Dunkl–Schrödinger operators with the reverse Hölder's class potentials in the rational Dunkl setting

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Modern Analysis Related to Root Systems with Applications
19.10.2021r.

1. Introduction

2. Classical Schrödinger operators with reverse Hölder's class potentials

3. Fefferman–Phong inequality in the rational Dunkl setting

4. Applications
   - Hardy spaces
   - Behavior of eigenvalues

Harmonic oscillator operator

Classical harmonic oscillator

$$-\Delta + \|\mathbf{x}\|^2, \quad \Delta = \sum_{j=1}^{N} \partial_j^2, \quad \mathbf{x} \in \mathbb{R}^N.$$

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$$-\Delta_k + \|\mathbf{x}\|^2, \;\; \Delta_k = \sum_{j=1}^{N} T_{e_j}^2.$$

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M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. 192 (1998), no. 3, 519–542.

Results in the harmonic analysis spirit

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Approach to the theory of Riesz transforms.

A. Nowak, K. Stempak, *Riesz transforms for the Dunkl harmonic oscillator*, Math. Z. 262 (2009), no. 3, 539–556.

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Riesz transforms are bounded on $L^p(dw)$, $1 < p < \infty$.

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Further results on Riesz transforms.

P. Boggarapu, S. Thangavelu, *Mixed norm estimates for the Riesz transforms associated to Dunkl harmonic oscillators*, Ann. Math. Blaise Pascal 22 (2015), no. 1, 89–120.

W. Nefzi, *Higher order Riesz transforms for the Dunkl harmonic oscillator*, Taiwanese J. Math. 19 (2015), no. 2, 567–583.

Possible direction of further studies

1. Replace $\|\mathbf{x}\|^2$ by potential $V(\mathbf{x})$ which preserve key properties of $\|\mathbf{x}\|^2$ in some sense.

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   3. spectral properties (e.g. distribution of eigenvalues).

On $\mathbb{R}^N$, $N \geq 3$, let us consider Schrödinger operator

$$\mathscr{L} = -\Delta + V(\mathbf{x}) = -\sum_{j=1}^{N} \partial_j^2 + V(\mathbf{x})$$

where $V \in L^2_{\text{loc}}(\mathbb{R}^N, d\mathbf{x})$ is a non-negative potential.

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3. non-negative polynomial (initially in Ch. Fefferman's work),
4. function of polynomial growth (e.g. laying between two non-negative polynomials).

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Reverse Hölder's class

Let $q > 1$. We say that V belongs to the reverse Hölder's class $\mathrm{RH}^q(d\mathbf{x})$ if there is a constant $C > 0$ such that the inequality

$$\left(\frac{1}{|B|}\int_B V(\mathbf{x})^q \, d\mathbf{x}\right)^{1/q} \leq C\frac{1}{|B|}\int_B V(\mathbf{x}) \, d\mathbf{x}$$

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Example

Non-negative polynomials belong to $\mathrm{RH}^q(d\mathbf{x})$ because any two norms on the finite–dimensional space of polynomials of degree at most d are equivalent.

Auxiliary function $\mathbf{m}$

$$\frac{1}{\mathbf{m}(\mathbf{x})} = \sup\left\{ r > 0 : \frac{1}{r^{N-2}} \int_{B(\mathbf{x},r)} V(\mathbf{y})\, d\mathbf{y} \le 1 \right\}.$$

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Example

If $V(\mathbf{x}) = \|\mathbf{x}\|^2$, then $\mathbf{m}(\mathbf{x}) \sim (1 + \|\mathbf{x}\|)^{-1}$.

Fefferman–Phong inequality

Assume that V belongs to the reverse Hölder's class $\mathrm{RH}^q(d\mathbf{x})$ with $q > \frac{N}{2}$. There is a constant $C > 0$ such that for all $f \in C_c^1(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \mathbf{m}(\mathbf{x})^2 |f(\mathbf{x})|^2 \, d\mathbf{x} \leq C \left(\sum_{j=1}^{N} \int_{\mathbb{R}^N} |\partial_j f(\mathbf{x})|^2 \, d\mathbf{x} + \int_{\mathbb{R}^N} V(\mathbf{x}) |f(\mathbf{x})|^2 \, d\mathbf{x} \right).$$

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1. Studying behaviour of the eigenvalues of $\mathscr{L}$ - box formula (Ch. Fefferman).

Applications of **m** and Fefferman–Phong inequality

1. Studying behaviour of the eigenvalues of $\mathscr{L}$ - box formula (Ch. Fefferman).
2. Estimating of the fundamental solution of $\mathscr{L}$ and studying L^p-bounds of the operators $\nabla\mathscr{L}^{i\gamma}, \nabla\mathscr{L}^{-1/2}, \nabla\mathscr{L}^{-1}\nabla, \nabla^2\mathscr{L}^{-1}$ (Z. Shen).

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3. Studying behaviour of the integral kernel $k_t(x,y)$ of the Schrödinger semigroup $e^{-t\mathscr{L}}$.
4. Hardy spaces associated with $\mathscr{L}$ and prove a local character of atoms (J. Dziubański, J. Zienkiewicz).

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Notation

- We work in $\mathbb{R}^N$, root system R and multiplicity function k are fixed,
- $\mathbf{N} = N + \sum_{\alpha \in R} k(\alpha)$,
- associated weighted measure $dw(\mathbf{x}) = w(\mathbf{x})\, d\mathbf{x}$, where $w(\mathbf{x}) = \Pi_{\alpha \in R} |\langle \mathbf{x}, \alpha \rangle|^{k(\alpha)}$,
- $w(B(\mathbf{x}, r))$ - measure of the Euclidean ball of center $\mathbf{x} \in \mathbb{R}^N$ and radius $r > 0$,
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Dunkl–Schrödinger operator

We will study

$$L = -\Delta_k + V(\mathbf{x}),$$

where Δ_k is generalized Laplace operator.

Reverse Hölder's class

We assume that $q > \max(1, \frac{\mathbf{N}}{2})$ and $V \geq 0$ belongs to the reverse Hölder's class $\mathrm{RH}^q(dw)$, that is, there is a constant $C_{\mathrm{RH}} > 0$ such that

$$\left(\frac{1}{w(B)} \int_B V(\mathbf{x})^q \, dw(\mathbf{x})\right)^{1/q} \leq C_{\mathrm{RH}} \frac{1}{w(B)} \int_B V(\mathbf{x}) \, dw(\mathbf{x}) \text{ for every ball } B.$$

Reverse Hölder's class and function m

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The auxiliary function $m(\mathbf{x})$

For $\mathbf{x} \in \mathbb{R}^N$ we define

$$\frac{1}{m(\mathbf{x})} = \sup\left\{r > 0 : \frac{r^2}{w(B(\mathbf{x}, r))}\int_{B(\mathbf{x},r)} V(\mathbf{y})\, dw(\mathbf{y}) \leq 1\right\}.$$

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Fefferman–Phong type inequality, A.H.

Assume that V belongs to the reverse Hölder's class $\mathrm{RH}^q(dw)$ with $q > \frac{\mathbf{N}}{2}$. There is a constant $C > 0$ such that for all $f \in C_c^1(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} m(\mathbf{x})^2 |f(\mathbf{x})|^2 \, dw(\mathbf{x}) \leq C \left(\sum_{j=1}^{N} \int_{\mathbb{R}^N} |T_j f(\mathbf{x})|^2 \, dw(\mathbf{x}) + \int_{\mathbb{R}^N} V(\mathbf{x}) |f(\mathbf{x})|^2 \, dw(\mathbf{x}) \right).$$

Main steps and difficulties in the proof 1/3

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In the classical proof, the equivalent characterization of the reverse Hölder's class in terms of A_p weights is used.

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In the Dunkl setting, the associated measure dw is not Lebesgue measure, so theory of A_p weights should be adapted.

It can be done be careful studying of the associated measure with the weight

$$w(\mathbf{x}) = \Pi_{\alpha \in R} |\langle \mathbf{x}, \alpha \rangle|^{k(\alpha)}.$$

Main steps and difficulties in the proof 2/3

The key element of the classical proof is Poincaré's inequality

$$\frac{1}{|B(x,r)|}\int_{B(x,R)} |f(y) - f_{B(x,r)}|^2 \, dy \leq \frac{Cr^2}{|B(x,r)|}\int_{B(x,r)} |\nabla f(y)|^2 \, dx.$$

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It is not clear what would be an adequate analog of the Poincaré's inequality in the Dunkl setting.

Instead of Poincaré inequality we use a version of pseudo–Poincaré's inequality.

- J. Dziubański, J. Zienkiewicz, *Hardy spaces H^1 for Schrödinger operators with certain potentials,* Studia Math. 164 (2004), 39–53.
- A. Velicu, *Sobolev-Type Inequalities for Dunkl Operators,* J. Funct. Anal. 279 (2020), no. 7, 108695, 37 pp.

The formula

$$T_j(fg)(\mathbf{x}) = f(\mathbf{x})T_j g(\mathbf{x}) + T_j f(\mathbf{x}) g(\mathbf{x})$$

holds just in specific cases e.g. if f or g is radial.

Main steps and difficulties in the proof 3/3

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Careful analysis of the function m should be done in the context of the Dunkl operators. Assume that $V \in \mathrm{RH}^q(dw)$, where $q > \max(1, \frac{\mathbf{N}}{2})$, and $V \geq 0$. There are constants $C, \kappa > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ we have

$$C^{-1}m(\mathbf{y}) \leq m(\mathbf{x}) \leq Cm(\mathbf{y}) \text{ if } \|\mathbf{x}-\mathbf{y}\| < m(\mathbf{x})^{-1},$$

$$m(\mathbf{y}) \leq Cm(\mathbf{x})(1 + m(\mathbf{x})\|\mathbf{x}-\mathbf{y}\|)^{\kappa},$$

$$m(\mathbf{y}) \geq C^{-1}m(\mathbf{x})(1 + m(\mathbf{x})\|\mathbf{x}-\mathbf{y}\|)^{-\frac{\kappa}{1+\kappa}}.$$

Application is inspired by

J. Dziubański, J. Zienkiewicz, *Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder's inequality*, Rev. Mat. Iberoamericana 15 (1999), no. 2, 279–296.

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Let $\{K_t\}_{t\geq 0}$ - semigroup generated by the Dunkl–Schrödinger operator.

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Let $\{K_t\}_{t \geq 0}$ - semigroup generated by the Dunkl–Schrödinger operator.

Maximal Hardy space

Let $f \in L^1(dw)$. We say that f belongs to the *Hardy space* H^1_L associated with operator L if and only if

$$f^*(\mathbf{x}) = \sup_{t>0} |K_t f(\mathbf{x})|$$

belongs to $L^1(dw)$.

Atomic Hardy space 1/2

We define a collection of dyadic cubes $\mathcal{Q}$ associated with the potential V by the following stopping-time condition:

$$Q \in \mathcal{Q} \iff Q \text{ is the maximal dyadic cube for which } \frac{d(Q)^2}{w(Q)} \int_Q V(\mathbf{y})\, dw(\mathbf{y}) \leq 1.$$

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Atom

A measurable function $a(\mathbf{x})$ is called an *atom for the Hardy space* $H^{1,\mathrm{at}}_{\mathcal{Q}}$ associated with the collection of cubes $\mathcal{Q}$ if

- (A) $\operatorname{supp} a \subseteq B(\mathbf{x}_0, r) \subseteq Q$ for some $Q \in \mathcal{Q}$, $\mathbf{x}_0 \in \mathbb{R}^N$, and $r > 0$,
- (B) $\sup_{\mathbf{y} \in \mathbb{R}^N} |a(\mathbf{y})| \leq w(B(\mathbf{x}_0, r))^{-1}$,
- (C) if $r < d(Q)/8$, then $\int_{\mathbb{R}^N} a(\mathbf{x})\, dw(\mathbf{x}) = 0$.

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Atomic Hardy space

The *atomic Hardy space* $H^{1,\mathrm{at}}_{\mathcal{Q}}$ associated with the collection $\mathcal{Q}$ is the space of functions $f \in L^1(dw)$ which admit a representation of the form

$$f(\mathbf{x}) = \sum_{j=1}^{\infty} c_j a_j(\mathbf{x}),$$

where $c_j \in \mathbb{C}$ and a_j are atoms for the Hardy space $H^{1,\mathrm{at}}_{\mathcal{Q}}$ such that $\sum_{j=1}^{\infty} |c_j| < \infty$.

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Theorem, A.H.

We have $H^{1,\mathrm{at}}_{\mathcal{Q}} = H^1_L$ (together with the relevant norm equivalence).

Behavior of eigenvalues 1/2

Application is inspired by

- C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. (N.S.) 9 (1983), no. 2, 129–206.
- K. Kurata, S. Sugano, *Fundamental solution, eigenvalue asymptotics and eigenfunctions of degenerate elliptic operators with positive potentials*, Studia Math. 138 (2000), no. 2, 101–119.

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For $a > 0$ we define

$$(\mathrm{Grid})_a = \{[0,a]^N + a\mathbf{n} \, : \, \mathbf{n} \in \mathbb{Z}^N\}.$$

Behavior of eigenvalues 2/2

$N(L,\lambda)$ - number of eigenvalues of the operator $L \leq \lambda$

Behavior of eigenvalues 2/2

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$N(L,\lambda)$ - number of eigenvalues of the operator $L \leq \lambda$

Theorem, A.H.

Assume that $V \in \mathrm{RH}^q(dw)$, where $q > \max(1, \frac{\mathbf{N}}{2})$, and $V \geq 0$. For $\lambda > 0$ we set

$$E_\lambda = \{\mathbf{x} \in \mathbb{R}^N \,:\, m(\mathbf{x}) \leq \sqrt{\lambda}\}.$$

Let $M(\lambda)$ denote the number of cubes K from the $(\mathrm{Grid})_{\lambda^{-1/2}}$ such that $K \cap E_\lambda \neq \emptyset$. There are constants $C_1, C_2, C_3 > 0$, which depend on R, N, q, k and the constant C_{RH} such that for all $\lambda > 0$ we have

$$M(C_1^{-1}\lambda) \leq N(L,\lambda) \leq C_2 M(C_3^{-1}\lambda).$$

Thank you for your attention.