Selected results in real harmonic analysis in the Dunkl setting

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Modern Analysis Related to Root Systems with Applications
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joint works with Jean-Phlippe Anker and Agnieszka Hejna

Goal

Our aim is to define and study objects associated with the Dunkl operators which are parallel to ones from the classical real harmonic and Fourier analysis.

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Important objects in the classical analysis:

- convolution: $f * g(x) = \int f(x-y)g(y)\,dy$ - **translation** invariant operators
- **Fourier** transform: $\widehat{f}(\xi) = \int f(x)e^{-i\xi\cdot x}\,dx$

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They allow us to:

- solve the heat equation $\partial_t u(x,t) = \Delta u(x,t)$, $u(x,0) = f$ or other differential equations
- define and study function spaces like L^p spaces, Hardy spaces, Besov spaces, Lipschitz spaces; via Littlewood-Paley theory, maximal functions, singular integrals
- potential theory - $(-\Delta)^{\beta}$, $(I-\Delta)^{\beta}$

- C.F. Dunkl, *Reflection groups and orthogonal polynomials on the sphere*, Math. Z. 197 (1988), no. 1, 33–60.
- C.F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. 311 (1989), no. 1, 167–183.
- C.F. Dunkl, *Integral kernels with reflection group invariance*, Canad. J. Math. 43 (1991), no. 6, 1213–1227.
- M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. 192 (1998), 519–542.
- M. Rösler, *Positivity of Dunkl's intertwining operator*, Duke Math. J. 98 (1999), no. 3, 445–463.
- M. Rösler, *A positive radial product formula for the Dunkl kernel*, Trans. Amer.Math. Soc. 355 (2003), no. 6, 2413–2438.
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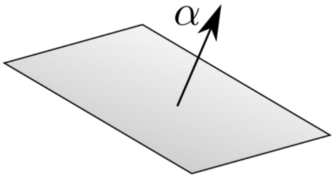
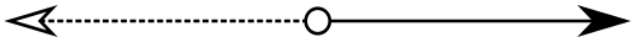
Root system

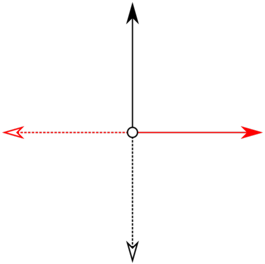
For $0 \neq \alpha \in \mathbb{R}^N$, let $\sigma_\alpha(x) = x - 2\frac{\langle x,\alpha\rangle}{\|\alpha\|^2}\alpha$ be the reflection in $\mathbb{R}^N$ with respect to $\alpha^\perp$



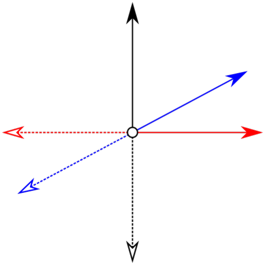
R is a root system in $\mathbb{R}^N$, finite set of of vectors α such that $\sigma_\alpha(R) = R$ for $\alpha \in R$
normalized $\|\alpha\|^2 = 2$
G- reflection group - finite group generated by σ_α, $\alpha \in R$
$k : R \to \mathbb{C}$ - multiplicity function, $k(\sigma_\alpha(\alpha')) = k(\alpha')$, for $\alpha, \alpha' \in R$, $k(\alpha) \geq 0$



A_1



$A_1 \times A_1$



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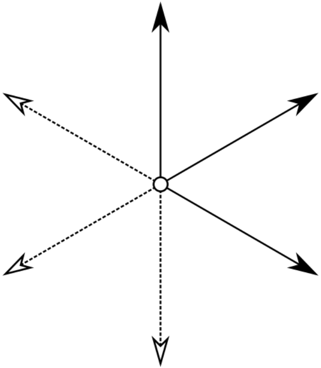
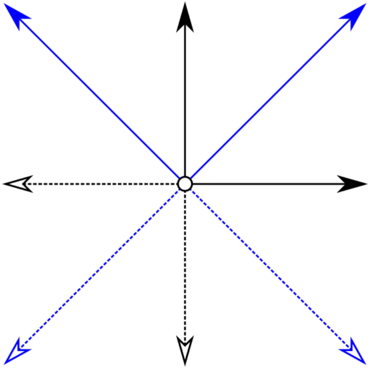
Examples of root systems



A_2



B_2

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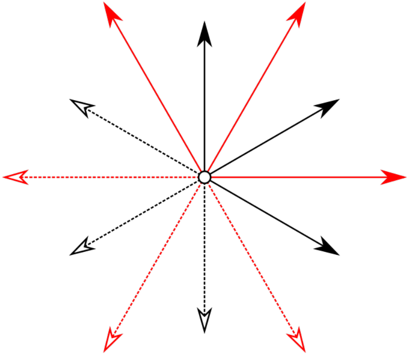
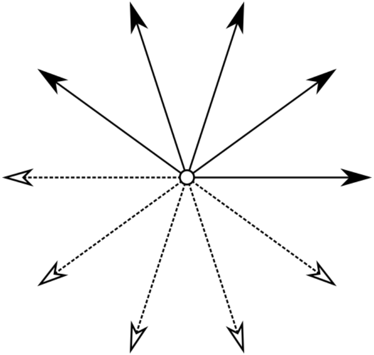


G_2



$I_2(5)$

Measure and the Dunkl operators

$dw(x) = \prod_{\alpha\in R} |\langle x, \alpha\rangle|^{k(\alpha)}\, dx$ - associated measure, $w(B(x,r)) \sim r^N \prod_{\alpha\in R}(|\langle x,\alpha\rangle| + r)^{k(\alpha)}$

w is doubling, that is, $w(B(x,2r)) \le C\, w(B(x,r))$

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$\mathbf{N} = N + \sum_{\alpha\in R} k(\alpha)$ homogeneous dimension
$w(B(tx,tr)) = t^{\mathbf{N}} w(B(x,r))$

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Dunkl operator

$$T_\eta f(x) = \partial_\eta f(x) + \sum_{\alpha\in R} \frac{k(\alpha)}{2} \langle \alpha,\eta\rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}$$

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$T_j = T_{e_j}$, where e_j is the canonical basis of $\mathbb{R}^N$.

$T_\eta T_\xi = T_\xi T_\eta$
T_η(Polynomial) = Polynomal of a lower degree
$\int (T_\eta f) g \, dw = -\int f (T_\eta g)\, dw$

Leibniz formula $T_\eta(fg) = (T_\eta f)g + f(T_\eta g)$ doesn't hold

unless either f or g is G invariant.

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Dunkl-Laplace operator

$$\Delta_k f(x) = \sum_{j=1}^{N} T_j^2$$

(does not depend on the choice of the basis)

Dunkl kernel

For fixed y, $E(x,y)$ is a unique solution of $T_\eta f(x) = \langle \eta, y\rangle f(x)$, $f(0)=1$.
$E(x,y)$ generalizes $\exp(\langle x,y\rangle)$ and has a unique extension to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$

Dunkl transform (generalization the Fourier transform)

$$\mathscr{F} f(\xi) = c_k^{-1} \int f(x) E(-i\xi, x)\, dw(x)$$

(M. de Jeu)

$$\|\mathscr{F} f\|_{L^2(dw)} = \|f\|_{L^2(dw)} \quad \text{and} \quad \mathscr{F}^{-1} f(\xi) = c_k^{-1} \int f(x) E(i\xi, x)\, dw(x)$$

Generalized translation

$$\tau_x f(y) = c_k^{-1} \int E(i\xi, x) \mathscr{F} f(\xi) E(i\xi, y)\, dw(\xi)$$

It is not known if τ_x is bounded on $L^p(dw)$.

τ_x is a contraction on $L^2(dw)$.

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Thangavelu-Xu

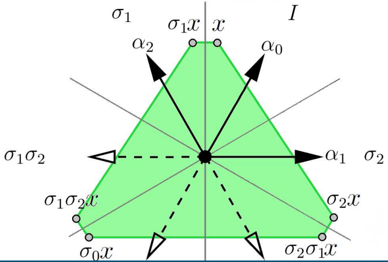
Moreover, $\|\tau_x f\|_{L^1} \leq \|f\|_{L^1}$ if f is radial.

(M. Rösler)

If $f(x) = \tilde{f}(|x|)$, then

$$\tau_x f(-y) = \int (\tilde{f}(A(x,y,\eta))\, d\mu_x(\eta), \quad A((x,y,\eta) = (\|x\|^2 + \|y\|^2 - 2\langle y,\eta\rangle)^{1/2},$$

μ_x is a probability measure with support in the convex hull of $\mathcal{O}(x) = \{\sigma(x) : \sigma \in G\}$.



Dunkl heat kernel

$$e^{t\Delta_k} f(x) = h_t * f(x) = \int_{\mathbb{R}^N} h_t(x,y) f(y)\, dw(y),$$

$$h_t(x) = c_k^{-1}(2t)^{-\mathbf{N}/t} e^{-\|x\|^2/4t},$$

$$h_t(x,y) = \tau_x(h_t)(-y) = c_k^{-1}(2t)^{-\mathbf{N}/2} e^{-(\|x\|^2+\|y\|^2)/4t} E\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big)$$

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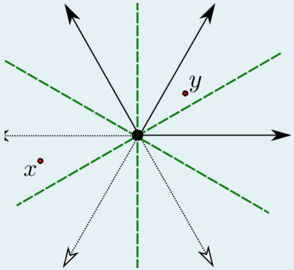
$$d(x,y)=\min\{\|\sigma(x)-y\| : \sigma\in G\} \text{ distance of } \mathcal{O}(x) \text{ to } \mathcal{O}(y)$$



Distance of orbits

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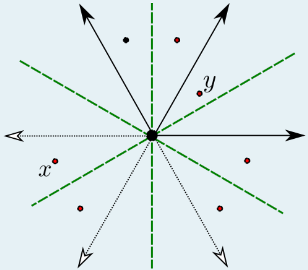

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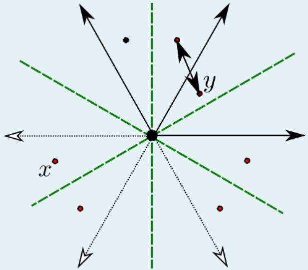

Thanks to

$$d(x,y) \leq A(x,y,\eta) \quad \text{for } \eta \in \operatorname{conv}\mathcal{O}(x).$$

and

$$\tau_x f(-y) = \int (\tilde{f}(A(x,y,\eta))\, d\mu_x(\eta),$$

one deduces

$$0 < h_t(x,y) \leq c t^{-\mathbf{N}/2} \exp(-d(x,y)^2/4t)$$

Motivation

Having the objects like the generalized convolution $*$, the Dunkl transform $\mathscr{F}$, the Dunkl-Laplace operator Δ_k,and the heat semigroup $e^{t\Delta_k}$, we may ask if there are theorems which are analogue to classical ones.

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Thangavelu-Xu (2005)

If $\chi(x) = \chi_{B(0,1)}(x)$, $\chi_t(x) = t^{-\mathbf{N}}\chi(x/t)$,
then $M_\chi f(x) = \sup_{t>0} |\chi_t * f(x)|$ is of weak-type (1.1) and bounded on $L^p(w)$, $1 < p \leq \infty$.
Heat semigroup approach in the spirit of Stein ("Topics in Harmonic Analysis ...").

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Question. **Can we deduce this theorem from estimates for $\tau_x\chi_t(-y)$ or from estimates of the heat kernel $h_t(x,y)$?**

Motivation cont.

Riesz transforms:

$$R_j f(x) = c T_j (-\Delta_k)^{-1/2} f(x) = c' \mathscr{F}^{-1}\Big(\frac{\xi_j}{\|\xi\|}\mathscr{F} f(\xi)\Big)(x).$$

(Thangavelu-Xu (2007)) in dimension 1 and Amri-Sifi (2012):

R_j are bounded on $L^p(dw)$ and of weak-type (1.1).

Questions: **Can we build theory of singular integrals of convolution type operators? Can we find conditions on $m : \mathbb{R}^N \to \mathbb{C}$ such that**

$$T_m f(x) = \mathscr{F}^{-1}(m(\xi)\mathscr{F} f(\xi))(x)$$

is bounded on L^p or weak type (1.1)?

Motivation cont.

Study of Δ_k harmonic functions

$$\Delta_k u = 0$$

(Gallardo-Rejeb)

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Generalized Cauchy-Riemann equations

Problem: **Investigate conjugate harmonic functions:** $\boldsymbol{u} = (u_0(t,x), u_1(t,x), \ldots, u_N(t,x))$**, that is,**

$$\sum_{j=0}^{N} T_j u = 0, \quad T_j u_\ell = T_\ell u_j, \quad T_0 = \partial_t, \quad 0 \le j, \ell \le N, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N,$$

$$\sup_{t>0} \int_{\mathbb{R}^N} |\boldsymbol{u}(t,x)| \, dw(x) < \infty,$$

and build theory of Hardy spaces in the spirit of Stein, Weiss, Fefferman, Coifman, Latter.

Problem. **Study Schrödinger operators**

$$-\Delta_k + V$$

in particular

$$-\Delta_k + \|x\|^2.$$

(Agnieszka Hejna talk)

Investigate higher order operators

$$\Delta_k^2, \quad \sum_{j=1}^{N} T_j^4$$

or their fractional powers

Estimates of generalized translations of radial functions

From the estimates

$$0 < h_t(x,y) \leq c\, t^{-\mathbf{N}/2} \exp(-d(x,y)^2/4t)$$

we cannot deduce that $M_h f(x) = \sup_{t>0} |h_t * f(x)|$ is bounded on L^p.

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Improvement (J.-Ph. Anker, J.D., A. Hejna)

$$h_t(x,y) \leq C\, w(B(x,\sqrt{t}))^{-1} \exp(-c\, d(x,y)^2/t)$$

$$\Longrightarrow M_h f(x) \leq C \sum_{\sigma \in G} M_{HL} f(\sigma(x)), \quad M_{HL} \text{ is the Hardy Littlewood max function}$$

$$M_{HL} f(x) = \sup_{B \ni x} \frac{1}{w(B)} \int_B |f(y)|\, dw(y).$$

Proof of the estimates

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Important properties

$$\tau_x f(-y) \le \tau_x g(-y) \quad \text{for radial } f \le g$$

$$\operatorname{supp} \tau_x f \subset \mathcal{O}(B(x,r)) \text{ for radial such that } f,\ \operatorname{supp} f \subset B(0,r)$$

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For a function f on $\mathbb{R}^N$, let

$$f_t(x) = t^{-\mathbf{N}} f(x/t), \quad f_t(x,y) = \tau_x(f_t)(-y).$$

The heat kernel does not fit to this notation - parabolic scaling.

Consider $p(x) = \tilde{p}(|x|)$, where $\tilde{p}(s) = c(1+s^2)^{-M/2}$ and $\int p(x)\,dw(x) = 1$.
Then $\tilde{p}'(s) \leq C(1+s^2)^{-(M+1)/2}$ and $|\nabla_y(\tilde{p}(A(x,y,\eta)))| \leq C\tilde{p}(A(x,y,\eta))$.
Consider $\bar{B} = \overline{B(y,1)}$ and let $y_0 \in \bar{B}$ be such that $p(x,y_0) = \sup_{y' \in \bar{B}} p(x,y') = K$.
Then $0 \leq p(x,y_0) - p(x,y') = \int \int_0^1 \frac{d}{ds}(p \circ A)(x, y' + s(y_0 - y')\,ds\,d\mu_x(\eta) \leq CK\|y_0 - y'\|$.
Thus $p(x,y') \geq K/2$ for $\|y_0 - y'\| \leq (2C)^{-1}$.
So, $1 \geq \int p(x,y')\,dy' \geq Kw(B(y_0,(2C)^{-1}))/2 \implies p(x,y) \leq K \lesssim w(B(y_0,(2C)^{-1})^{-1} \sim w(B(y,1))^{-1}$

Now if q is radial and

$$|q(x)| \leq (1+|x|^2)^{-M/2}(1+|x|^2)^{-\ell}$$

then

$$|q(x,y)| \leq \int (1 + A(x,y,\eta)^2)^{-\ell} p(A(x,y,\eta))\,d\mu_x(\eta) \lesssim (1 + d(x,y)^2)^{-\ell} w(B(y,1))^{-1}.$$

The factor $(1+|x|^2)^{-\ell}$ can be replaced by $e^{-c|x|^2}$.

Estimates of translations of radial function - improvement

The estimate $h_t(x,y) \leq C w(B(x,\sqrt{t}))^{-1} \exp(-cd(x,y)^2/t)$ is symmetric with respect to G.
We expect that the main mass of $y \mapsto h_t(x,y)$ **is concentrated near** x.

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This result, with an outline of a proof which uses a Poincaré inequality, was announced W. Hebisch.

If f is radial and $|f(x)| \lesssim (1+|x|)^{-\mathbf{N}-\ell-\varepsilon}$, then

$$|\tau_x(f_t)(-y)| = |f_t(x,y)| \lesssim \left(1+\frac{\|x-y\|}{t}\right)^{-2} (w(B(x,t))^{-1}\left(1+\frac{d(x,y)}{t}\right)^{-\ell}.$$

This type of approach allows us to apply methods of analysis on spaces of homogeneous type. For example:

(J.-Ph. Anker, J.D., A. Hejna)

build theory of H^1 spaces in the Dunkl setting and prove characterizations by:

- boundary values of conjugate $(\partial_t^2+\Delta_k)$-harmonic functions + $L^1(dw)$ condition
- maximal function: $\sup_{t>0}|h_t * f|) \in L^2(dw)$
- Riesz transforms: $R_j f = T_j(-\Delta_k)^{-1/2} f \in L^1(dw)$
- square functions: $(\int_0^\infty |t\partial_t h_t * f|^2 \frac{dt}{t})^{1/2} \in L^1(dw)$
- atomic decomposition: a is atom if supp $a \subset B$, $\|a\|_\infty \le w(B)^{-1}$, $\int a\,dw = 0$.

Generalized translation of non-radial functions

Our goal is to investigate

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Good information.

We can apply the formula of Rölser:

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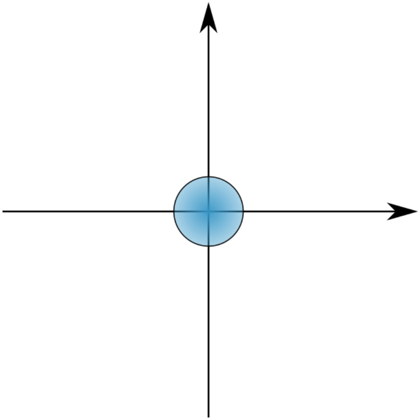
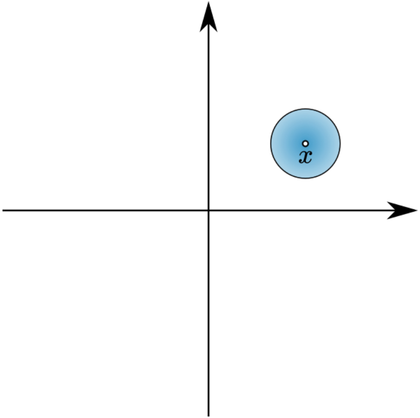
Suppose that $f \in L^2(dw)$ is such that $\operatorname{supp} f \subseteq B(0, r)$.



Support property of the generalized translation

Suppose that $f \in L^2(dw)$ is such that $\operatorname{supp} f \subseteq B(0,r)$.
If we consider $f_x = f(x - \cdot)$, then $\operatorname{supp} f_x \subseteq B(x,r)$.



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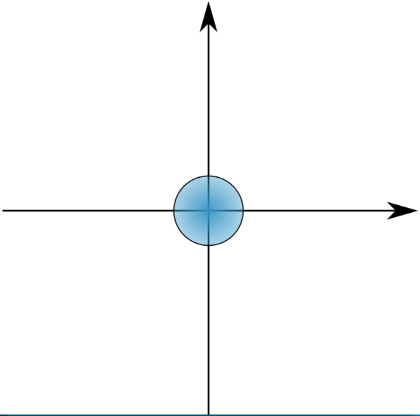
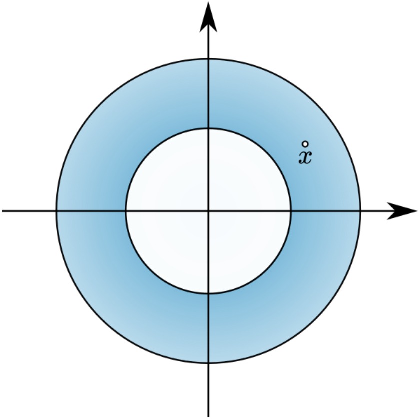
If we consider $f_x = f(x - \cdot)$, then $\operatorname{supp} f_x \subseteq B(x,r)$.
Question: What about $\operatorname{supp} \tau_x f(-\cdot)$?



Results of Amri, Anker and Sifi (Paley-Wiener approach) assert:

$$\operatorname{supp} f \subset B(0,r) \implies \operatorname{supp} \tau_x f(-\cdot) \subseteq \{y : \|x\| - r \leq \|y\| \leq \|x\| + r\}.$$



Support property of the generalized translation

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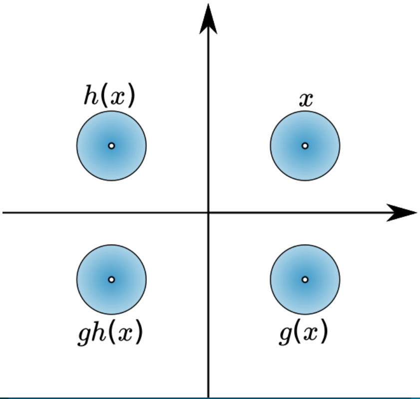
Results of Rösler imply that if f is **radial**, then

$$\operatorname{supp} f \subset B(0,r) \implies \operatorname{supp} \tau_x f(-\cdot) \subseteq \mathcal{O}(B(x,r)) = \bigcup_{g\in G} B(g(x),r) = \{y : d(x,y) < r\}.$$



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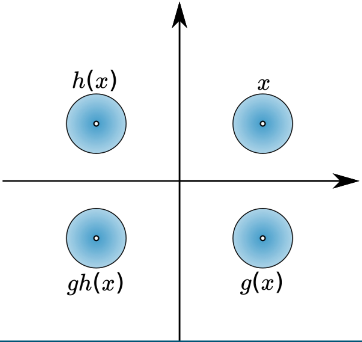
Theorem (J.D., A. Hejna)

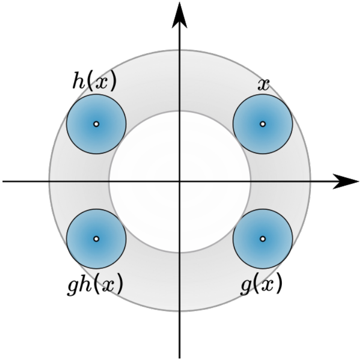
Let $f \in L^2(dw)$, $\operatorname{supp} f \subseteq B(0,r)$, and $x \in \mathbb{R}^N$. Then

$$\operatorname{supp} \tau_x f(-\cdot) \subseteq \mathcal{O}(B(x,r)) = \{y : d(x,y) \leq r\}.$$



Difference

The measure of $\mathcal{O}(B(x,r))$ is **much smaller** than the measure of $\{y : \|x\| - r \leq \|y\| \leq \|x\| + r\}$.



If $r = 1$ and $dw = dx$, then $|\mathcal{O}(B(x,1))| = \text{const}$, while $|\text{anulus}| \sim \|x\|^{N-1}$ for $\|x\|$ large.

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Lemma (J.D., A. Hejna)

Let ϕ be radial continuous function, $\operatorname{supp}\phi \subset B(0,r_1)$ and let $f \in L^1(dw)$, $\operatorname{supp} f \subset B(0,r_2)$. Then

$$\|\tau_x(\phi * f)\|_{L^1(dw)} \leq C(r_1(r_1+r_2))^{\mathbf{N}/2}\|\phi\|_{L^\infty}\|f\|_{L^1(dw)}$$

Note. The estimate does not depend on x.

Theorem (J.D., A. Hejna)

If $g \in \mathscr{S}(\mathbb{R}^N)$, then $\|\tau_x g\|_{L^1(dw)} \le C$.

Theorem (J.D., A. Hejna)

Let $\varphi \in \mathscr{S}(\mathbb{R}^N)$. Then

$$|\varphi_t(x,y)| \le C_M w(B(x,t))^{-1}\left(1 + \frac{d(x,y)}{t}\right)^{-M}.$$

Consequently, the maximal function

$$M_\varphi f = \sup_{t>0} |\varphi_t * f|$$

is of weak-type (1.1) and bounded on L^p for $1 < p \le \infty$.

Improvement (A. Hejna)

$$|\varphi_t(x,y)| \leq C_M w(B(x,t))^{-1}\left(1+\frac{\|x-y\|}{t}\right)^{-1}\left(1+\frac{d(x,y)}{t}\right)^{-M}.$$

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Theorem (J.D. A. Hejna)

Assume that m - not necessarily radial, satisfies Hörmander's condition

$$\sup_{t>0} \|\psi(\cdot)m(t\cdot)\|_{W_2^s} < \infty$$

for certain $s > \mathbf{N}$, where $\psi \in C_c^\infty(\mathbb{R}^N)$ is radial supported by an annuls.
Then the Dunkl multiplier operator

$$\mathcal{T}_m f = \mathcal{F}^{-1}(m\mathcal{F} f),$$

is of weak-type $(1,1)$, bounded on $L^p(dw)$, and on the Hardy space $H^1_{\Delta_k}$.

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Theorem (J.D., A. Hejna)

Let $n = \lfloor \frac{\mathbf{N}}{2} \rfloor + 2$. Assume that $K \in C^n(\mathbb{R}^N \setminus \{0\})$, satisfies:

$$\left| \int_{a<\|x\|<b} K(\mathbf{x})\, dw(x) \right| < \infty,$$

$$\left| \partial^\beta K(x) \right| \leq C_\beta \|x\|^{-\mathbf{N}-|\beta|} \text{ for all } x \in \mathbb{R}^N \setminus \{0\},\ |\beta| \leq n;$$

$$\lim_{\varepsilon \to 0} \int_{\varepsilon<\|\mathbf{x}\|<1} K(x)\, dw(x) = L.$$

Then the operator

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B(0,\varepsilon)} \tau_x K(-y) f(y)\, dw(y)$$

is weak type (1.1), bounded on $L^p(dw)$, $1 < p < \infty$, and bounded on H^1.

Application 4. Square functions

Assume that ϕ - not necessarily radial, smooth enough with certain decay. We are able to establish upper and lower $L^p(dw)$-bounds for square functions, including e.g.

$$S_{\nabla_k,\phi}f(x) := \Big(\int_0^\infty \|t\nabla_k(\phi_t * f)(x)\|^2 \frac{dt}{t}\Big)^{1/2}.$$

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J.D., A. Hejna

$$\|S_{\nabla_k,\phi}f\|_{L^p(dw)} \leq C\|f\|_{L^p(dw)},$$

$$\|f\|_{L^p(dw)} \leq C\|S_{\nabla_k,\phi}f\|_{L^p(dw)},$$

the lower bound under the assumption that $\mathscr{F}\phi$ does not vanish along any direction, $(\forall \xi \neq 0)(\exists t > 0)(\mathscr{F}\phi(t\xi) \neq 0)$.

Proof

For the upper bound we use a vector valued Calderón-Zygmund approach.

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Lower bound:

$$\int_{\mathbb{R}^N}\int_0^\infty t^2\langle\nabla_k(\phi_t*f)(\mathbf{x}),\nabla_k(\phi_t*g)(\mathbf{x})\rangle\frac{dt}{t}\,dw(\mathbf{x})=\int_{\mathbb{R}^N}\mathscr{F}f(\xi)\overline{\mathscr{F}g(\xi)}c_\phi(\xi)\,dw(\xi), \qquad (\star)$$

$$c_\phi(\xi)=c_k\int_0^\infty t^2\|\xi\|^2|\mathscr{F}\phi(t\xi)|^2\frac{dt}{t},\ 0<\delta<c_\phi<C,$$

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$$\left|\int f\bar{g}\,dw\right|=\left|\int\mathscr{F}f\overline{\mathscr{F}g}c_\phi^{-1}c_\phi dw\right|=\left|\int\mathscr{F}f\overline{\mathscr{F}(T_{c_\phi^{-1}}g)}c_\phi dw\right|$$

$(\star)$ + Hölder ineq. $\le\|S_{\nabla_k,\phi}f\|_{L^p(dw)}\|S_{\nabla_k,\phi}(T_{c_\phi^{-1}}g)\|_{L^{p'}(dw)}$

upper bounds $\le C\|S_{\nabla_k,\phi}f\|_{L^p(dw)}\|(T_{c_\phi^{-1}}g)\|_{L^{p'}(dw)}$

multiplier thm. $\le C'\|S_{\nabla_k,\phi}f\|_{L^p(dw)}\|g\|_{L^{p'}(dw)}.$

For $\ell_0 \in \mathbb{N}$, we study the semigroup generated by

$$D_{\ell_0} = (-1)^{\ell_0+1} \sum_{j=1}^{N} T_{\zeta_j}^{2\ell_0}$$

and prove the estimates for the integral kernel

$$|u_{\{t\}}(x,y)| \leq C w(B(x, t^{1/(2\ell_0)}))^{-1} \exp\left(-c \frac{d(x,y)^{2\ell_0/(2\ell_0-1)}}{t^{1/(2\ell_0-1)}}\right).$$

Thank you