Hausdorff dimension of collision times for the multivariate Bessel process

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Preliminaries - Bessel process type A_{N-1}

The multivariate Bessel process of type A_{N-1} is given via the SDE

$$\mathrm{d}X_{t,i}^{A} = \mathrm{d}B_{t,i} + k \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mathrm{d}t}{X_{t,i}^{A} - X_{t,j}^{A}}$$

for $X_0^A = x$ and $k > 0$, where $i = 1, \dots, N$.
This process lives on the Weyl-Chamber

$$W_{A_{N-1}} := \{x \in \mathbb{R}^N \mid x_1 < x_2 < \cdots < x_N\}.$$

[Demni 2009]:

- $k > \frac{1}{2}$: No collisions with $\partial W_{A_{N-1}}$.
- $k < \frac{1}{2}$: First collision time is a.s. finite.

Preliminaries - multivariate Bessel process type A_1

Question: How frequently does the process hit the border of the Weyl chamber?

Problem: $\lambda^1(X^{-1}(\partial W_{A_{N-1}})) = 0$ and $|X^{-1}(\partial W_{A_{N-1}})| = \infty$ a.s.

Recall: $W_{A_{N-1}} := \{x \in \mathbb{R}^N \mid x_1 < x_2 < \cdots < x_N\}$.

Dimension $N = 2$:

- $Y_{1,t} := \frac{1}{\sqrt{2}}(X_{1,t} - X_{2,t})$ classical one dim. Bessel process,
- $Y_{2,t} := \frac{1}{\sqrt{2}}(X_{1,t} + X_{2,t})$ Brownian motion,
- $X_{1,t} = X_{2,t} \Leftrightarrow Y_{1,t} = 0$.

Answer [Liu and Xiao 1998]: The Hausdorff dimension of $Y_1^{-1}(0) = X^{-1}(\partial W_{A_1})$ is $\frac{1}{2} - k$.

Aim: Derive the Hausdorff dimension of $X^{-1}(\partial W_{A_{N-1}})$.

Hausdorff dimension

The Hausdorff measure for $\alpha > 0$ is defined as

$$m_\alpha(E) := \lim_{\varepsilon \to 0} \inf\{\sum_{i=1}^{\infty} (2r_i)^\alpha : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), 0 \leq r_i < \varepsilon\}.$$

Lemma [Adler 1981, 8.1 Hausdorff dimension]

For any set $E \subset \mathbb{R}^n$ there exists a unique number $\alpha^\star$, called the Hausdorff dimension of E, for which

$$\alpha < \alpha^\star \Rightarrow m_\alpha(E) = \infty, \quad \alpha > \alpha^\star \Rightarrow m_\alpha(E) = 0.$$

This number is denoted by $\dim(E)$ *and satisfies*

$$\begin{aligned} \alpha^\star = \dim(E) &= \sup\{\alpha > 0 : m_\alpha(E) = \infty\} \\ &= \inf\{\alpha > 0 : m_\alpha(E) = 0\}. \end{aligned}$$

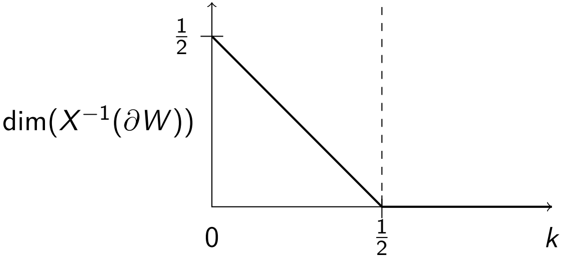
Main result

Theorem [Andraus, Hufnagel 2021]

The Hausdorff dimension is

$$\dim(X^{-1}(\partial W_{A_{N-1}})) = \frac{1}{2} - k$$

P^x-almost surely for $x \in \overline{W}_{A_{N-1}}$ and $k < \frac{1}{2}$.



Main difference to [Liu and Xiao 1998]

[Liu and Xiao 1998]: $B(0,r) := \left\{ x \in \mathbb{R}^N \mid \|x\|_2 \leq r \right\}$.

Edge set with thickness r:

$$\begin{aligned} E^r_{A_{N-1}} &:= \left\{ x \in \overline{W}_{A_{N-1}} \mid \mathrm{dist}_{\|.\|_1}(x, \partial W_{A_{N-1}}) \leq r \right\} \\ &= \left\{ x \in \overline{W}_{A_{N-1}} \mid \exists\, i \in \{1, \ldots, N-1\} : 0 \leq x_{i+1} - x_i \leq r \right\}. \end{aligned}$$

Covering: $X^{-1}(\partial W_{A_{N-1}}) = X^{-1}(E^0_{A_{N-1}}) \subset X^{-1}(E^r_{A_{N-1}})$.

Problem: $E^r_{A_{N-1}}$ not bounded.

Crucial point: We extend to a bigger, complicated set.

Solution: Auxiliary stopping time $T_R := \inf\{t > 0 : \|X_t\|_2 > R\}$.

Sketch of proof: Upper bound

$$\mathbb{E}^x(m_{\frac{1}{2}-k}(X^{-1}(\partial W_{A_{N-1}}) \cap [\varepsilon, t \cap T_R])) < \infty \Rightarrow$$
$$\dim(X^{-1}(\partial W_{A_{N-1}}) \cap [\varepsilon, t \cap T_R]) \leq \frac{1}{2} - k$$

Lemma

Given $R > r > 0$, for every $\varepsilon > 0$, there exists a constant $C(N, R, k, \varepsilon)$ such that

$$\mathbb{P}^x[\exists s \in [t_1, t_2 \wedge T_R] : X_s \in E^r_{A_{N-1}}] \leq C(t_2 - t_1)^{k+1/2}$$

for every $t_2 > t_1 > \varepsilon$ and $x \in E^r_{A_{N-1}}$.

Sketch of proof.

- $C_1 r^{2k+1} \leq \mathbb{P}^x(X_1 \in E^r_{A_{N-1}}) \leq C_2 r^{2k+1}$
- $\frac{1}{2}$-semi stable: $X_{ct} \sim c^{\frac{1}{2}} X_t$
- strong Markov property

□

Sketch of proof: Lower bound

Lemma [Adler 1981, 8.1 Hausdorff dimension]

Let A be a compact subset of $\mathbb{R}^n$. Suppose there exists a positive measure μ with

$$\mu(\mathbb{R}^n) = \mu(A) = 1$$

and a finite α such that the energy integrals satisfy

$$I_\beta(\mu) = \int_A \int_A \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{\|x-y\|^\beta} < \infty$$

for all $\beta < \alpha$. Then, $\dim(A) \geq \alpha$.

References

R. Adler, *The Geometry of Random Fields.* Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), 1981.

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N. Demni, "Radial Dunkl processes: Existence, uniqueness and hitting time", *Comptes Rendus de l'Académie des Sciences de Paris*, Série I, vol. 347, pp. 1125-1128, 2009.

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Appendix - Bessel process type B_N

The multivariate Bessel process of type B_N is given via the SDE

$$\mathrm{d}X_{t,i}^B = \mathrm{d}B_{t,i} + k_1 \frac{\mathrm{d}t}{X_{t,i}^B} + k_2 \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{1}{X_{t,i}^B - X_{t,j}^B} + \frac{1}{X_{t,i}^B + X_{t,j}^B} \right) \mathrm{d}t,$$

for $k_1, k_2 > 0$, where $i = 1, \ldots, N$.
This process lives on the Weyl-Chamber

$$W_{B_N} := \{x \in \mathbb{R}^N \mid 0 < x_1 < x_2 < \cdots < x_N\}.$$

[Demni 2009]:

- $\min(k_1, k_2) > \frac{1}{2}$: No collisions with ∂W_{B_N}.
- $\min(k_1, k_2) < \frac{1}{2}$: First collision time is a.s. finite.

Result:

$$\dim(X^{-1}(\partial W_{B_N})) = \frac{1}{2} - \min(k_1, k_2).$$