Dunkl processes - freezing, jumps and collisions

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Modern Analysis Related to Root Systems with Applications
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1.1 Notations and definitions: (rational) Dunkl operators

Consider $x \in \mathbb{R}^N$ and a function $f : \mathbb{R}^N \to \mathbb{C}$. Take $\beta > 0$ and $\kappa > 0$. T_i is defined by [Dunkl (1989)]

$$T_i f(x) := \frac{\partial}{\partial x_i} f(x) + \frac{\beta}{2} \sum_{\alpha \in R_+} \kappa(\alpha) \frac{f(x) - f(\sigma_\alpha x)}{\alpha \cdot x} \alpha_i$$

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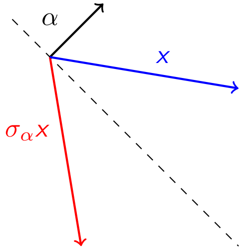
Reflection operator:

$$\sigma_\alpha x = x - 2 \frac{\alpha \cdot x}{\|\alpha\|^2} \alpha.$$



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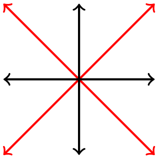
Root system R: finite set of roots $\alpha \in \mathbb{R}^N$,

$$R = \{\alpha \in \mathbb{R}^N | \forall \xi \in R, \sigma_\alpha \xi \in R\}.$$

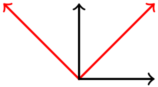
We denote the rank of R by d_R. Positive subsystem R_+: choose an arbitrary vector m s.t. $m \cdot \alpha \neq 0$. Then,

$$R_+ = \{\alpha \in R | m \cdot \alpha > 0\}.$$

R generates a reflection group, W.



Example: $R = B_2$



Example: $R_+ = B_{2,+}$

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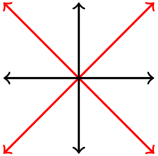
Multiplicities $\kappa(\alpha) > 0$: parameters assigned to each disjoint orbit of the roots under the action of W, i.e., $\forall \alpha, \xi \in R$,

$$\kappa(\sigma_\alpha \xi) = \kappa(\xi).$$

We impose the condition that at least one multiplicity is equal to 1. We also define the sum

$$\gamma = \sum_{\alpha \in R_+} \kappa(\alpha).$$



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Note that we have set

$$k(\alpha) = \frac{\beta}{2} \kappa(\alpha).$$

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Intertwining operator V_β

V_β: linear, degree-preserving functional uniquely defined by $V_\beta[1] = 1$ and

$$T_i V_\beta[f(x)] = V_\beta\left[\frac{\partial}{\partial x_i} f(x)\right].$$

Form of V_β is unknown in general. Developments: [Deleaval-Demni-Youssfi (2015)], [De Bie-Lian (2021)]. Comment by M. Rösler: formula for A_{N-1} by P. Sawyer.}

1.2 Dunkl processes

[Rösler-Voit (1998)] Dunkl process: Markov process with *Dunkl Laplacian* as generator. Transition density $p(t, y|x)$ satisfies:

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Explicitly,

$$\frac{\partial}{\partial t}p(t,y|x)=\frac{1}{2}\Delta p(t,y|x)+\frac{\beta}{2}\sum_{\alpha\in R_+}\kappa(\alpha)\Big(\frac{\alpha\cdot\nabla}{\alpha\cdot x}-\frac{\|\alpha\|^2}{2}\frac{1-\sigma_\alpha}{(\alpha\cdot x)^2}\Big)p(t,y|x).$$

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Radial Dunkl process: invariant under the action of W (W-*invariant*), jump term vanishes [Biane-Bougerol-O'Connell, Gallardo-Yor (2005)].

$$\hat{p}(t, y|x) = \sum_{\rho \in W} p(t, y|\rho x).$$

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Radial Dunkl processes of type A_{N-1} and B_N $\to$ Dyson's model and the Wishart-Laguerre processes (matrix-valued processes).

1.3 Scaled Dunkl processes

$$p(t,y|x) = w_\beta\left(\frac{y}{\sqrt{t}}\right)\frac{\mathrm{e}^{-(\|x\|^2+\|y\|^2)/2t}}{c_\beta t^{N/2}} V_\beta \exp\left(\frac{x\cdot y}{t}\right).$$

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Observation: setting $y = \sqrt{\beta t}Y$,

$$(\beta t)^{N/2} p(t,\sqrt{\beta t}Y|x) \quad \overset{t\to\infty}{\longrightarrow} \quad \frac{e^{-\beta F_R(Y)}}{z_\beta},\ F_R(Y) = \frac{\|Y\|^2}{2} - \sum_{\alpha\in R_+}\kappa(\alpha)\log|\alpha\cdot Y|.$$

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β: inverse temperature.

2.1 Freezing

$\beta \to \infty$ in a radial Dunkl process

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A_{N-1} case, pointwise limit

$$\sum_{\rho \in S_N} V_\beta[\exp(\rho x \cdot y)] = N!\,{}_0\mathcal{F}_0^{(2/\beta)}(x,y) = \sum_{n=0}^{\infty} \sum_{\lambda:\sum_l \lambda_l = n} \frac{N!}{\lambda!} \frac{\mathcal{C}_\lambda^{(2/\beta)}(x)\mathcal{C}_\lambda^{(2/\beta)}(y)}{\mathcal{C}_\lambda^{(2/\beta)}(1,\ldots,1)}$$ [Baker-Forrester (1997)]

$$\lim_{\beta\to\infty} N!\,{}_0\mathcal{F}_0^{(2/\beta)}(x,y) = N! \exp\left(\frac{1}{N}\sum_{j=1}^{N} x_j \sum_{l=1}^{N} y_l\right)$$ [A-Katori-Miyashita (2012)]

$$\left(\sum_{j=1}^{N} x_j = \sum_{l=1}^{N} y_l = 0\right) \quad \lim_{\beta\to\infty} N!\,{}_0\mathcal{F}_0^{(2/\beta)}(\sqrt{\beta}x, y) = N! \exp\left(\frac{\|x\|^2\|y\|^2}{N(N-1)}\right)$$ [A-Voit (2019B)]

2.2 Freezing

B_N case, $\kappa_1 = \nu, \kappa_2 = 1$

$$\lim_{\beta\to\infty} \sum_{\rho\in W_{B_N}} V_\beta[\exp(\rho x \cdot \sqrt{\beta} y)] = 2^N N! \exp\left(\frac{\|x\|^2 \|y\|^2}{2N(N+\nu-1)}\right)$$ [A-Katori-Miyashita (2014)]

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General case for $x, y \in \mathrm{span}(R)$, pointwise limit

$$\lim_{\beta\to\infty} \sum_{\rho\in W} V_\beta[\exp(\rho x \cdot \sqrt{\beta} y)] = |W| \exp\left(\frac{\|x\|^2\|y\|^2}{2\gamma}\right)$$ [A-Miyashita (2015)]

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2.4 Freezing

Fixed initial point x, radial Dunkl process $(X_t)_{t\geq 0}$

$$\lim_{\beta\to\infty} \frac{X_t}{\sqrt{\beta t}} = z \quad \text{weakly.}$$

Vector z given by

- zeros of $H_N(x)$ for A_{N-1} [A-Katori-Miyashita (2012)],
- sq. roots of zeros of $L_N^{(\nu-1)}(x)$ for B_N [A-Katori-Miyashita (2014)],
- $z = \sum_{\alpha\in R_+} \alpha \frac{\kappa(\alpha)}{\alpha\cdot z}$ for R [A-Miyashita (2015)].

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Scaled initial point $x=x_1+\sqrt{\beta}x_2$, x_2 at interior of Weyl chamber

$$\lim_{\beta\to\infty}\frac{X_t}{\sqrt{\beta}}=\Phi(t,x_2) \quad \text{loc. unif. with } \Phi \text{ deterministic [A-Voit (2019A)].}$$

Also, $\Phi(t,x_2)$ approaches $z\sqrt{t+\|x_2\|^2/\gamma}$ as $t\to\infty$.

Scaled initial point $x = x_1 + \sqrt{\beta}x_2$, x_2 at interior of Weyl chamber

$$\lim_{\beta\to\infty} \frac{\frac{X_t}{\sqrt{\beta}} - \Phi(t, x_2)}{1/\sqrt{\beta}} = W_t \quad \text{loc. unif. [Voit-Woerner (2020)].}$$

Process $(W_t)_{t\geq 0}$ is Gaussian with correlated increments.

2.5 Freezing - fluctuations

Scaled initial point $x = x_1 + \sqrt{\beta} x_2$, x_2 at interior of Weyl chamber

$$\lim_{\beta\to\infty} \frac{\frac{X_t}{\sqrt{\beta}} - \Phi(t, x_2)}{1/\sqrt{\beta}} = W_t \quad \text{loc. unif. [Voit-Woerner (2020)].}$$

Process $(W_t)_{t\geq 0}$ is Gaussian with correlated increments.

Fixed initial point x

$$\lim_{\beta\to\infty} \frac{X(t)/\sqrt{\beta t} - z}{1/\sqrt{\beta}} = \xi \quad \text{weakly.}$$

$\xi \sim \mathcal{N}(0, \Sigma)$, covariance matrix depends on R. Laguerre case [Voit (2019)], Hermite case [A-Voit (2019B)], similar static result for Jacobi case [Hermann-Voit (2019)].

2.6 Freezing - covariance matrices

Hermite case:

$$[\Sigma_H^{-1}]_{i,j} := \delta_{ij} + \delta_{ij} \sum_{l=1:l\neq i} \frac{1}{(z_{H,i} - z_{H,l})^2} - \frac{1-\delta_{ij}}{(z_{H,i} - z_{H,j})^2}.$$

Laguerre case:

$$[\Sigma_L^{-1}]_{i,j} := \delta_{ij} + \frac{\delta_{ij}\nu}{z_{L,i}^2} + \delta_{ij} \sum_{l=1:l\neq i}^{N} \left[\frac{1}{(z_{L,i} - z_{L,l})^2} + \frac{1}{(z_{L,i} + z_{L,l})^2}\right] - \frac{1-\delta_{ij}}{(z_{L,i} - z_{L,j})^2} + \frac{1-\delta_{ij}}{(z_{L,i} + z_{L,j})^2}.$$

Jacobi case:

$$[\Sigma_J^{-1}]_{i,j} := \frac{\hat{\alpha}+1}{2} \frac{\delta_{ij}}{(1 - z_{J,i})^2} + \frac{\hat{\beta}+1}{2} \frac{\delta_{ij}}{(1 + z_{J,i})^2} + \sum_{l=1:l\neq i} \frac{\delta_{ij}}{(z_{J,i} - z_{J,l})^2} - \frac{1-\delta_{ij}}{(z_{J,i} - z_{J,j})^2}.$$

We derived Σ using de Boor-Saff dual polynomials [A-Hermann-Voit (2021)]. See K. Hermann's talk.

2.7 Freezing - (non-radial) Dunkl processes

$\beta \to \infty$ then $N \to \infty$

- Radial type $A_{N-1} \to$ Wigner semicircle.
- Radial type $B_N \to$ Marchenko-Pastur distribution. Comment by M. Voit: these two first results are known, see the textbook by Anderson, Guionnet and Zeitouni.
- Full Dunkl type $B_N \to$ non-symmetric semicircle-like distributions.

This is accomplished using free convolutions [Voit-Woerner (2020)]. See J.H.C. Woerner's talk.

3.1 Dunkl jump processes

- First studied by Gallardo, Yor, Chybiryakov (2006∼2008).
- $(X_t)_{t\geq 0}$: Dunkl process, $(\hat{X}_t)_{t\geq 0}$: radial process. Whenever $\beta\kappa(\alpha) > 1$ for all α, there exists ρ_t s.t.

$$X_t = \rho_t \hat{X}_t.$$

- ρ_t is the jump process, a doubly stochastic Poisson process on the Weyl group W.
- Jump probability from x:

$$p(dt, y|x) = \frac{\beta}{2} \sum_{\alpha\in R_+} k(\alpha) \frac{\|\alpha\|^2}{2} \frac{\delta(\sigma_\alpha y - x)}{(\alpha \cdot x)^2} dt.$$

3.2 Dunkl jump processes - dynamics

- Jump transition rates: Dunkl process starts at x_0, Weyl Chamber C_W,

$$\lambda_\beta(t,\alpha|x_0) := \frac{\beta\|\alpha\|^2}{4}\int_{C_W}\frac{k(\alpha)}{(\alpha\cdot x)^2}\hat{p}(t,x|x_0)dx, \qquad \Lambda_\beta(t|x_0) := \sum_{\alpha\in R+}\lambda_\beta(t,\alpha|x_0).$$

- $P_\beta^{\mathcal{J}}(t,\tau|x_0)$: prob. that $\rho_t = \tau \in W$, starting from x_0. Then,

$$\frac{\partial}{\partial t}P_\beta^{\mathcal{J}}(t,\tau|x_0) = \sum_{\alpha\in R_+}\lambda_\beta(t,\alpha|x_0)P_\beta^{\mathcal{J}}(t,\tau\sigma_\alpha|x_0) - \Lambda_\beta(t|x_0)P_\beta^{\mathcal{J}}(t,\tau|x_0).$$

Note that the radial part affects the jump part, but not the other way around!

3.3 Dunkl jump processes - convergence to uniform probability

[A 2020]

For $x_0 \neq 0$ and $\beta > 1$, there exists $0 < r \leq 1$ s.t.

$$P_\beta^{\mathcal{J}}(t,\tau|x_0) = \frac{1}{|W|} + \frac{C(\tau,x_0)}{t^r} + o(t^{-r}).$$

In particular, for the A_{N-1} case

$$r = r(\beta) = \frac{1}{2} + o(\beta^{-1}).$$

- These results follow from comparing the dynamics of $P_\beta^{\mathcal{J}}(t,\tau|x_0)$ to those of a similar process with transitions given by $\lambda_\beta(t,\alpha|0)$, which has nicer properties than $\lambda_\beta(t,\alpha|x_0)$.
- Dynamical exponents follow from perturbation theory on $\lim_{\beta\to\infty}\lambda_\beta(t,\alpha|0)$ and integrable systems.
- $\lim_{\beta\to\infty}\lambda_\beta(t,\alpha|0)$ closely resemble the inv. covariance matrices Σ^{-1} of [A-Hermann-Voit (2021)].

3.4 Dunkl jump processes - phase transition

[A 2020]

Set $\beta > 1$ and $\{\mathrm{x}_0^{(N)} \in C_{W_R^N}\}_{N=2}^{\infty}$ s.t. $\|\mathrm{x}_0^{(N)}\| \leq K$ with $K > 0$ fixed. Fixing $t = N$ we have

$$\lim_{N\to\infty} \frac{1}{N}\Lambda_\beta^{(R)}(N|\mathrm{x}_0^{(N)}) = \frac{\beta}{c_R(\beta-1)}, \tag{1}$$

for $R = A_{N-1}$ and B_N with $\kappa(\alpha) = \beta/2$ for all α and $c_{A_{N-1}} = 8$ and $c_{B_N} = 4$, respectively.

- This shows a phase transition at $\beta = 1$ for the total jump rate per particle.
- The calculation of $\Lambda_\beta^{(R)}(N|0)$ is straightforward, as one can show that $\Lambda_\beta^{(R)}(1|0) = \beta|R_+|/4(\beta-1)$.
- In [Demni (2009)], it is shown that there are collisions a.s. when $\beta < 1$.
- Recall that $\lambda_\beta(t,\alpha|\mathrm{x}_0) = \frac{\beta\|\alpha\|^2}{4}\int_{C_W}\frac{k(\alpha)}{(\alpha\cdot x)^2}\hat{p}(t,x|\mathrm{x}_0)dx$! The phase transition is driven by collisions!

4. Concluding remarks

Jumps

- No formulation for $\beta < 1$
- Behavior clearly tied to collisions

Collisions

- Characterization of collisions after the first one?
- Frequency, β-dependence
- See N. Hufnagel's talk

Freezing

- Several new and exciting results
- Heckman-Opdam case, use of freezing to find martingales of eigenfunctions of generator [Rösler-Voit 2021]
- Dynamical results for compact cases?

Thank you!

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