From Kajihara's transformation formula to deformed Macdonald–Ruijsenaars and Noumi–Sano operators

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Calogero–Moser–Sutherland models

The original models

Calogero (1971–75), Sutherland (1971–72).

Identical quantum particles in one-dimension, interacting through an inverse square pair potential:

$$H_N = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\kappa(\kappa+1) \sum_{1\leq i<j\leq N} V(x_i - x_j) \quad (N \in \mathbb{N})$$

w/ potential function

$$V(z) = \begin{cases} 1/z^2 & \text{(rational)} \\ 1/\sin^2 z & \text{(trigonometric)} \\ \wp(z) & \text{(elliptic)} \end{cases}$$

The original models

Associated integrable system (N commuting PDOs):

$$H_N^{(r)} = \sum_{i=1}^{N} \left(-\mathrm{i}\frac{\partial}{\partial x_i}\right)^r + \text{l.o.t.} \quad (r = 1, \ldots, N),$$

w/ $H_N^{(2)} = H_N$.

- Moser (1975) proved integrability at the classical level by obtaining Lax representations.
- Olshanetsky & Perelomov (1977) established quantum integrability.
- Joint eigenfunctions: Bessel- (rat.) and Heckman–Opdam hypergeometric functions (trig.) associated w/ A_{N-1}, Jack polynomials (trig.), Baker–Akhiezer functions (rat./trig.),...

Root system generalisations

Input data:

- a finite collection of vectors $\mathcal{A} \subset \mathbb{R}^N$,
- a ‘multiplicity’ function $\kappa : \mathcal{A} \to \mathbb{C}, \alpha \mapsto \kappa_\alpha$.

Consider the associated Schrödinger operator

$$H_{\mathcal{A}} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{\alpha \in \mathcal{A}} \kappa_\alpha(\kappa_\alpha + 1)(\alpha, \alpha) V((\alpha, x)).$$

- Integrable when $\mathcal{A} = R_+$ and κ is Weyl group invariant for a root system R of type $A_{N-1}, B_N, \ldots, E_8$ (Debiard, Heckman & Opdam, Olshanetsky–Perelomov, Oshima & Sekiguchi,...) or BC_N (Inozemtsev, Oshima & Sekiguchi).
- Joint eigenfunctions: Bessel- and Heckman–Opdam hypergeometric functions as well as Jacobi polynomials associated w/ R, Baker–Akhiezer functions,...

Deformed models

Exist collections of vectors $\mathcal{A} \neq R_+$ such that $H_{\mathcal{A}}$ is integrable!

Ex: Type $A(n,m)$ in $\mathbb{R}^{n+m}$, given by vectors and 'multiplicities'

$$e_i - e_j, \ \ \kappa_{e_i - e_j} = \kappa, \quad 1 \leq i < j \leq n;$$

$$\sqrt{\kappa} e_i - \sqrt{\kappa} e_j, \ \ \kappa_{e_i - e_j} = \kappa^{-1}, \quad n+1 \leq i < j \leq n+m;$$

$$e_i - \sqrt{\kappa} e_j, \ \ \kappa_{e_i - \sqrt{\kappa} e_j} = 1, \quad 1 \leq i \leq n, \ n+1 \leq j \leq n+m;$$

(where $\mathcal{A} \cup (-\mathcal{A})$ can be viewed as a deformation of a root system of type A_{n+m-1}).

Deformed models

Introducing

$$y_i = \sqrt{\kappa} x_{n+i} \quad (i = 1, \ldots, m),$$

we get

$$H_{n,m} = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \kappa \sum_{i=1}^{m} \frac{\partial^2}{\partial y_i^2} + \kappa(\kappa+1) \sum_{1 \leq i < j \leq n} V(x_i - x_j)$$
$$+ (\kappa+1) \sum_{i=1}^{n} \sum_{j=1}^{m} V(x_i - y_j) + (1 + 1/\kappa) \sum_{1 \leq i < j \leq m} V(y_i - y_j).$$

(When $m = 0$ or $n = 0$ we recover ordinary CMS ops: $H_{n,0}(\kappa) = H_n(\kappa)$ and $H_{0,m}(\kappa) = \kappa H_m(1/\kappa)$.)

Deformed models

- Chalykh, Feigin & Veselov (1998) proved integrability when $m = 1$ and V is rational/trigonometric.
- For $n, m \in \mathbb{N}$ arbitrary and V trigonometric, the operator was introduced and studied by Sergeev (2001). Integrability proved by Sergeev & Veselov (2004).
- Khodarinova (2005) established integrability for $m = 1$ and V elliptic.
- There are intimate connections w/
  - Lie superalgebras (Sergeev, Seergev & Veselov),
  - Cherednik algebras (Feigin),
  - β-ensembles of random matrices (Desrosiers & Liu),
  - CFT and the fractional quantum Hall effect (Atai & Langmann),
  - ...

Relativistic generalisations

Ruijsenaars (1987) introduced relativistic (quantum A_{N-1}) Calogero–Moser–Sutherland models.

Integrable system of commuting difference ops:

$$D_N^{(r)} = \sum_{\substack{I\subset\{1,\dots,N\} \\ |I|=r}} \prod_{i\in I, j\notin I} \frac{[x_i - x_j + \kappa]}{[x_i - x_j]} \cdot T_x^{\delta I} \quad (r = 1, \dots, N),$$

where

$$[z] = \begin{cases} z & \text{(rational)} \\ \sin z & \text{(trigonometric)} \\ Ce^{cz^2}\sigma(z \mid \omega_1, \omega_2) & \text{(elliptic)} \end{cases}$$

and $T_x^{\delta I} = \prod_{i\in I} T_{x_i}^{\delta}$, w/

$$T_{x_i}^{\delta} f(x_1, \dots, x_i, \dots, x_n) = f(x_1, \dots, x_i + \delta, \dots, x_n).$$

Relativistic generalisations

- Calogero–Moser–Sutherland operators obtained (formally and up to a change in gauge) in the limit $\delta \to 0$.
- The difference ops

$$H_N := D_N^{(1)}(x) + D_N^{(1)}(-x) \quad \text{(time transl.)}$$
$$P_N := D_N^{(1)}(x) - D_N^{(1)}(-x) \quad \text{(space transl.)}$$
$$B_N := -\sum_{i=1}^{n} x_i \quad \text{(Lorentz boost)}$$

  yield a representation of the Lie alg. of the Poincaré group in $1+1$ dimensions; see Ruijsenaars (1987).
- Intimate connections w/ integrable (quantum) field theories. (For example, when $[z] = \sinh(\pi z/\omega)$ joint eigenfuncs. of $D_N^{(r)}$ reproduce scattering in the quantum sine-Gordon model (for suitable δ, κ); see H. & Ruijsenaars (2020).)

Noumi–Sano ops

Noumi & Sano (2020) introduced the difference ops

$$H_N^{(r)} = \sum_{\substack{\mu\in\mathbb{N}^N \\ |\mu|=r}} \prod_{1\leq i<j\leq N} \frac{[x_i - x_j + (\mu_j - \mu_j)\delta]}{[x_i - x_j]} \cdot \prod_{i,j=1}^{N} \frac{[x_i - x_j + \kappa]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}} \cdot T_x^{\delta\mu} \quad (r \in \mathbb{N})$$

and proved:

- For $K \in \mathbb{N}$,

$$\sum_{r+s=K} (-1)^r [r\kappa + s\delta] D_N^{(r)} H_N^{(s)} = 0 \quad \text{(Wronski relations)},$$

- $\mathbb{C}\left[H_N^{(1)}, H_N^{(2)}, \ldots\right] = \mathbb{C}\left[D_N^{(1)}, \ldots, D_N^{(N)}\right]$.

(Notation: $[z]_k = [z][z+\delta]\cdots[z+(k-1)\delta]$ and $T_x^{\delta\mu} = \prod_{i=1}^N \left(T_{x_i}^{\delta}\right)^{\mu_i}$.)

Deformed Ruijsenaars and Noumi–Sano ops

Ruijsenaars and Noumi–Sano operators can be unified in a family of commuting difference operators in two sets of variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$:

$$D^{(r)}_{n,m} = \sum_{\substack{I \subset \{1,\ldots,n\}, \mu \in \mathbb{N}^m \\ |I|+|\mu|=r}} C_{I,\mu}(x,y) T_x^{\delta I} T_y^{-\kappa\mu} \quad (r \in \mathbb{N}),$$

w/ coefficients

$$C_{I,\mu}(x,y) = (-1)^{|I|} \prod_{i \in I, j \notin I} \frac{[x_i - x_j + \kappa]}{[x_i - x_j]}$$

$$\cdot \prod_{1 \le i < j \le n} \frac{[x_i - x_j - (\mu_i - \mu_j)\kappa]}{[x_i - x_j]} \cdot \prod_{i,j=1}^{m} \frac{[y_i - y_j - \delta]_{\mu_i}}{[y_i - y_j - \kappa]_{\mu_i}}$$

$$\cdot \prod_{i=1}^{n} \left(\prod_{j \in I} \frac{[x_i - y_j + \delta]}{[x_i - y_j - \mu_i \kappa]} \prod_{j \notin I} \frac{[x_i - y_j + \kappa]}{[x_i - y_j - (\mu_i - 1)\kappa]} \right).$$

Deformed Ruijsenaars and Noumi–Sano ops

- Deformed Calogero–Moser–Sutherland operators can be obtained as limiting cases ($\delta, \kappa \to 0$).
- Chalykh (2000, 2002) introduced such deformations of rat./trig. Ruijsenaars operators in $n+1$ variables.
- The trig. instance of $D_{n,m}^{(1)}$ due to Sergeev & Veselov (2009).
- Feigin and Silantyev (2014) obtained the trig. ops $D_{n,m}^{(r)}$ for all $r \in \mathbb{N}$ and proved commutativity using DAHA techniques.
- The elliptic operator $D_{n,m}^{(1)}$ was first considered by Atai, H. & Langmann (2014), who established a corresponding kernel function identity.

Main results

Trigonometric case:

- We give a new proof of integrability, based on Kajihara's transformation formula for multiple basic hypergeometric series associated with A-type root systems.
- We show that the Ruijsenaars and Noumi–Sano ops are simultaneously diagonalised by the super-Macdonald polynomials.

Elliptic case:

- We prove integrability, which was not previously known. The proof is based on identities for theta functions closely related to transformation formulae for multiple elliptic hypergeometric series.

Trigonometric deformed models

Multiplicative notation

We take $[z] = \sin z$.

Let

$$z_i = e^{2\pi \mathrm{i} x_i} \quad (i = 1, \ldots, n) \quad w_j = e^{2\pi \mathrm{i} y_j} \quad (j = 1, \ldots, m).$$

Then, additive shifts $x_i \mapsto x_i + \delta$ and $y_j \mapsto y_j - \kappa$ correspond to $z_j \mapsto q z_j$ and $w_j \mapsto t^{-1} w_j$, respectively, where

$$q = e^{2\pi \mathrm{i}\delta}, \quad t = e^{2\pi \mathrm{i}\kappa}.$$

Hence, our operators become linear combinations of

$$T^I_{q,z} T^{-\mu}_{t,w} : f(z_1, \ldots, z_n; w_1, \ldots, w_m) \\ \mapsto f(q^{I_1} z_1, \ldots, q^{I_n} z_n; t^{-\mu_1} w_1, \ldots, t^{-\mu_m} w_m),$$

where

$$I \subset \{0,1\}^n, \quad \mu \in \mathbb{N}^m.$$

Multiplicative notation

Specifically,

$$D_{n,m}^{(r)} = \sum_{\substack{I\subset\{1,\ldots,n\},\mu\in\mathbb{N}^n \\ |I|+|\mu|=r}} C_{I,\mu}(z,w) T_{q,z}^{I} T_{t,w}^{-\mu} \quad (r\in\mathbb{N}),$$

w/ coefficients

$$\begin{aligned} C_{I,\mu}(z,w) = (q^m t^{-n})^{|\mu|}(-1)^{|I|} t^{\binom{|I|}{2}-|I|(n-1)} \prod_{\substack{1\leq i,j\leq n \\ i\in I; j\notin I}} \frac{tz_i - z_j}{z_i - z_j} \\ \cdot \frac{\Delta(t^{-\mu}w)}{\Delta(w)} \prod_{i,j=1}^{m} \frac{(w_i/qw_j;t^{-1})_{\mu_i}}{(w_i/tw_j;t^{-1})_{\mu_i}} \cdot \\ \cdot \prod_{i=1}^{m}\left(\prod_{j\in I}\frac{1-qw_i/z_j}{1-t^{-\mu_i}w_i/z_j} \cdot \prod_{j\notin I}\frac{1-tw_i/z_j}{1-t^{1-\mu_i 1}w_i/z_j}\right), \end{aligned}$$

where

$$\Delta(z) = \prod_{1\leq i<j\leq n}(z_i - z_j), \quad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i).$$

Multiplicative notation

Setting $m=0$, we recover the q-difference operators found in Macdonald's 1995 book 'Symmetric functions and Hall polynomials':

$$D_{n,0}^{(r)} = D_n^{(r)} := t^{\binom{r}{2}-r(n-1)} \sum_{\substack{I\subset\{1,\dots,n\} \\ |I|=r}} \prod_{i\in I, j\notin I} \frac{tz_i - z_j}{z_i - z_j} \cdot \prod_{i\in I} T_{q,z_i}.$$

Macdonald polynomials:

- $P_\lambda(z) = m_\lambda(z) + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu(z)$,
- $D_n^{(r)} P_\lambda(z) = e_r(q^\lambda t^\delta) P_\lambda(z)$.

(Notation: $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n)$ partitions, $\delta = (0, -1, \dots, 1-n)$, $m_\lambda(z) = \sum_{a\in S_n(\lambda)} z_1^{a_1} \cdots z_n^{a_n}$ and $e_r(z) = \sum_{1\leq i_1<\cdots<i_r\leq n} z_{i_1} \cdots z_{i_r}$.)

Kernel identities

Consider the generating series

$$\mathcal{D}_{n,m}(z,w;u) = \frac{(q^m t^{-n} u; t^{-1})_\infty}{(u; t^{-1})_\infty} \sum_{r=0}^{\infty} u^r D_{n,m}^{(r)}(z,w).$$

Theorem (H., Langmann, Noumi & Rosengren)

For $|q| < 1$ and $|t| > 1$, the function

$$\Phi_{n,m;N,M}(z,w;Z,W) = \prod_{i=1}^{n}\prod_{j=1}^{N} \frac{(z_i Z_j; q)_\infty}{(t^{-1} z_i Z_j; q)_\infty} \cdot \prod_{i=1}^{m}\prod_{j=1}^{M} \frac{(w_i W_j; t^{-1})_\infty}{(q w_i W_j; t^{-1})_\infty} \cdot \prod_{i=1}^{n}\prod_{j=1}^{M}(1 - z_i W_j) \cdot \prod_{i=1}^{m}\prod_{j=1}^{N}(1 - w_i Z_j).$$

satisfies

$$\mathcal{D}_{n,m}(z,w;u)\Phi_{n,m;N,M}(z,w;Z,W) = \mathcal{D}_{N,M}(Z,W;u)\Phi_{n,m;N,M}(z,w;Z,W).$$

Kernel identities

When $m = M = 0$, we recover Macdonald's (reproducing) kernel function:

$$\Phi_{n,0;N,0}(z;Z) = \prod_{i=1}^{n}\prod_{j=1}^{N} \frac{(z_i Z_j;q)_\infty}{(t^{-1}z_i Z_j;q)_\infty} = \Pi_{n,N}(t^{-1}z,Z)$$
$$= \sum_\lambda b_\lambda P_\lambda(z) P_\lambda(Z).$$

(The corresponding kernel identity is established in Macdonald's book.)

Kernel identities

Our proof relies on Kajihara's (2014) transformation formula

$$\phi^{K,L}\left(\begin{matrix} a_1,\ldots,a_K \\ X_1,\ldots,X_K \end{matrix}\middle| \begin{matrix} b_1Y_1,\ldots,b_LY_L \\ cY_1,\ldots,cY_L \end{matrix}; u\right)$$
$$= \frac{(\alpha\beta u/c^L;q)_\infty}{(u;q)_\infty}\phi^{L,K}\left(\begin{matrix} c/b_1,\ldots,c/b_L \\ Y_1,\ldots,Y_L \end{matrix}\middle| \begin{matrix} cX_1/a_1,\ldots,cX_K/a_K \\ cX_1,\ldots,cX_K \end{matrix}; \alpha\beta u/c^L\right),$$

where $\alpha = a_1\cdots a_K$ and $\beta = b_1\cdots b_L$, for Kajihara and Noumi's (2013) multiple basic hypergeometric series

$$\phi^{K,L}\left(\begin{matrix} a_1,\ldots,a_K \\ X_1,\ldots,X_K \end{matrix}\middle| \begin{matrix} b_1,\ldots,b_L \\ c_1,\ldots,c_L \end{matrix}; u\right)$$
$$= \sum_{\gamma\in\mathbb{N}^K} u^{|\gamma|}\frac{\Delta(q^\gamma X)}{\Delta(X)}\prod_{i,j=1}^{K}\frac{(a_jX_i/X_j;q)_{\gamma_i}}{(qX_i/X_j;q)_{\gamma_i}}\cdot\prod_{i=1}^{K}\prod_{k=1}^{L}\frac{(X_ib_k;q)_{\gamma_i}}{(X_ic_k;q)_{\gamma_i}}.$$

Kernel identities

When $K = L = 1$, the latter reduces (essentially) to the basic hypergeometric series

$$ {}_2\phi_1 = \left[\begin{matrix} a, b \\ c \end{matrix}; q, u\right] \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(q;q)_k (c;q)_k} u^k $$

and the former to Heine's q-analogue of Euler's transformation formula for ${}_2F_1$:

$$ {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, u\right] = \frac{(abu/c;q)_\infty}{(u;q)_\infty} {}_2\phi_1 \left[\begin{matrix} c/a, c/b \\ c \end{matrix}; q, abu/c\right], $$

We obtain our theorem by taking $K = n + m$, $L = N + M$ and specialising:

$$ X_i = z_i,\ a_i = t\ \ (i = 1, \ldots, n);\ \ X_{n+i} = w_i,\ a_{n+i} = q^{-1}\ \ (i = 1, \ldots, m); $$
$$ Y_j = Z_j,\ b_j = t\ \ (j = 1, \ldots, N);\ \ Y_{N+j} = W_j,\ b_{N+j} = q^{-1}\ \ (j = 1, \ldots, M). $$

Commutativity

Using double affine Hecke algebra techniques, Feigin & Silantyev (2014) proved:

Theorem

For all $r, s \in \mathbb{N}$*,*

$$\left[D^{(r)}_{n,m}, D^{(s)}_{n,m}\right] = 0.$$

We give a new proof, based on kernel identities and commutativity of ordinary Macdonald–Ruijsenaars operators.

Commutativity

Taking $M = 0$ in kernel ids, we get

$$\mathcal{D}_{n,m}(z, w; u)\Phi_{n,m;N,0}(z, w; Z) = \mathcal{D}_N(Z; u)\Phi_{n,m;N,0}(z, w; Z),$$

where

$$\mathcal{D}_N(Z; u) = \frac{(t^{-N}u; t^{-1})_\infty}{(u; t^{-1})_\infty} \sum_{r=0}^{\infty} u^r D_N^{(r)}(z).$$

From the well-known commutativity of the Macdonald–Ruijsenaars operators $D_N^{(r)}(Z)$, we infer

$$\begin{aligned}\mathcal{D}_{n,m}(z, w; u)\mathcal{D}_{n,m}(z, w; v)\Phi_{n,m;N,0}(z, w; Z) \\ &= \mathcal{D}_N(Z; v)\mathcal{D}_N(Z; u)\Phi_{n,m;N,0}(z, w; Z) \\ &= \mathcal{D}_N(Z; u)\mathcal{D}_N(Z; v)\Phi_{n,m;N,0}(z, w; Z) \\ &= \mathcal{D}_{n,m}(z, w; v)\mathcal{D}_{n,m}(z, w; u)\Phi_{n,m;N,0}(z, w; Z).\end{aligned}$$

Commutativity

Comparing coefficients of $u^r v^s$, we obtain

$$\left[D_{n,m}^{(r)}(z,w), D_{n,m}^{(s)}(z,w)\right]\Phi_{n,m;N,0}(z,w;Z) = 0.$$

Commutativity is now a direct consequence of the following lemma.

Lemma

Let $L_{n,m}(z,w)$ be a difference operator in (z,w) of the form

$$L_{n,m}(z,w) = \sum_{\substack{\mu\in\mathbb{N}^n,\nu\in\mathbb{N}^m \\ |\mu|+|\nu|\leq d}} a_{\mu,\nu}(z,w) T_{q,z}^{\mu} T_{t,w}^{-\nu},$$

with meromorphic coefficients $a_{\mu,\nu}(z,w)$ and $d\in\mathbb{N}$. If $L_{n,m}(z,w)\Phi_{n,m;N,0}(z,w;Z) = 0$ for all $N\in\mathbb{N}^*$, then $L_{n,m}(z,w)\equiv 0$ as a difference operator.

Joint eigenfunctions

Sergeev & Veselov (2009) introduced the so-called Super-Macdonald polynomials, which can be defined by

$$\Phi_{n,m;N,0}(z,w;Z) = \prod_{i=1}^{n}\prod_{j=1}^{N}\frac{(z_iZ_j;q)_\infty}{(t^{-1}z_iZ_j;q)_\infty}\prod_{i=1}^{m}\prod_{j=1}^{N}(1-w_iZ_j)$$
$$= \sum_{\lambda} t^{-|\lambda|}b_\lambda SP_\lambda(z,w)P_\lambda(Z).$$

Theorem

As long as $q^it^j \neq 1$ *for all* $i,j\in\mathbb{N}$ *w/* $i+j\geq 1$*, we have*

$$\mathcal{D}_{n,m}(z,w;u)SP_\lambda(z,w) = SP_\lambda(z,w)E^{\natural}_{n,m}(q^\mu, t^{-\nu-(n^m)};u),$$

where $\mu=(\lambda_1,\ldots,\lambda_n)$*,* $\nu=(\lambda_{n+1},\lambda_{n+2},\ldots)'$ *and*

$$E^{\natural}_{n,m}(x,y;u) = \prod_{i=1}^{n}\frac{1-x_it^{1-i}u}{1-t^{1-i}u}\cdot\prod_{j=1}^{m}\frac{(t^{-n}q^ju;t^{-1})_\infty}{(y_jq^ju;t^{-1})_\infty}\frac{(y_jq^{j-1}u;t^{-1})_\infty}{(t^{-n}q^{j-1}u;t^{-1})_\infty}.$$

Joint eigenfunctions

We deduce the result from

$$\mathcal{D}_{n,m}(z,w;u)\Phi_{n,m;N,0}(z,w;Z) = \mathcal{D}_N(Z;u)\Phi_{n,m;N,0}(z,w;Z)$$

and well-known eigenvalues of $\mathcal{D}_N(Z;u)$.

From the above definition, we also recover Sergeev & Veselov's (2009) expression

$$SP_\lambda(z,w;q,t) = \sum_{\nu\subseteq\lambda}(-t)^{|\nu|}b_{\nu'}(t,q)P_{\lambda/\nu}(z;q,t)P_{\nu'}(w;t,q).$$

Elliptic deformed models

Reminder: Deformed elliptic Ruijsenaars ops

Let $[z] = Ce^{cz^2}\sigma(z \mid \omega_1, \omega_2)$ and consider

$$D_{n,m}^{(r)} = \sum_{\substack{I\subset\{1,\dots,n\},\mu\in\mathbb{N}^m\\ |I|+|\mu|=r}} C_{I,\mu}(x,y) T_x^{\delta I} T_y^{-\kappa\mu} \quad (r\in\mathbb{N}),$$

w/ coefficients

$$C_{I,\mu}(x,y) = (-1)^{|I|} \prod_{i\in I, j\notin I} \frac{[x_i - x_j + \kappa]}{[x_i - x_j]}$$
$$\cdot \prod_{1\le i<j\le n} \frac{[x_i - x_j - (\mu_i - \mu_j)\kappa]}{[x_i - x_j]} \cdot \prod_{i,j=1}^{m} \frac{[y_i - y_j - \delta]_{\mu_i}}{[y_i - y_j - \kappa]_{\mu_i}}$$
$$\cdot \prod_{i=1}^{n} \left(\prod_{j\in I} \frac{[x_i - y_j + \delta]}{[x_i - y_j - \mu_i\kappa]} \prod_{j\notin I} \frac{[x_i - y_j + \kappa]}{[x_i - y_j - (\mu_i - 1)\kappa]} \right).$$

Commutativity

Theorem

We have

$$\left[D_{n,m}^{(r)}, D_{n,m}^{(s)}\right] = 0$$

for all $r, s \in \mathbb{N}$.

Commutativity

There are two main steps in our proof.

Step 1: We reduce $\left[D_{n,m}^{(r)}, D_{n,m}^{(s)}\right] = 0$ to the identities

$$S_r = S_{|\lambda|+m-r}, \quad \lambda \in \mathbb{N}^n, \ \ 0 \leq r \leq |\lambda|,$$

for

$$S_r = \sum_{\substack{0\leq \mu_j \leq \lambda_j, 1\leq j\leq n \\ P\subset\{1,\dots,m\}, |\mu|+|P|=r}} \prod_{i\in P, j\notin P} \frac{[y_i - y_j - \delta][y_i - y_j + \delta - \kappa]}{[y_i - y_j][y_i - y_j - \kappa]}$$

$$\cdot \prod_{i,j=1}^{n} \left(\frac{[x_i - x_j + \delta]_{\mu_i - \mu_j}}{[x_i - x_j + \kappa]_{\mu_i - \mu_j}} \frac{[x_i - x_j + \kappa]_{\mu_i} [x_i - x_j - \lambda_j \delta]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i} [x_i - x_j - (\lambda_j - 1)\delta - \kappa]_{\mu_i}} \right)$$

$$\cdot \prod_{i=1}^{n} \left(\prod_{j\in P} \frac{[x_i - y_j + \lambda_i \delta][x_i - y_j + (\mu_i - 1)\delta + \kappa]}{[x_i - y_j + \mu_i \delta][x_i - y_j + (\lambda_i - 1)\delta + \kappa]} \right.$$

$$\left. \cdot \prod_{j\notin P} \frac{[x_i - y_j - \delta][x_i - y_j + \mu_i \delta - \kappa]}{[x_i - y_j - \kappa][x_i - y_j + (\lambda_i - 1)\delta]} \right).$$

Commutativity

We note that

$$\prod_{i,j=1}^{n} \frac{[x_i - x_j + \delta]_{\mu_i - \mu_j}}{[x_i - x_j + \kappa]_{\mu_i - \mu_j}} = \prod_{1 \le i < j \le n} \left(\frac{[x_i - x_j + (\mu_i - \mu_j)\delta]}{[x_i - x_j]} \frac{[x_i - x_j + \delta - \kappa]_{\mu_i - \mu_j}}{[x_i - x_j + \kappa]_{\mu_i - \mu_j}} \right),$$

where factors of the form $[x_i - x_j + (\mu_i - \mu_j)\delta]$ are typical of elliptic hypergeometric series related to root systems of type A. In fact, $S_k = S_{|\lambda|+m-r}$ is essentially equivalent to an elliptic hypergeometric transformation formula due to Langer, Schlosser and Warnaar (2009).

Commutativity

Step 2: We obtain the identity $S_k = S_{|\lambda|+m-r}$ by *multiple principal specialization* in

$$\sum_{\substack{I\subset\{1,\dots,N\}\\|I|=r}} \prod_{i\in I, j\notin I} \frac{[z_i-z_j-a][z_i-z_j-b]}{[z_i-z_j][z_i-z_j-a-b]} = \sum_{\substack{I\subset\{1,\dots,N\}\\|I|=N-r}} \prod_{i\in I, j\notin I} \frac{[z_i-z_j-a][z_i-z_j-b]}{[z_i-z_j][z_i-z_j-a-b]}.$$

The latter identity is due to Ruijsenaars (1987).
(He used it to prove commutativity for his elliptic difference operators.)

Specifically, we take $N = |\lambda| + m$, $a = \delta$, $b = \kappa - \delta$ and set

$$(z_1,\dots,z_N) = (x_1, x_1+\delta, \dots, x_1+(\lambda_1-1)\delta, \dots, x_n, x_n+\delta, \dots, x_n+(\lambda_n-1)\delta, y_1, \dots, y_m).$$

References

The talk was based on the following papers:

M. Hallnäs, E. Langmann, M. Noumi, H. Rosengren (2021)
From Kajihara's transformation formula to deformed Macdonald–Ruijsenaars and Noumi–Sano operators
arXiv:2105.01936

M. Hallnäs, E. Langmann, M. Noumi, H. Rosengren (2021)
Higher order deformed elliptic Ruijsenaars operators
arXiv:2105.02536

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