Hypergeometric functions of type BC and standard multiplicities

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Modern analysis related to root systems with applications
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Spherical functions and hypergeometric functions

Two extensions of Harish-Chandra's theory of spherical functions on G/K:

• Heckman-Opdam hypergeometric functions
Introduced and studied by Heckman and Opdam.
Significant contributions by Cherednik, Schapira etc...

• τ-spherical functions, $\tau \in \widehat{K}$
Harmonic analysis on homogenous vector bundles on G/K
Relevant to our results: van Dijk-Pasquale(1999), Shimeno(1994), Heckman-Schlichtkrull (1994-book), Oda-Shimeno(2019)

When τ is one dimensional,
$L^1(G//K;\tau) = \{f \in L^1(G) : \ f(k_1 g k_2) = \tau(k_1^{-1}) f(g) \tau(k_2^{-1})\}$
is commutative

(G, K, τ) is a Gelfand triple
Characters of $L^1(G//K;\tau)$ are τ-spherical functions:

τ-spherical functions = Heckman-Opdam hypergeometric function $F_\lambda(m)$ multiplied with a cosh-like factor

m can assume non-positive values.

Standard multiplicities

Let Σ be a BC-root system in $\mathfrak{a}^*$. Positive roots of the form $\Sigma^+ = \Sigma_{\mathrm{s}}^+ \sqcup \Sigma_{\mathrm{m}}^+ \sqcup \Sigma_{\mathrm{l}}^+$, where

$$\Sigma_{\mathrm{s}}^+ = \left\{\frac{\beta_j}{2} : 1 \leq j \leq r\right\}, \quad \Sigma_{\mathrm{m}}^+ = \left\{\frac{\beta_j \pm \beta_i}{2} : 1 \leq i < j \leq r\right\}$$

$$\Sigma_{\mathrm{l}}^+ = \{\beta_j : 1 \leq j \leq r\}$$

Let $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}})$ be a (standard) multiplicity function on Σ satisfying $m_{\mathrm{m}} > 0, m_{\mathrm{s}} > 0, m_{\mathrm{s}} + 2m_{\mathrm{l}} > 0$

Some results

Let $F_\lambda(m, x)$ and $G_\lambda(m, x)$ be the corresponding Heckman-Opdam hypergeometric functions for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Modifying Schapira's methods we have

Theorem

1. F_λ *and* G_λ *are strictly positive for* $\lambda \in \mathfrak{a}$
2. $|F_\lambda| \leq F_{Re\ \lambda}, \quad |G_\lambda| \leq G_{Re\ \lambda}$
3. $\max\{|F_\lambda(x)|, |G_\lambda(x)|\} \leq \sqrt{|W|}e^{\max_{w \in W}(w\lambda)(x)}$ *for all* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ *and* $x \in \mathfrak{a}$,
4. $F_{\lambda+\mu}(x) \leq F_\mu(x)e^{\max_{w \in W}(w\lambda)(x)}$ *and* $G_{\lambda+\mu}(x) \leq G_\mu(x)e^{\max_{w \in W}(w\lambda)(x)}$ *for all* $\lambda \in \mathfrak{a}^*$, $\mu \in \overline{(\mathfrak{a}^*)^+}$ *and* $x \in \mathfrak{a}$,

(Last one follows from a Lemma of Koornwinder-Rösler-Voit)

Harish-Chandra series

For a non-generic point $\lambda = \lambda_0$ a series expansion for $F_\lambda(m)$ was obtained by following the steps given below:

Step I: List the possible singularities of the c-function and the coefficients $\Gamma_\mu(m;\lambda)$ at $\lambda = \lambda_0$.

Step II: Identify a polynomial p so that

$$\lambda \to p(\lambda)\Big(\sum_{w \in W} c(m; w\lambda) e^{(\lambda - \rho(m))(x)} \sum_{\mu \in 2\Lambda} \Gamma_\mu(m;\lambda) e^{-\mu(x)}\Big)$$

is holomorphic in a neighborhood of λ_0.

Step III: Write $F_{\lambda_0}(m) = a\,\partial(\pi)(pF_\lambda(m))|_{\lambda=\lambda_0}$ where $\partial(\pi)$ is the differential operator corresponding to the highest degree homogenous term in p and a, is a non-zero constant. This gives the series expansion of $F_{\lambda_0}(m)$.

Sharp Asymptotics

Theorem

$$F_{\lambda_0}(m;x) \asymp \Big[\prod_{\alpha \in \Sigma^0_{\lambda_0}} (1+\alpha(x)) \Big] e^{(\lambda_0 - \rho(m))(x)}$$

for $\lambda_0 \in \overline{(\mathfrak{a}^*)^+}$ *where* $\Sigma^0_{\lambda_0} = \{\alpha \in \Sigma^+_{\mathrm{s}} \cup \Sigma^+_{\mathrm{m}} : \langle \alpha, \lambda_0 \rangle = 0\}$.

Analogue of Helgason-Johnson theorem:

Theorem

Let m be a standard multiplicity function. Then $F_\lambda(m)$ *is bounded if and only if* $\lambda \in C(\rho(m)) + i\mathfrak{a}^*$, *where* $C(\rho(m))$ *is the convex hull of the set* $\{w\rho(m) : \; w \in W\}$.

Two parameter deformation of m

Triple-$(\mathfrak{a}, \Sigma, m)$
Σ-BC root system with $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}})$ (non-negative).
For any two parameters $\ell, \widetilde{\ell}$ we define a deformation $m(\ell, \widetilde{\ell})$ of m as follows:

$$m_\alpha(\ell, \widetilde{\ell}) = \begin{cases} m_{\mathrm{s}} + 2\ell & \text{if } \alpha \in \Sigma_{\mathrm{s}} \\ m_{\mathrm{m}} + 2\widetilde{\ell} & \text{if } \alpha \in \Sigma_{\mathrm{m}} \\ m_{\mathrm{l}} - 2\ell & \text{if } \alpha \in \Sigma_{\mathrm{l}} \, . \end{cases}$$

Applications

Define

$$u(x) = \prod_{j=1}^{r} \cosh\left(\frac{\beta_j(x)}{2}\right)$$

$$v(x) = \prod_{1 \leq i < j \leq r} \cosh\left(\frac{\beta_j(x) - \beta_i(x)}{2}\right) \cosh\left(\frac{\beta_j(x) + \beta_i(x)}{2}\right)$$

For $\xi \in \mathfrak{a}$ and $\ell, \widetilde{\ell} \in \mathbb{R}$ we define the Cherednik operator $T_{\ell,\widetilde{\ell},\xi}$ by

$$T_{\ell,\widetilde{\ell},\xi}(m) = u^{-\ell} v^{-\widetilde{\ell}} \circ T_{\xi}(m(\ell, \widetilde{\ell})) \circ u^{\ell} v^{\widetilde{\ell}}$$

$(\ell, \widetilde{\ell})$-Heckman-Opdam Laplacian,

$$T_{\ell,\widetilde{\ell},p_L}(m) = \sum_{j=1}^{r} T_{\ell,\widetilde{\ell},\xi_j}(m)^2 = u^{-\ell} v^{-\widetilde{\ell}} \circ T_{p_L}(m(\ell)) \circ u^{\ell} v^{\widetilde{\ell}}$$

$(\ell, \widetilde{\ell})$-hypergeometric functions

Define $(\ell, \widetilde{\ell})$-hypergeometric functions $F_{\ell,\widetilde{\ell},\lambda}(m)$ by the equations

$$D_{\ell,\widetilde{\ell},p}(m)f = p(\lambda)f \quad p \in S(\mathfrak{a}_{\mathbb{C}})^W$$

Theorem

$F_{\ell,\widetilde{\ell},\lambda}(m) = F_{-\ell+m_{\mathrm{l}}-1,\widetilde{\ell},\lambda}(m)$

Geometric case: $m_{\mathrm{l}} = 1$, even in ℓ

One has

$$F_{\ell,\widetilde{\ell},\lambda}(m) = u^{-\ell}\, v^{-\widetilde{\ell}}\, F_\lambda(m(\ell, \widetilde{\ell}))$$

$m(\ell, \widetilde{\ell})$ is a standard multiplicity function if $\ell \in (-\frac{m_{\mathrm{s}}}{2}, \frac{m_{\mathrm{s}}}{2} + m_{\mathrm{l}})$. Earlier results apply.

Bounded $(\ell, \widetilde{\ell})$-hypergeometric functions

Theorem

Assume that $m_{\mathrm{l}} \geq 1$ *and* $\widetilde{\ell} \geq 0$, $\ell \in (-\frac{m_{\mathrm{s}}}{2}, \frac{m_{\mathrm{s}}}{2} + m_{\mathrm{l}})$. *Then,* $F_{\ell,\widetilde{\ell},\lambda}(m)$ *is bounded if and only if* $\lambda \in C(\rho(m(2\widetilde{\ell}))) + i\mathfrak{a}^*_{\mathbb{C}}$, *where* $C(\rho(m(2\widetilde{\ell})))$ *is the convex hull of the set* $\{w\rho(m(2\widetilde{\ell})) : \ w \in W\}$.

Here $m(2\widetilde{\ell}) = m(0, 2\widetilde{\ell})$

Includes many cases of small K-types (Shimeno-Oda)

Sketch of the proof

$$|F_{\ell,\widetilde{\ell},\lambda}(m;x)| \leq F_{\ell,\widetilde{\ell},\mathrm{Re}\,\lambda}(m;x) = u^{-\ell}(x)v^{-\widetilde{\ell}}(x)F_{\mathrm{Re}\,\lambda}(m(\ell,\widetilde{\ell});x),$$

the maximum of $|F_{\ell,\widetilde{\ell},\lambda}(m;x)|$ is attained at $\{w\rho(m(2\widetilde{\ell})) : \ w \in W\}$.

$$|F_{\ell,\lambda}(m;x)| \leq u^{-\ell}(x)v^{-\widetilde{\ell}}(x)F_{\rho(m(2\widetilde{\ell}))}(m(\ell,\widetilde{\ell});x).$$

$$\rho(m(2\widetilde{\ell})) = \rho(m(\ell,\widetilde{\ell})) + \frac{\ell}{2}\sum_{j=1}^{r}\beta_j + \frac{\widetilde{\ell}}{2}\sum_{1\leq i<j\leq r}(\beta_j \pm \beta_i).$$

$\rho(m(\ell,\widetilde{\ell})) \in \overline{(\mathfrak{a}^*)^+}$. Can apply $F_{\lambda+\mu}(x) \leq F_\mu(x)e^{\max_{w\in W}(w\lambda)(x)}$ and $F_{\rho(m(\ell,\widetilde{\ell}))}(m(\ell,\widetilde{\ell});x) = 1$.

Thank you!