, d\sigma, \quad f \in C_0^\infty(X),$$

where $q_\sigma = \mathcal{H}^{-1}(e^{(i-\sigma)(\lambda^2+\rho^2)^{1/2}})$ a K-bi-invariant kernel and

$$T_{q_\sigma}(f)(x) = (f * q_\sigma)(x) = \int_G q_\sigma(y^{-1}x) f(y) dy.$$

Case α=1: Part at infinity

- Cartan decomposition of G:

$$(\gamma y)^{-1}x = k_\gamma \exp(t_\gamma H_0) k'_\gamma, \quad \|H_0\| = 1,$$

$$\rightsquigarrow q_\sigma((\gamma y)^{-1}x) = q_\sigma(\exp t_\gamma H_0).$$

- Giulini-Meda on $X = G/K$:

$$|q_\sigma(\exp tH_0)| \lesssim \begin{cases} \sigma(t+1)^{-3/2}e^{-2\rho t}, & \sigma > 1,\ t > 0, \\ (t+1)^{-3/2}e^{-2\rho t}, & 0 < \sigma \leq 1,\ t > 2. \end{cases}$$

- At infinity, interested at $t > 2$.
- On our class of M, $\widehat{q}_\sigma(x,y) = \sum_{\gamma\in\Gamma} q_\sigma(x,\gamma y)$ is well defined:

$$\sum_{\{\gamma\in\Gamma:\ d(x,\gamma y)>2\}} e^{-2\rho d(x,\gamma y)} \leq P_{2\rho}(x,y) < +\infty.$$

Case $\alpha = 1$

- So, $\widehat{T}_{q_\sigma} = \int_M \widehat{q}_\sigma(x,y) f(y) dy$ is defined as an integral operator on M.
- Giulini-Meda: $\|q_\sigma\|_{L^1(G/K)} \lesssim \begin{cases} \sigma & \sigma > 1, \\ \sigma^{(1-n)/2} & 0 < \sigma \leq 2. \end{cases}$
- The above imply $\widehat{T}_{q_\sigma}$ bounded on $L^\infty(M)$.
- $\|\widehat{T}_{q_\sigma}\|_{L^2(M)\to L^2(M)} \leq e^{-\lambda_0 \sigma}$, where λ_0 the bottom of the spectrum on M.
- $\lambda_0(M) = \begin{cases} \rho^2 & \text{if } 0 \leq \delta(\Gamma) \leq \rho \\ \rho^2 - (\delta(\Gamma) - \rho)^2 & \text{if } \rho \leq \delta(\Gamma) \leq 2\rho, \end{cases}$
  positive iff $0 < \delta(\Gamma) < 2\rho$.

Interpolation between $L^2(M)$ and $L^\infty(M)$:

$$\begin{aligned}\|\widehat{T}_{1,\beta}\|_{L^p(M)\to L^p(M)} &\leq \frac{1}{\Gamma(\beta)}\int_0^\infty \sigma^{\beta-1}\|\widehat{T}_{q_\sigma}\|_{L^p(M)\to L^p(M)}\, d\sigma \\ &\leq \frac{1}{\Gamma(\beta)}\int_0^1 \sigma^{\beta-1}\sigma^{(1-n)(1/2-1/p)}\, d\sigma \\ &+ \frac{1}{\Gamma(\beta)}\int_1^\infty \sigma^{\beta-1}e^{-k_p\sigma}\, d\sigma,\ k_p > 0.\end{aligned}$$

The second integral above is finite, while the first is convergent provided that

$$\beta > (n-1)(1/2-1/p).$$

Thank you for your attention!