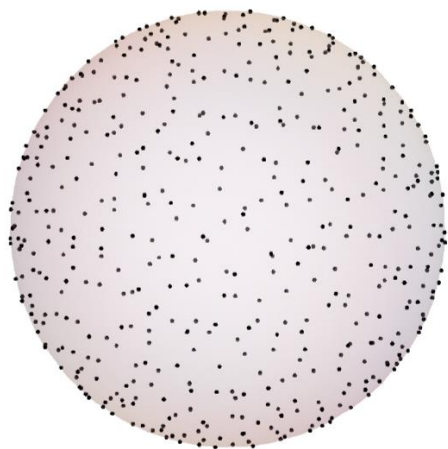# Orthogonal Theta Functions Associated with Affine Root Systems and Determinantal Point Processes

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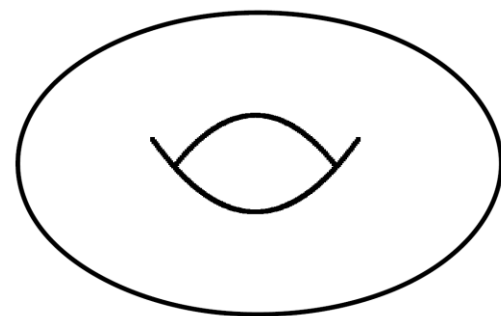**Conference**

Modern Analysis Related to Root Systems with Applications

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CIRM - Luminy

Centre International de Rencontres Mathématiques

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1. R_n Theta Functions of Rosengren and Schlosser

- We write the complex plane which is punctured at the origin as

$$\mathbb{C}^{\times} := \mathbb{C} \setminus \{0\} = \{\zeta \in \mathbb{C} : 0 < |\zeta| < \infty\}.$$

- Let $p \in \mathbb{C}$ be a fixed number so that $0 < |p| < 1$. The theta function with argument $\zeta \in \mathbb{C}^{\times}$ and nome p is defined by

$$\theta(\zeta; p) = \prod_{j=0}^{\infty} \left(1 - \zeta p^j\right)\left(1 - \frac{p^{j+1}}{\zeta}\right).$$

- By this definition, we can readily see that $\lim_{p \to 0} \theta(\zeta; p) = 1 - \zeta$.

This implies that $\lim_{p \to 0} \dfrac{\theta(q^n; p)}{1 - q} = \dfrac{1 - q^n}{1 - q} =: [n]_q$ (q-analogue of $n \in \mathbb{N}$).

If we consider $\theta(aq^n; p)$ with $a = e^{-2i\alpha}$ and $q = e^{-2i\phi}$, $\alpha, \phi \in [0, 2\pi)$, $i := \sqrt{-1}$,

$$\lim_{p \to 0} \frac{\theta(aq^n; p)}{2i\sqrt{aq^n}} = \sin(\alpha + n\phi) \quad \text{(triginometric function)}.$$

(q, p)-analogues
theta functions

elliptic extensions

q-analogues
triginometric functions

q-extensions

classical numbers
polynomials

- The fact $\lim\limits_{p \to 0} \theta(\zeta; p) = 1 - \zeta$ suggests that the theta function $\theta(\zeta; p)$ is an elliptic analogue of a linear function of ζ.

- What is the **elliptic analogue of a polynomial of ζ?**

- It might be given by a product of θ's. But should notice the equalities

$$\theta(\zeta^k; p^k) = \prod_{j=0}^{k-1} \theta(\zeta \omega_k^j; p), \quad \theta(\zeta; p) = \prod_{j=0}^{k-1} \theta(\zeta p^j; p^k), \quad k \in \mathbb{N},$$

  Here ω_k denotes a primitive k-th root of unity.
  The degree of product of θ's depends on a choice of nome.

- In order to define a degree of products of θ's with respect to a specified nome, **Rosengren and Schlosser (2006)** generalized the notion of the **quasi-periodicity** of the theta function, $\quad \theta(p\zeta; p) = -\dfrac{1}{\zeta}\theta(\zeta; p).$

- We notice that we have also the **inversion formula** $\theta(1/\zeta; p) = -\dfrac{1}{\zeta}\theta(\zeta; p)$, and the combination of these two proves the **periodicity** of the theta function, $\theta(p/\zeta; p) = \theta(\zeta, p).$

Definition 1.1 (Rosengren and Schlosser (2006)) Assume that $f(\zeta)$ is holomorphic in $\mathbb{C}^\times$. Then if there is a parameter $r \in \mathbb{C}^\times$ and f satisfies the equality,

$$f(p\zeta) = \frac{(-1)^n}{r\zeta^n} f(\zeta),$$

then f is said to be an A_{n-1} **theta function** of **norm** r. The space of all A_{n-1} theta functions with nome p and norm r is denoted by $\mathcal{E}_{p,r}^{A_{n-1}}$.

[RS06] Rosengren, H., Schlosser, M.: Elliptic determinant evaluations and the Macdonald identities for affine root systems. Compositio Math. 142, 937–961 (2006)

- By this definition, we can say that $\theta(\zeta; p)$, satisfying the quasi-periodicity, $\theta(p\zeta; p) = -\frac{1}{\zeta}\theta(\zeta; p)$, is an A_0 theta function of norm $r = 1$.

- **The following is proved.**

Lemma 1.2 (Tarasov and Varchenko (1997)) The space $\mathcal{E}_{p,r}^{\boldsymbol{A}_{n-1}}$ is n **dimensional** and $\{\psi_j^{\boldsymbol{A}_{n-1}}(\zeta;p,r)\}_{j=1}^n$ defined below form a **basis**. **For** $j = 1, \ldots, n$,

$$\psi_j^{\boldsymbol{A}_{n-1}}(\zeta;p,r) := \zeta^{j-1}\theta(p^{j-1}(-1)^{n-1}r\zeta^n;p^n)$$

$$= \zeta^{j-1}\prod_{k=0}^{n-1}\theta(\alpha^{j-1}\beta\omega_n^k\zeta;p),$$

where α and β are complex numbers such that $\alpha^n = p$, $\beta^n = (-1)^{n-1}r$, respectively, and ω_n is a primitive n-th root of unity.

[TV97] Tarasov, V., Varchenko, A.: Geometry of q-hypergeometric functions, quantum affine algebras and elliptic quantum groups. Astérisque <u>246</u>, 135 pages (1997)

- **By** $\lim\limits_{p \to 0} \theta(\zeta; p) = 1 - \zeta$, **we see that**

$$\psi_j^{\boldsymbol{A}_{n-1}}(\zeta; 0, r) := \lim_{p \to 0} \psi_j^{\boldsymbol{A}_{n-1}}(\zeta; p, r) = \begin{cases} 1 - (-1)^{n-1} r \zeta^n, & j = 1, \\ \zeta^{j-1}, & j = 2, \ldots, n. \end{cases}$$

- **Hence** $\mathcal{E}_{0,r}^{\boldsymbol{A}_{n-1}}$ **spanned by them is a space of polynomials of degree** n **in the form**

$$c_0 + c_1 \zeta + \cdots + c_n \zeta^n \quad \text{with} \quad \frac{c_n}{c_0} \equiv (-1)^{n-1} r.$$

It implies that $\dim \mathcal{E}_{0,r}^{\boldsymbol{A}_{n-1}} = n$.

- **It is easy to verify that** $\det\limits_{1 \le j,k \le n} \left[\psi_j^{\boldsymbol{A}_{n-1}}(\zeta_k; 0, r) \right] = \left(1 - r \prod\limits_{\ell=1}^{n} \zeta_\ell \right) W^{\boldsymbol{A}_{n-1}}(\boldsymbol{\zeta})$,

where $W^{\boldsymbol{A}_{n-1}}(\boldsymbol{\zeta}) := \prod\limits_{1 \le j < k \le n} (\zeta_k - \zeta_j), \quad \boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$

(Vandermonde determinant =Weyl denominator of type $\boldsymbol{A}_{n-1}$**).**

- **If we set** $r = 0$ **as well as** $p = 0$, $\mathcal{E}_{0,0}^{\boldsymbol{A}_{n-1}}$ **is the space of all polynomials of degree** $n-1$ **without any restriction on coefficients. In this case, the above is reduced to** $\det\limits_{1 \le j,k \le n} \left[\zeta_k^{j-1} \right] = \prod\limits_{1 \le j < k \le n} (\zeta_k - \zeta_j)$ **which is known as the** **Weyl denominator formula of type** $\boldsymbol{A}_{n-1}$**.**

- The elliptic extension of $W^{A_{n-1}}$ is defined as follows.

Definition 1.3 (Macdonald (1972)) For $\zeta = (\zeta_1, \dots, \zeta_n) \in (\mathbb{C}^\times)^n$, the Macdonald denominator of type A_{n-1} is defined as

$$M^{A_{n-1}}(\zeta; p) := \prod_{1 \le j < k \le n} \zeta_k \theta(\zeta_j/\zeta_k; p).$$

- It is easy to confirm that $\lim_{p \to 0} M^{A_{n-1}}(\zeta; p) = W^{A_{n-1}}(\zeta)$.

- The elliptic extension of the Weyl denominator formula of type A_{n-1} was proved by Rosengre and Schlosser [RS06]. See also Proposition 5.6.3 on page 216 of the textbook of Forrester [For10].

Proposition 1.4 (Rosengren and Schlosser (2006))

$$\det_{1 \le j,k \le n} \left[\psi_j^{A_{n-1}}(\zeta_k; p, r) \right] = \frac{(p; p)_\infty^n}{(p^n; p^n)_\infty^n} \theta\left(r \prod_{\ell=1}^n \zeta_\ell; p \right) M^{A_{n-1}}(\zeta; p).$$
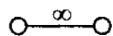
[Mac72] Macdonald, I. G.: Affine root systems and Dedekind's η-function. Invent. Math. <u>15</u>, 91–143 (1972)

[For10] Forrester, P. J.: Log-Gases and Random Matrices. Princeton University Press, Princeton, NJ, (2010)
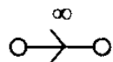
- **There are seven infinite families of irreducible reduced affine root systems,** which are denoted by A, B, $B^\vee$, C, $C^\vee$, BC and D [Mac72].

Type

$A_1 = A_1^\vee$

$BC_1 = BC_1^\vee$

$A_l = A_l^\vee$ $(l \geqq 2)$

B_l $(l \geqq 3)$

$B_l^\vee$ $(l \geqq 3)$
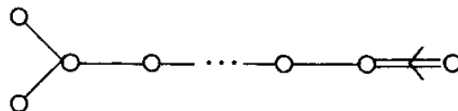
C_l $(l \geqq 2)$

$C_l^\vee$ $(l \geqq 2)$
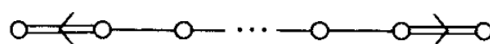
$BC_l = BC_l^\vee$ $(l \geqq 2)$
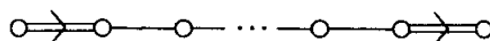
$D_l = D_l^\vee$ $(l \geqq 4)$

[Mac72] Macdonald, I. G.: Affine root systems and Dedekind's η-function. Invent. Math. <u>15</u>, 91–143 (1972)

- There are seven infinite families of irreducible reduced affine root systems, which are denoted by A, B, $B^\vee$, C, $C^\vee$, BC and D [Mac72].
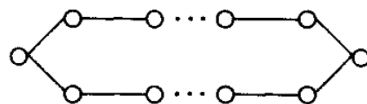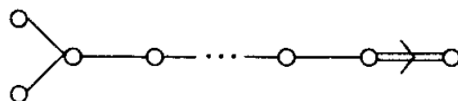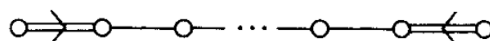
- Associated with them, Rosengren and Schlosser [RS06] defined seven types of theta functions. (The A_{n-1} theta function was already explained.)

Definition 1.5 (Rosengren and Schlosser (2006)) Assume that $f(\zeta)$ is holomorphic in $\mathbb{C}^\times$. For $R_n = B_n$, $B_n^\vee$, C_n, $C_n^\vee$, BC_n, D_n, we call f an R_n theta function if the following are satisfied,

$$f(p\zeta) = -\frac{1}{p^{n-1}\zeta^{2n-1}}f(\zeta), \quad f(1/\zeta) = -\frac{1}{\zeta}f(\zeta), \quad \text{if } R_n = B_n,$$

$$f(p\zeta) = -\frac{1}{p^n\zeta^{2n}}f(\zeta), \quad f(1/\zeta) = -f(\zeta), \quad \text{if } R_n = B_n^\vee,$$

$$f(p\zeta) = \frac{1}{p^{n+1}\zeta^{2n+2}}f(\zeta), \quad f(1/\zeta) = -f(\zeta), \quad \text{if } R_n = C_n,$$

$$f(p\zeta) = \frac{1}{p^{n-1/2}\zeta^{2n}}f(\zeta), \quad f(1/\zeta) = -\frac{1}{\zeta}f(\zeta), \quad \text{if } R_n = C_n^\vee,$$

$$f(p\zeta) = \frac{1}{p^n\zeta^{2n+1}}f(\zeta), \quad f(1/\zeta) = -\frac{1}{\zeta}f(\zeta), \quad \text{if } R_n = BC_n,$$

$$f(p\zeta) = \frac{1}{p^{n-1}\zeta^{2n-2}}f(\zeta), \quad f(1/\zeta) = f(\zeta), \quad \text{if } R_n = D_n.$$

The space of all R_n theta functions with nome p is denoted by $\mathcal{E}_p^{R_n}$.

- **In order to clarify the common structure, we introduce the notations,**

$$\mathcal{N} = \mathcal{N}^{\boldsymbol{R}_n} := \begin{cases} 2n - 1, & \boldsymbol{R}_n = \boldsymbol{B}_n, \\ 2n, & \boldsymbol{R}_n = \boldsymbol{B}_n^\vee, \boldsymbol{C}_n^\vee, \\ 2n + 2, & \boldsymbol{R}_n = \boldsymbol{C}_n, \\ 2n + 1, & \boldsymbol{R}_n = \boldsymbol{B}\boldsymbol{C}_n, \\ 2n - 2, & \boldsymbol{R}_n = \boldsymbol{D}_n, \end{cases}$$

$$a = a^{\boldsymbol{R}_n} := \begin{cases} 1, & \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{C}_n^\vee, \boldsymbol{B}\boldsymbol{C}_n, \\ 0, & \boldsymbol{R}_n = \boldsymbol{B}_n^\vee, \boldsymbol{C}_n, \boldsymbol{D}_n, \end{cases}$$

$$\sigma_1 = \sigma_1^{\boldsymbol{R}_n} := \begin{cases} -1, & \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^\vee, \\ 1, & \boldsymbol{R}_n = \boldsymbol{C}_n, \boldsymbol{C}_n^\vee, \boldsymbol{B}\boldsymbol{C}_n, \boldsymbol{D}_n. \end{cases}$$

$$\sigma_2 = \sigma_2^{\boldsymbol{R}_n} := \begin{cases} -1, & \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^\vee, \boldsymbol{C}_n, \boldsymbol{C}_n^\vee, \boldsymbol{B}\boldsymbol{C}_n, \\ 1, & \boldsymbol{D}_n. \end{cases}$$

- **Then the above equations are simply expressed as**

$$f(p\zeta) = \sigma_1 \frac{f(\zeta)}{p^{(\mathcal{N}-a)/2}\zeta^{\mathcal{N}}}, \qquad f(1/\zeta) = \sigma_2 \frac{1}{\zeta^a} f(\zeta).$$

- **In addition to the above, we put**

$$\alpha_j = \alpha_j^{\boldsymbol{R_n}} := \begin{cases} j - n, & \boldsymbol{R_n} = \boldsymbol{B_n}, \boldsymbol{C_n^\vee}, \boldsymbol{BC_n}, \boldsymbol{D_n}, \\ j - n - 1, & \boldsymbol{R_n} = \boldsymbol{B_n^\vee}, \boldsymbol{C_n}, \end{cases}$$

$$\beta_j(p) = \beta_j^{\boldsymbol{R_n}}(p) := -\sigma_1 p^{\alpha_j + (\mathcal{N} - a)/2}, \quad \boldsymbol{R_n} = \boldsymbol{B_n}, \boldsymbol{B_n^\vee}, \boldsymbol{C_n}, \boldsymbol{C_n^\vee}, \boldsymbol{BC_n}, \boldsymbol{D_n},$$

$$j = 1, \dots, n.$$

Lemma 1.6 (Rosengren and Schlosser (2006)) For $\boldsymbol{R_n} = \boldsymbol{B_n}$, $\boldsymbol{B_n^\vee}$, $\boldsymbol{C_n}$, $\boldsymbol{C_n^\vee}$, $\boldsymbol{BC_n}$, $\boldsymbol{D_n}$, the space $\mathcal{E}_p^{\boldsymbol{R_n}}$ is n dimensional and a basis is formed by

$$\psi_j^{\boldsymbol{R_n}}(\zeta; p) = \zeta^{\alpha_j} \theta(\beta_j(p)\zeta^{\mathcal{N}}; p^{\mathcal{N}}) + \sigma_2 \zeta^{-(\alpha_j - a)} \theta(\beta_j(p)\zeta^{-\mathcal{N}}; p^{\mathcal{N}}), \quad j = 1, \dots, n.$$

- **The explicit expressions are given below;**

$$\psi_j^{\boldsymbol{B_n}}(\zeta; p) := \zeta^{j-n} \theta(p^{j-1}\zeta^{2n-1}; p^{2n-1}) - \zeta^{n+1-j} \theta(p^{j-1}\zeta^{1-2n}; p^{2n-1}),$$

$$\psi_j^{\boldsymbol{B_n^\vee}}(\zeta; p) := \zeta^{j-n-1} \theta(p^{j-1}\zeta^{2n}; p^{2n}) - \zeta^{n+1-j} \theta(p^{j-1}\zeta^{-2n}; p^{2n}),$$

$$\psi_j^{\boldsymbol{C_n}}(\zeta; p) := \zeta^{j-n-1} \theta(-p^j\zeta^{2n+2}; p^{2n+2}) - \zeta^{n+1-j} \theta(-p^j\zeta^{-2n-2}; p^{2n+2}),$$

$$\psi_j^{\boldsymbol{C_n^\vee}}(\zeta; p) := \zeta^{j-n} \theta(-p^{j-1/2}\zeta^{2n}; p^{2n}) - \zeta^{n+1-j} \theta(-p^{j-1/2}\zeta^{-2n}; p^{2n}),$$

$$\psi_j^{\boldsymbol{BC_n}}(\zeta; p) := \zeta^{j-n} \theta(-p^j\zeta^{2n+1}; p^{2n+1}) - \zeta^{n+1-j} \theta(-p^j\zeta^{-2n-1}; p^{2n+1}),$$

$$\psi_j^{\boldsymbol{D_n}}(\zeta; p) := \zeta^{j-n} \theta(-p^{j-1}\zeta^{2n-2}; p^{2n-2}) + \zeta^{n-j} \theta(-p^{j-1}\zeta^{-2n+2}; p^{2n-2}).$$

- In addition to the Macdonald denominator of type A_{n-1} given by Definition 1.3, the following other six kinds of Macdonald denominators are defined.

Definition 1.7 (Macdonald (1972)) For $\zeta = (\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^\times)^n$, let

$$M^{B_n}(\zeta; p) := \prod_{\ell=1}^{n} \theta(\zeta_\ell; p) \prod_{1 \leq j < k \leq n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j/\zeta_k; p),$$

$$M^{B_n^\vee}(\zeta; p) := \prod_{\ell=1}^{n} \zeta_\ell^{-1} \theta(\zeta_\ell^2; p^2) \prod_{1 \leq j < k \leq n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j/\zeta_k; p),$$

$$M^{C_n}(\zeta; p) := \prod_{\ell=1}^{n} \zeta_\ell^{-1} \theta(\zeta_\ell^2; p) \prod_{1 \leq j < k \leq n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j/\zeta_k; p),$$

$$M^{C_n^\vee}(\zeta; p) := \prod_{\ell=1}^{n} \theta(\zeta_\ell; p^{1/2}) \prod_{1 \leq j < k \leq n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j/\zeta_k; p),$$

$$M^{BC_n}(\zeta; p) := \prod_{\ell=1}^{n} \theta(\zeta_\ell; p) \theta(p\zeta_\ell^2; p^2) \prod_{1 \leq j < k \leq n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j/\zeta_k; p),$$

$$M^{D_n}(\zeta; p) := \prod_{1 \leq j < k \leq n} \zeta_j^{-1} \theta(\zeta_j \zeta_k; p) \theta(\zeta_j/\zeta_k; p).$$

They are called the Macdonald denominators of type R_n for $R_n = B_n$, $B_n^\vee$, C_n, $C_n^\vee$, BC_n, D_n, respectively.

- **The above are regarded as the elliptic extensions of the Weyl denominators of types B_n, C_n and D_n,**

$$W^{B_n}(\boldsymbol{\zeta}) = \det_{1 \leq j,k \leq n} \left[\zeta_k^{j-n} - \zeta_k^{n+1-j} \right] = \prod_{\ell=1}^{n} \zeta_\ell^{1-n}(1-\zeta_\ell) \prod_{1 \leq j < k \leq n} (\zeta_k - \zeta_j)(1 - \zeta_j\zeta_k)$$

$$= \lim_{p \to 0} M^{B_n}(\boldsymbol{\zeta}; p) = \lim_{p \to 0} M^{C_n^\vee}(\boldsymbol{\zeta}; p) = \lim_{p \to 0} M^{BC_n}(\boldsymbol{\zeta}; p),$$

$$W^{C_n}(\boldsymbol{\zeta}) = \det_{1 \leq j,k \leq n} \left[\zeta_k^{j-n-1} - \zeta_k^{n+1-j} \right] = \prod_{\ell=1}^{n} \zeta_\ell^{-n}(1-\zeta_\ell^2) \prod_{1 \leq j < k \leq n} (\zeta_k - \zeta_j)(1 - \zeta_j\zeta_k)$$

$$= \lim_{p \to 0} M^{B_n^\vee}(\boldsymbol{\zeta}; p) = \lim_{p \to 0} M^{C_n}(\boldsymbol{\zeta}; p),$$

$$W^{D_n}(\boldsymbol{\zeta}) = \det_{1 \leq j,k \leq n} \left[\zeta_k^{j-n} + \zeta_k^{n-j} \right] = 2 \prod_{\ell=1}^{n} \zeta_\ell^{1-n} \prod_{1 \leq j < k \leq n} (\zeta_k - \zeta_j)(1 - \zeta_j\zeta_k)$$

$$= 2 \lim_{p \to 0} M^{D_n}(\boldsymbol{\zeta}; p).$$

Notice the degeneracy in the limit $p \to 0$.

- Rosengren and Schlosser proved the following.

Proposition 1.8 (Rosengren and Schlosser (2006)) The following equalities hold for $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^\times)^n$,

$$\det_{1 \leq j,k \leq n} \left[\psi_j^{B_n}(\zeta_k; p) \right] = \frac{2(p;p)_\infty^n}{(p^{2n-1}; p^{2n-1})_\infty^n} M^{B_n}(\boldsymbol{\zeta}; p),$$

$$\det_{1 \leq j,k \leq n} \left[\psi_j^{B_n^\vee}(\zeta_k; p) \right] = \frac{2(p^2; p^2)_\infty (p;p)_\infty^{n-1}}{(p^{2n}; p^{2n})_\infty^n} M^{B_n^\vee}(\boldsymbol{\zeta}; p),$$

$$\det_{1 \leq j,k \leq n} \left[\psi_j^{C_n}(\zeta_k; p) \right] = \frac{(p;p)_\infty^n}{(p^{2n+2}; p^{2n+2})_\infty^n} M^{C_n}(\boldsymbol{\zeta}; p),$$

$$\det_{1 \leq j,k \leq n} \left[\psi_j^{C_n^\vee}(\zeta_k; p) \right] = \frac{(p^{1/2}; p^{1/2})_\infty (p;p)_\infty^{n-1}}{(p^{2n}, p^{2n})_\infty^n} M^{C_n^\vee}(\boldsymbol{\zeta}; p),$$

$$\det_{1 \leq j,k \leq n} \left[\psi_j^{BC_n}(\zeta_k; p) \right] = \frac{(p;p)_\infty^n}{(p^{2n+1}; p^{2n+1})_\infty^n} M^{BC_n}(\boldsymbol{\zeta}; p),$$

$$\det_{1 \leq j,k \leq n} \left[\psi_j^{D_n}(\zeta_k; p) \right] = \frac{4(p;p)_\infty^n}{(p^{2n-2}; p^{2n-2})_\infty^n} M^{D_n}(\boldsymbol{\zeta}; p).$$

[RS06] Rosengren, H., Schlosser, M.: Elliptic determinant evaluations and the Macdonald identities for affine root systems. Compositio Math. 142, 937–961 (2006)

2. Orthogonality of R_n Theta Functions

- Since the theta function $\theta(\zeta; p)$ is holomorphic for $\zeta \in \mathbb{C}^\times$, it allows the Laurent expansion,

$$\theta(\zeta; p) = \frac{1}{(p; p)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n p^{\binom{n}{2}} \zeta^n,$$

where

$$(p; p)_\infty := \prod_{j=1}^{\infty} (1 - p^j).$$

- From now on we will assume $p \in \mathbb{R}$ and $r \in \mathbb{R}$. That is,

$$p \in (0, 1), \qquad r \in \mathbb{R} \setminus \{0\}.$$

- In this case, $\overline{\theta(\zeta; p)} = \theta(\bar{\zeta}; p)$.

2.1 Orthonormal A_{n-1} Theta Function

- We write the unit circle on the complex plane as

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} = \mathbb{R}/2\pi\mathbb{Z} \text{ (one-dimensional torus)}.$$

Each point in $\mathbb{T}$ is expressed by $e^{ix}, x \in [0, 2\pi)$.

- For the space $\mathcal{E}_{p,r}^{\boldsymbol{A}_{n-1}}$ of the $\boldsymbol{A}_{n-1}$ theta functions, we introduce the following inner product,

$$\langle f, g \rangle_{\mathbb{T}} := \frac{1}{2\pi} \int_{|\zeta|=1} f(\zeta)\overline{g(\zeta)}\ell(d\zeta)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix})\overline{g(e^{ix})}dx, \quad f, g \in \mathcal{E}_{p,r}^{\boldsymbol{A}_{n-1}},$$

where ℓ denotes the arc length measure on $\mathbb{T}$ normalized as $\ell(\mathbb{T}) = 2\pi$.

Proposition 2.1 Let $p, \widehat{p} \in (0, 1)$ and $r, \widehat{r} \in \mathbb{R} \setminus \{0\}$. Then

$$\langle \psi_j^{A_{n-1}}(\cdot; p, r), \psi_k^{A_{n-1}}(\cdot; \widehat{p}, \widehat{r}) \rangle_{\mathbb{T}} = h_j^{A_{n-1}}(p, \widehat{p}, r\widehat{r}) \delta_{jk},$$

for $j, k = 1, \ldots, n$, where

$$h_j^{A_{n-1}}(p, \widehat{p}, r\widehat{r}) = \frac{((p\widehat{p})^n; (p\widehat{p})^n)_\infty}{(p^n; p^n)_\infty (\widehat{p}^n, \widehat{p}^n)_\infty} \theta(-(r\widehat{r})(p\widehat{p})^{j-1}; (p\widehat{p})^n).$$

<u>**Proof.**</u>   We apply the Laurent expansion of the theta function and obtain

$$\psi_j^{A_{n-1}}(e^{ix}; p, r) = \frac{e^{i(j-1)x}}{(p^n; p^n)_\infty} \sum_{\ell \in \mathbb{Z}} (-1)^\ell p^{n\binom{\ell}{2}} (-1)^{(n-1)\ell} r^\ell p^{(j-1)\ell} e^{in\ell x},$$

$$\overline{\psi_k^{A_{n-1}}(e^{ix}; \widehat{p}, \widehat{r})} = \frac{e^{-i(k-1)x}}{(\widehat{p}^n; \widehat{p}^n)_\infty} \sum_{m \in \mathbb{Z}} (-1)^m \widehat{p}^{n\binom{m}{2}} (-1)^{(n-1)m} \widehat{r}^m \widehat{p}^{(k-1)m} e^{-inmx}.$$

The inner product of them includes the integrals

$$I_{jk,\ell m}^{A_{n-1}} := \frac{1}{2\pi} \int_0^{2\pi} e^{i\{(j-k)+n(\ell-m)\}x} dx$$

as $\langle \psi_j^{A_{n-1}}(\cdot; p, r), \psi_k^{A_{n-1}}(\cdot; \widehat{p}, \widehat{r}) \rangle_{\mathbb{T}}$

$$= \frac{1}{(p^n; p^n)_\infty (\widehat{p}^n; \widehat{p}^n)_\infty} \sum_{\ell \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} (-1)^{n(\ell+m)} p^{n\binom{\ell}{2}} \widehat{p}^{n\binom{m}{2}} r^\ell \widehat{r}^m p^{(j-1)\ell} \widehat{p}^{(k-1)m} I_{jk,\ell m}^{A_{n-1}}.$$

It is easy to verify that

$$\psi_j^{\boldsymbol{A}_{n-1}}(\zeta; p, r) := \zeta^{j-1}\theta(p^{j-1}(-1)^{n-1}r\zeta^n; p^n)$$

$$I_{jk,\ell m}^{\boldsymbol{A}_{n-1}} := \frac{1}{2\pi}\int_0^{2\pi} e^{i\{(j-k)+n(\ell-m)\}x}dx = \mathbf{1}((j-k)+n(\ell-m)=0).$$

Since $\ell, m \in \mathbb{Z}$, while $j, k \in \{1, \ldots, n\}$, $|j-k| \leq n-1 < n$ and thus the nonzero condition of $I_{jk,\ell m}^{\boldsymbol{A}_{n-1}}$ is satisfied if and only if $j - k = 0$ and $\ell - m = 0$. That is, we have the equalities,

$$I_{jk,\ell m}^{\boldsymbol{A}_{n-1}} = \delta_{jk}\delta_{\ell m} \quad \text{for } j, k \in \{1, \ldots, n\}, \; \ell, m \in \mathbb{Z}.$$

Therefore, we obtain

$$\langle \psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r), \psi_k^{\boldsymbol{A}_{n-1}}(\cdot; \widehat{p}, \widehat{r})\rangle_{\mathbb{T}}$$

$$= \frac{\delta_{jk}}{(p^n; p^n)_\infty(\widehat{p}^n; \widehat{p}^n)_\infty}\sum_{\ell \in \mathbb{Z}}(p\widehat{p})^{n\binom{\ell}{2}}(r\widehat{r})^\ell(p\widehat{p})^{(j-1)\ell}.$$

Again we use the Laurent-series expression of the theta function and the assertion is proved. ∎

- When $p \neq \widehat{p}$, $r \neq \widehat{r}$, we have two distinct sets of functions $\{\psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r)\}_{j=1}^n$ and $\{\psi_j^{\boldsymbol{A}_{n-1}}(\cdot; \widehat{p}, \widehat{r})\}_{j=1}^n$. In such a general case, the property

$$\langle \psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r), \psi_k^{\boldsymbol{A}_{n-1}}(\cdot; \widehat{p}, \widehat{r})\rangle_{\mathbb{T}} = h_j^{\boldsymbol{A}_{n-1}}(p, \widehat{p}, r\widehat{r})\delta_{jk}, \; j, k = 1, \ldots, n,$$

shall be called **biorthononality**.

- As a special case with $p = \widehat{p}$ and $r = \widehat{r}$, $\{\psi_j^{\boldsymbol{A}_{n-1}}(\cdot; p, r)\}_{j=1}^n$ makes an **orthogonal basis of $\mathcal{E}_{p,r}^{\boldsymbol{A}_{n-1}}$**. Here we consider this simplified situation.

- But now we replace $x \in \mathbb{R}$ by a complex argument $z = x + iy$, $x, y \in \mathbb{R}$.

Lemma 2.2 Let $p \in (0, 1)$ and $r \in \mathbb{R} \setminus \{0\}$. For $j, k = 1, \ldots, n$, the orthogonal relations hold;

$$\frac{1}{2\pi} \int_0^{2\pi} \psi_j^{\boldsymbol{A}_{n-1}}(e^{i(x+iy)}; p, r)\overline{\psi_k^{\boldsymbol{A}_{n-1}}(e^{i(x+iy)}; p, r)}dx = \widetilde{h}_j^{\boldsymbol{A}_{n-1}}(y; p, r)\delta_{jk},$$

with $\quad \widetilde{h}_j^{\boldsymbol{A}_{n-1}}(y; p, r) = e^{-2(j-1)y}\frac{(p^{2n}; p^{2n})_\infty}{(p^n; p^n)_\infty^2}\theta(-r^2 e^{-2ny}p^{2(j-1)}; p^{2n}).$

Proof. Just note $\psi_j^{\boldsymbol{A}_{n-1}}(e^{i(x+iy)}; p, r) = e^{-(j-1)y}\psi_j^{\boldsymbol{A}_{n-1}}(e^{ix}; p, re^{-ny})$. $\blacksquare$

- **For the nome $p \in \mathbb{C}$, $|p| < 1$, we define the nome modular parameter τ by**

$$p = e^{2\pi i \tau} =: p(\tau).$$

Lemma 2.3 (Jacobi's imaginary transformation) We define

$$\widetilde{p} := p(-1/\tau) = e^{-2\pi i/\tau}.$$

Then the following equality holds for $\zeta \in \mathbb{C}$,

$$\theta(e^{i\zeta}; p) = e^{\pi i/4} \frac{\widetilde{p}^{1/8}(\widetilde{p}; \widetilde{p})_\infty}{p^{1/8}(p; p)_\infty} \tau^{-1/2} e^{-i\zeta^2/4\pi\tau} e^{i\zeta/2} e^{-i\zeta/2\tau} \theta(e^{-i\zeta/\tau}; \widetilde{p}).$$

- Since we have assumed $p \in (0,1)$, τ is pure imaginary and written as $\tau = i|\tau|$; that is, $\quad p = e^{-2\pi|\tau|}$.

- Moreover, we fix the norm r as

$$r = (-1)^n e^{-n\pi|\tau|} = (-1)^n p^{n/2}.$$

- Then we find the following;

$$\widetilde{h}_j^{A_{n-1}}(y; p, r) = C_1(n|\tau|)(e^{-\pi/n|\tau|}; e^{-\pi/n|\tau|})_\infty e^{2\pi|\tau|(j-1)^2/n + ny^2/2\pi|\tau|}$$

$$\times \theta(C_2(n|\tau|, (j-1+n/2)|\tau|)e^{-iy/|\tau|}; e^{-\pi/n|\tau|}), \quad j = 1, \ldots, n,$$

where

$$C_1(t) := \frac{1}{\sqrt{2t}} \frac{e^{-\pi/t}}{(e^{-2\pi t}; e^{-2\pi t})_\infty^2}, \quad C_2(t, s) := e^{-(4is-1)\pi/2t}.$$

- **There are three important points;**

$$\widetilde{h}_j^{A_{n-1}}(y;p,r) \propto \exp\left(\frac{2\pi|\tau|}{n}(j-1)^2 + \frac{n}{2\pi|\tau|}y^2\right)\theta\left(C_2 e^{-iy/|\tau|}; e^{-\pi/n|\tau|}\right).$$

Lemma 2.4 Assume $p = e^{-2\pi|\tau|}$ and $r = (-1)^n p^{n/2}$. **Then**

$$\frac{1}{2\pi|\tau|}\int_0^{2\pi|\tau|} e^{-ny^2/2\pi|\tau|}\widetilde{h}_j^{A_{n-1}}(y;p,r)dy = C_1(n|\tau|)e^{2\pi|\tau|(j-1)^2/n}.$$

<u>**Proof.**</u> When C does not depend on y,

$$\frac{1}{2\pi|\tau|}\int_0^{2\pi|\tau|}\theta(Ce^{\pm iy/|\tau|};p)dy = \frac{1}{(p;p)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k p^{\binom{k}{2}}C^k \times \frac{1}{2\pi|\tau|}\int_0^{2\pi|\tau|} e^{\pm iyk/|\tau|}dy$$

$$= \frac{1}{(p;p)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k p^{\binom{k}{2}}C^k\delta_{k0} = \frac{1}{(p;p)_\infty}.$$

Then the assertion is immediately obtained. ∎

- With $p = e^{-2\pi|\tau|}$ and $C_1(t) = \dfrac{1}{\sqrt{2t}} \dfrac{e^{-\pi/t}}{(e^{-2\pi t}; e^{-2\pi t})_\infty^2}$, define

$$\Psi_j^{A_{n-1}}(z) = \Psi_j^{A_{n-1}}(z; |\tau|)$$

$$:= \frac{e^{-\pi|\tau|(j-1)^2/n}}{\sqrt{C_1(n|\tau|)}} e^{-ny^2/4\pi|\tau|} \psi_j^{A_{n-1}}(e^{iz}; p, (-1)^n p^{n/2}), \quad z = x + iy \in \mathbb{C}.$$

- Consider a rectangular domain in $\mathbb{C}$,

$$D_{(2\pi, 2\pi|\tau|)} := \left\{ z = x + iy \in \mathbb{C} : 0 \le x < 2\pi, 0 \le y < 2\pi|\tau| \right\}.$$

- Introduce the inner product for holomorphic functions f, g on $D_{(2\pi, 2\pi|\tau|)}$;

$$\langle f, g \rangle_{D_{(2\pi, 2\pi|\tau|)}} := \frac{1}{|D_{(2\pi, 2\pi|\tau|)}|} \int_{D_{(2\pi, 2\pi|\tau|)}} f(z)\overline{g(z)}dz$$

$$= \frac{1}{(2\pi)^2|\tau|} \int_0^{2\pi} dx \int_0^{2\pi|\tau|} dy\, f(x + iy)\overline{g(x + iy)}.$$

The above results are summarized as follows.

Proposition 2.5 $\quad \langle \Psi_j^{A_{n-1}}, \Psi_k^{A_{n-1}} \rangle_{D_{(2\pi, 2\pi|\tau|)}} = \delta_{jk}, \quad j, k = 1, \ldots, n.$

2.2 Other Orthonormal R_n Theta Functions

- **Let**

$$J(j) = J^{\boldsymbol{R}_n}(j) := \alpha_j - a/2 = \begin{cases} j - n - 1/2, & \boldsymbol{R}_n = \boldsymbol{B}_n,\, \boldsymbol{C}_n^\vee,\, \boldsymbol{BC}_n, \\ j - n - 1, & \boldsymbol{R}_n = \boldsymbol{B}_n^\vee,\, \boldsymbol{C}_n, \\ j - n, & \boldsymbol{R}_n = \boldsymbol{D}_n, \end{cases}$$

where $a = a^{\boldsymbol{R}_n}$ and $\alpha_j = \alpha_j^{\boldsymbol{R}_n}$ are defined above, and

$$c_j = c_j^{\boldsymbol{R}_n} := \begin{cases} 1, & j = 1, \ldots, n, \quad \boldsymbol{R}_n = \boldsymbol{C}_n,\, \boldsymbol{C}_n^\vee,\, \boldsymbol{BC}_n, \\ \begin{cases} 2, & j = 1, \\ 1, & j = 2, \ldots, n, \end{cases} & \boldsymbol{R}_n = \boldsymbol{B}_n,\, \boldsymbol{B}_n^\vee, \\ \begin{cases} 2, & j = 1, n, \\ 1, & j = 2, \ldots, n - 1, \end{cases} & \boldsymbol{R}_n = \boldsymbol{D}_n. \end{cases}$$

Proposition 2.6 For $R_n = B_n$, $B_n^\vee$, C_n, $C_n^\vee$, BC_n, D_n, define

$$\Psi_j^{R_n}(z) = \Psi_j^{R_n}(z; |\tau|) := \frac{e^{-\pi|\tau|J(j)^2/\mathcal{N}}}{\sqrt{2c_j C_1(\mathcal{N}|\tau|)}} e^{-\mathcal{N}y^2/4\pi|\tau|+ay/2} \psi_j^{R_n}(e^{iz}; p),$$

$j = 1, \ldots, n$, where $a = a^{R_n}$, $\mathcal{N} = \mathcal{N}^{R_n}$, $c_j = c_j^{R_n}$), and $J(j) = J^{R_n}(j)$ are given above. Then

$$\langle \Psi_j^{R_n}, \Psi_k^{R_n} \rangle_{D_{(2\pi, 2\pi|\tau|)}} = \delta_{jk}, \quad j, k = 1, \ldots, n.$$

2.3 Doubly-Quasi-Periodicity

- The **orthonormal functions** $\{\Psi_j^{R_n}\}_{j=1}^n$ have the following **doubly-quasi-periodicity** and can be extended for $z \in \mathbb{C}$.

Lemma 2.7 Assume $p = e^{-2\pi|\tau|}$. The following hold,

$$\Psi_j^{R_n}(z + 2\pi) = \Psi_j^{R_n}(z),$$

$$\Psi_j^{A_{n-1}}(z + 2\pi i|\tau|) = e^{-inx}\Psi_j^{R_n}(z), \quad \Psi_j^{R_n}(z + 2\pi i|\tau|) = \sigma_1 e^{-i\mathcal{N}x}\Psi_j^{R_n}(z),$$

$j = 1, \ldots, n$, where $\mathcal{N} = \mathcal{N}^{R_n}$ and $\sigma_1 = \sigma_1^{R_n}$ as above.

Proof. The periodicity with period 2π is obvious, since $\{\Psi_j^{R_n}(z)\}_{j=1}^n$ are functions of e^{iz}. For $\psi_j^{A_{n-1}}(e^{iz}; p, (-1)^n p^{n/2}) \in \mathcal{E}_{p,(-1)^n p^{n/2}}^{A_{n-1}}$, $\psi_j^{R_n}(e^{iz}; p) \in \mathcal{E}_p^{R_n}$,

$$\psi_j^{A_{n-1}}(e^{i(z+2\pi|\tau|i)}; p, (-1)^n p^{n/2}) = \psi_j^{A_{n-1}}(pe^{iz}; p, (-1)^n p^{n/2})$$

$$= \frac{(-1)^n}{(-1)^n p^{n/2}(e^{iz})^n}\psi_j^{A_{n-1}}(e^{iz}; p, (-1)^n p^{n/2}) = e^{n\pi|\tau|}e^{ny}e^{-inx}\psi_j^{A_{n-1}}(e^{iz}; p, r),$$

$$\psi_j^{R_n}(e^{i(z+2\pi|\tau|i)}; p) = \psi_j^{R_n}(pe^{iz}; p)$$

$$= \sigma_1 \frac{1}{p^{(\mathcal{N}-a)/2}(e^{iz})^{\mathcal{N}}}\psi_j^{R_n}(e^{iz}; p) = \sigma_1 e^{(\mathcal{N}-a)\pi|\tau|}e^{\mathcal{N}y}e^{-i\mathcal{N}x}\psi_j^{R_n}(e^{iz}; p),$$

for others. Irrelevant factors are cancelled by
$$e^{-n(y+2\pi|\tau|)^2/4\pi|\tau|} = e^{-n\pi|\tau|}e^{-ny}e^{-ny^2/4\pi|\tau|} \text{ and}$$
$$e^{-\mathcal{N}(y+2\pi|\tau|)^2/4\pi|\tau|+a(y+2\pi|\tau|)/2} = e^{-(\mathcal{N}-a)\pi|\tau|}e^{-\mathcal{N}y}e^{-\mathcal{N}y^2/4\pi|\tau|+ay/2}. \blacksquare$$

3. Determinantal Point Processes (DPPs) on a 2-Dim Torus and their Infinite Particle Limits

3.1 A Brief Review of DPPs

- Let a space S be a subset of $\mathbb{R}^d$ with $d \in \mathbb{N}$ equipped with a reference measure λ.

- A random point process with n points, $n \in \mathbb{N}$, on a space S is a statistical ensemble of nonnegative integer-valued Radon measures

$$\Xi(\cdot) = \sum_{j=1}^{n} \delta_{X_j}(\cdot).$$

- Here $\delta_y(\cdot), y \in S$ denotes the delta measure such that $\delta_y(\{x\}) = 1$ if $x = y$ and $\delta_y(\{x\}) = 0$ otherwise.

- In general, the configuration space of point process is given by

$$\mathbf{Conf}(S) := \left\{ \xi = \sum_i \delta_{x_i} : x_i \in S, \, \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$

- The distribution of points $\{X_j\}_{j=1}^n$ on S is governed by a probability measure **P**. We assume **P** has probability density **p** with respect to $\lambda(d\mathbf{x})$.

- That is, for the set of points $\mathbf{X} := (X_1, \dots, X_n)$,
$$\mathbf{P}(\mathbf{X} \in d\mathbf{x}) = \mathbf{p}(\mathbf{x})\lambda(d\mathbf{x}), \quad d\mathbf{x} \subset S^n.$$
Since the labeling order of n points $\{x_j\}_{j=1}^n$ is irrelevant for point configuration $\xi = \sum_{j=1}^n \delta_{x_j}$, the probability density should be normalized as
$$\frac{1}{n!} \int_{S^n} \mathbf{p}(\mathbf{x})\lambda(d\mathbf{x}) = 1.$$

- The point process is denoted by a triplet $(\Xi, \mathbf{p}, \lambda(dx))$.

- For $(\Xi, \mathbf{p}, \lambda(dx))$, the m-point correlation function, $1 \leq m \leq n$, is defined by

$$\rho_m(x_1, \dots, x_m) = \frac{1}{(n-m)!} \int_{S^{n-m}} \mathbf{p}(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \prod_{j=m+1}^n \lambda(dx_j),$$

$(x_1, \dots, x_m) \in S^m$. By definition, for any $m \in \{1, \dots, n\}$, the correlation function ρ_m is a symmetric function on S^m;
$$\rho_m(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = \rho_m(x_1, \dots, x_m) \quad \text{for all } \sigma \in \mathfrak{S}_m.$$

- Let $\mathcal{B}_c(S)$ be the set of all bounded measurable complex functions on S of compact support. For $\xi \in \mathbf{Conf}(S)$ and $\phi \in \mathcal{B}_c(S)$ we set

$$\langle \xi, \phi \rangle := \int_S \phi(x)\, \xi(dx) = \sum_{j=1}^n \phi(x_j).$$

- Then the expectation of the random variable $\langle \Xi, \phi \rangle$ with respect to $\mathbf{P}$ is given by

$$\mathbf{E}[\langle \Xi, \phi \rangle] = \int_S \phi(x)\rho_1(x)\lambda(dx).$$

In other words, the first correlation function $\rho_1(x)$ gives the **density of point at $x \in S$** with respect to the reference measure $\lambda(dx)$.

- For $2 \le m \le n$, from $\xi \in \mathbf{Conf}(S)$ we define $\xi_m := \sum_{i_1,\dots,i_m : i_j \ne i_k, j \ne k} \delta_{x_{i_1}} \cdots \delta_{x_{i_m}}$. Then for all $\phi \in \mathcal{B}_c(S^m)$,

$$\mathbf{E}[\langle \Xi_m, \phi \rangle] = \int_{S^m} \phi(x_1,\dots,x_m)\rho_m(x_1,\dots,x_m) \prod_{j=1}^m \lambda(dx_j).$$

- **With $\phi \in \mathcal{B}_c(S)$, $k \in \mathbb{R}$, the characteristic function of $(\Xi, \mathbf{p}, \lambda(dx))$ is defined by**

$$\Psi[\phi; \kappa] := \mathbf{E}\left[e^{\kappa\langle\Xi,\phi\rangle}\right] = \frac{1}{n!}\int_{S^n} e^{\kappa\langle\xi,\phi\rangle}\mathbf{p}(\mathbf{x})\lambda(d\mathbf{x}),$$

which can be regarded as the Laplace transform of the probability density function p.

- **Put**

$$\chi(x) = \chi(x; \kappa) := 1 - e^{\kappa\phi(x)}.$$

Then we can show that

$$\Psi[\phi; \kappa] = 1 + \sum_{m=1}^{n}(-1)^m\frac{1}{m!}\int_{S^m}\rho_m(x_1,\ldots,x_m)\prod_{k=1}^{m}\left\{\chi(x_k)\lambda(dx_k)\right\}.$$

This expression means that the characteristic function is regarded as the generating function of correlation functions.

- If every correlation function is expressed by a determinant in the form

$$\rho_m(x_1, \ldots, x_m) = \det_{1 \le j,k \le m} [K(x_j, x_k)], \quad m = 1, \ldots, n,$$

with a two-point continuous function $K(x, y)$, $x, y \in S$, then the point process is said to be a **determinantal point process (DPP)** and K is called the **correlation kernel**.
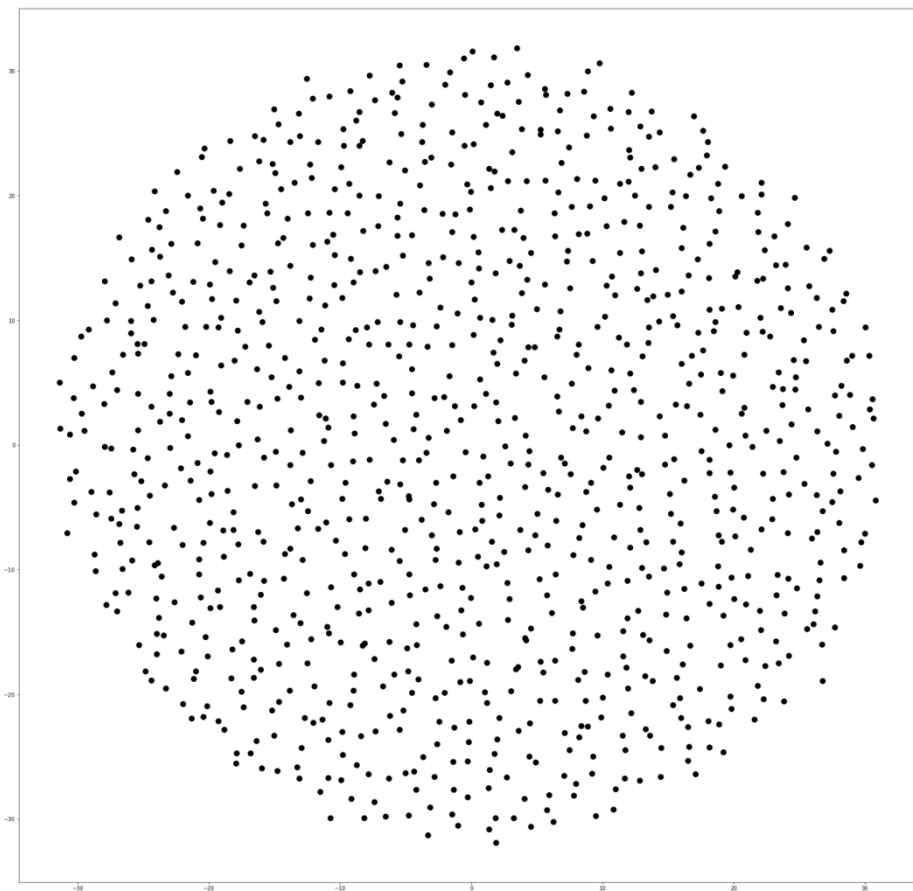
- In particular, the density of point with respect to λ on S is given by
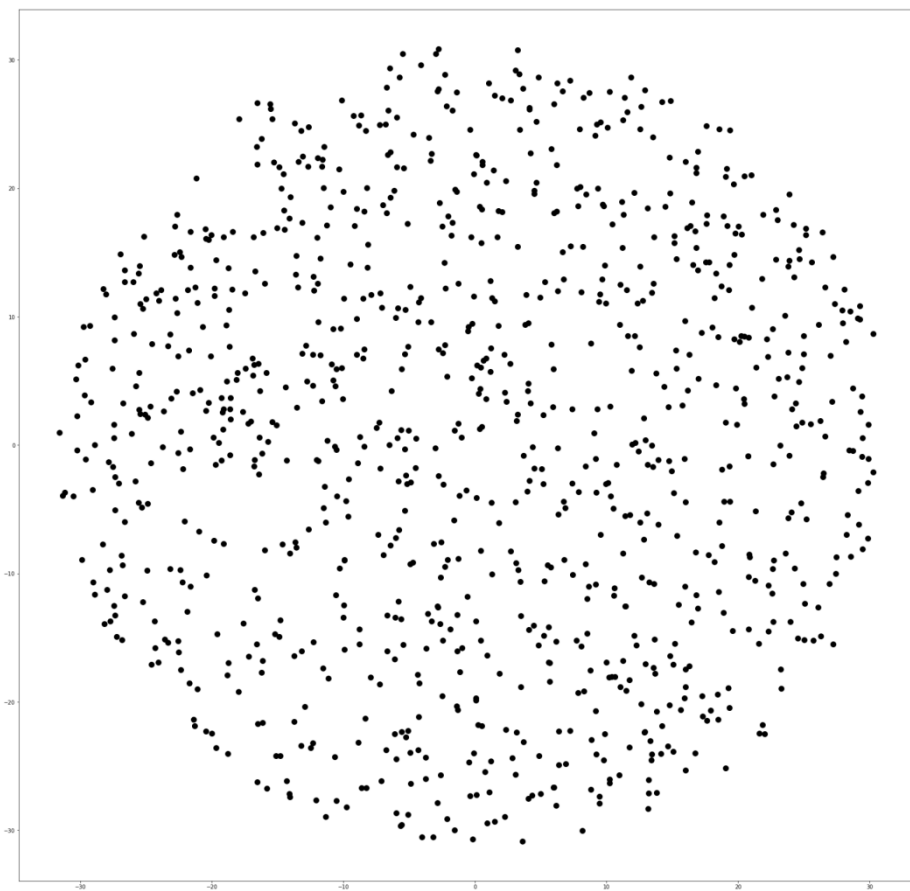
$$\rho_1(x) = K(x, x), \quad x \in S.$$

- The characteristic function is given by

$$\Psi[\phi; \kappa] = 1 + \sum_{m=1}^{n} (-1)^m \frac{1}{m!} \int_{S^m} \det_{1 \le j,k \le m} [K(x_j, x_k)\chi(x_k)] \prod_{\ell=1}^{m} \lambda(dx_\ell)$$

$$= \operatorname*{Det}_{(S,\lambda), x, y \in S} \left[\delta(x - y) - K(x, y)\chi(y) \right] \ (\textbf{Fredholm determinant}).$$

- We denote the DPP by a **triplet** $(\Xi, K, \lambda(dx))$.

an example of DPP (Ginibre DPP)   **Poisson point process**

(Computer simulation by T. Matsui (Chuo U.))
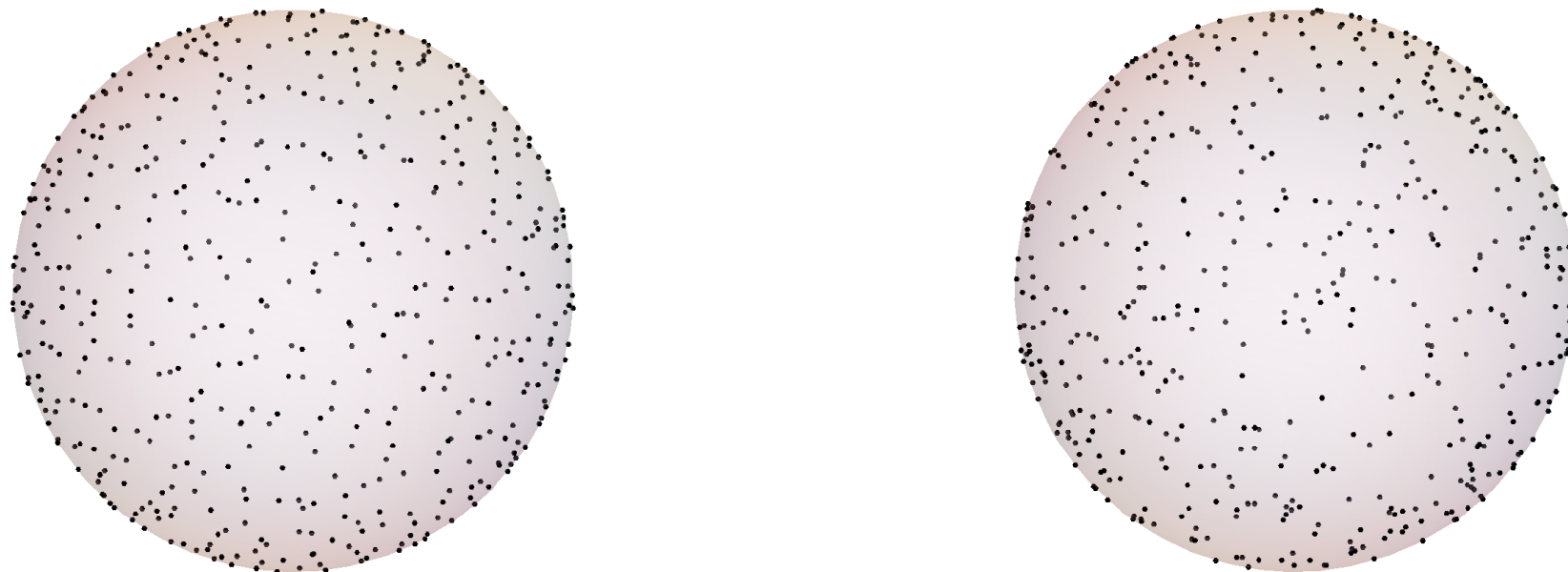
another example of DPP

Figure: Spherical ensemble (left) and Poisson (right) ($N = 500$)

(Computer simulation by T. Shirai (Kyushu U.))

- The following fact of DPPs is proved using a basic property of determinant.

Lemma 3.1 Consider a non-vanishing function $f : S \to \mathbb{C}$. Even if the correlation kernel $K(x, y)$ is transformed as

$$K(x, y) \to K_f(x, y) := f(x)K(x, y)\frac{1}{f(y)}, \quad x, y \in S,$$

all correlation functions are the same and hence

$$(\Xi, K, \lambda(dx)) \stackrel{(\text{law})}{=} (\Xi, K_f, \lambda(dx)).$$

- The above transformation is called the **Gauge transformation** and the above property of DPP is referred to **Gauge invariance**.

Theorem 3.2 Fix $n \in \mathbb{N}$ and assume that a set of functions $\{f_j\}_{j=1}^n$ on S with a reference measure $\lambda(dx)$ satisfies the orthogonality relation,

$$\int_S f_j(x)\overline{f_k(x)}\lambda(dx) = h_j \delta_{jk} \quad j, k \in \{1, \ldots, n\}.$$

Then we can define a point process with n particles on S such that the probability density function with respect to $\lambda(dx)$ is given by

$$\mathbf{p}(\mathbf{x}) = \frac{1}{Z} \left| \det_{1 \le j,k \le n} [f_j(x_k)] \right|^2, \quad \mathbf{x} \in S^n,$$

where $1/Z$ is a normalization factor so that $(1/n!) \int_{S^n} \mathbf{p}(\mathbf{x})\lambda(d\mathbf{x}) = 1$. Then this is a DPP $(\Xi, K, \lambda(dx))$ such that the correlation kernel is given by

$$K(x, y) = \sum_{\ell=1}^n \frac{1}{h_\ell} f_\ell(x)\overline{f_\ell(y)}, \quad x, y \in S.$$

- **This theorem is well known in random matrix theory. For example, see Appendix C in [K19].**

[K19] Katori, M.: Macdonald denominators for affine root systems, orthogonal theta functions, and elliptic determinantal point processes. J. Math. Phys. <u>60</u>, 013301/1-27 (2019)

Theorem 3.2 Fix $n \in \mathbb{N}$ and assume that a set of functions $\{f_j\}_{j=1}^n$ on S with a reference measure $\lambda(dx)$ satisfies the orthogonality relation,

$$\int_S f_j(x)\overline{f_k(x)}\lambda(dx) = h_j \delta_{jk} \quad j, k \in \{1, \ldots, n\}.$$

Then we can define a point process with n particles on S such that the probability density function with respect to $\lambda(dx)$ is given by

$$\mathbf{p}(\mathbf{x}) = \frac{1}{Z} \left| \det_{1 \le j,k \le n} [f_j(x_k)] \right|^2, \quad \mathbf{x} \in S^n,$$

where $1/Z$ is a normalization factor so that $(1/n!) \int_{S^n} \mathbf{p}(\mathbf{x})\lambda(d\mathbf{x}) = 1$. Then this is a DPP $(\Xi, K, \lambda(dx))$ such that the correlation kernel is given by

$$K(x, y) = \sum_{\ell=1}^n \frac{1}{h_\ell} f_\ell(x)\overline{f_\ell(y)}, \quad x, y \in S.$$

- A general framework to construct DPPs based on the notion of **partial isometry** is given in [K-Shirai21] not only for DPPs with finite number of particles $n \in \mathbb{N}$, but also for **DPPs with an infinite number of particles.**

[K-Shirai21] Katori, M., Shirai, T.: Partial isometries, duality, and determinantal point processes. Random Matrices: Theory and Applications. 2250025, 70 pages (2021)

- So far we have fixed $n \in \mathbb{N}$ for each system. We can consider a series of systems with increasing n; increasing the number of particles for DPPs.

- According to the change of n, we change the scale of coordinates, which is called dilatation of DPPs.

Definition 3.3 For a DPP $(\Xi, K, \lambda(dx))$ with $\Xi = \sum_j \delta_{X_j}$ on a space S, given a factor $c > 0$,

$$c \circ \Xi := \sum_j \delta_{cX_j},$$

$$c \circ K(x, y) := K\left(\frac{x}{c}, \frac{y}{c}\right), \quad x, y \in cS := \{cx : x \in S\},$$

$$c \circ \lambda(dx) := \lambda(dx/c).$$

Then the dilatation by factor c of the DPP is defined by
$$c \circ (\Xi, K, \lambda(dx)) := (c \circ \Xi, c \circ K, c \circ \lambda(dx)).$$

- Notice the following equivalence. For $g \in \mathcal{B}_c(S)$ such that $g : S \to (0, \infty)$,
$$(\Xi, K(x, y), g(x)\lambda(dx)) \overset{(\text{law})}{=} (\Xi, \sqrt{g(x)}K(x, y)\sqrt{g(y)}, \lambda(dx)).$$

- I will give limit theorems for DPPs at the end of this talk.

- Consider a DPP which depends on a continuous parameter, or a series of DPPs labeled by a discrete parameter (e.g., the number of points $n \in \mathbb{N}$), and describe the system by $(\Xi, K_p, \lambda_p(dx))$ with the continuous or discrete parameter p.

- If $(\Xi, K_p, \lambda_p(dx))$ converges to a DPP, $(\Xi, K, \lambda(dx))$, as $p \to \infty$, weakly in the vague topology, we write this limit theorem as

$$(\Xi, K_p, \lambda_p(dx)) \overset{p \to \infty}{\Longrightarrow} (\Xi, K, \lambda(dx)).$$

- The weak convergence of DPPs is verified by the uniform convergence of the kernel $K_p \to K$ on each compact set $C \subset S \times S$ [ST03].

[ST03] T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point process, J. Funct. Anal. 205 (2003) 414–463 (2003)

3.2 DPPs on a 2-dim Torus

$$\Psi_j^{R_n}(z + 2\pi i | \tau|) = \sigma_1 e^{-i\mathcal{N}x} \Psi_j^{R_n}(z)$$

- For $R_n = A_{n-1}$, B_n, $B_n^\vee$, C_n, $C_n^\vee$, BC_n, D_n, we put

$$k^{R_n}(z, z') := \sum_{j=1}^{n} \Psi_j^{R_n}(z)\overline{\Psi_j^{R_n}(z')}, \quad z, z' \in D_{(2\pi, 2\pi|\tau|)}.$$

By Lemma 2.7, the following double periodicity is proved ($\mathcal{N}^{A_{n-1}} := n$),

$$k^{R_n}(z + 2\pi, z' + 2\pi) = k^{R_n}(z, z'),$$

$$k^{R_n}(z + 2\pi|\tau|i, z' + 2\pi|\tau|i) = \frac{e^{-i\mathcal{N}x}}{e^{-i\mathcal{N}x'}} k^{R_n}(z, z')$$

$$\simeq k^{R_n}(z, z') \quad \text{(by Gauge invariance)}, \quad z, z' \in \mathbb{C}.$$

- Now we apply Theorem 3.2 to our seven types of orthonormal theta functions. Then we obtain seven types of DPPs on $\mathbb{C}$ such that their correlation kernels are given by k^{R_n} satisfying the above double periodicity.

- In other words, if we define the two-dimensional torus denoted as

$$\mathbb{T}^2 := \{z \in \mathbb{C} : z + 2\pi = z, z + 2\pi|\tau|i = z\} \simeq (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi|\tau|\mathbb{Z}),$$

then we have the seven types of DPPs on $\mathbb{T}^2$.

two-dimensional torus
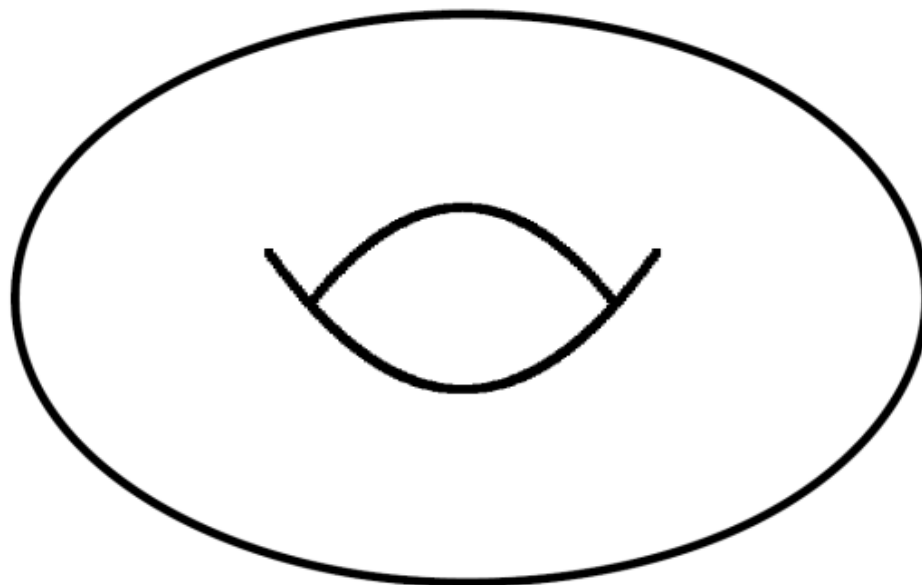
$$\mathbb{T}^2 = \mathbb{T}^2_{|\tau|} := \{z \in \mathbb{C} : z + 2\pi = z, z + 2\pi|\tau|i = z\}$$

$$\simeq (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi|\tau|\mathbb{Z}),$$

Theorem 3.4 The seven types of point processes

$$\Xi_{\mathbb{T}^2}^{R_n}(\cdot) = \sum_{j=1}^{n} \delta_{X_j^{R_n}}(\cdot),$$

for $R_n = A_{n-1}$, B_n, $B_n^\vee$, C_n, $C_n^\vee$, BC_n, D_n are well defined on $\mathbb{T}^2$ so that they have probability densities $\mathbf{p}_{\mathbb{T}^2}^{R_n}(\mathbf{z})$

$$\mathbf{p}_{\mathbb{T}^2}^{R_n}(\mathbf{z}) = \frac{1}{Z^{R_n}} \left| \det_{1 \le j,k \le n} [\Psi_j^{R_n}(z_k)] \right|^2, \quad \mathbf{z} = (z_1, \dots z_n) \in (\mathbb{T}^2)^n,$$

with respect to the Lebesgue measure $d\mathbf{z} = \prod_{j=1}^{n} d\Re z_j \, d\Im z_j$ on $(\mathbb{T}^2)^n$. Here $1/Z^{R_n}$ are the normalization factors such that $(1/n!) \int_{(\mathbb{T}^2)^n} \mathbf{p}_{\mathbb{T}^2}^{R_n}(\mathbf{z}) d\mathbf{z} = 1$. They are DPPs with the correlation kernels

$$K_{\mathbb{T}^2}^{R_n}(z, z') := \frac{1}{(2\pi)^2 |\tau|} \sum_{j=1}^{n} \Psi_j^{R_n}(z) \overline{\Psi_j^{R_n}(z')}, \quad z, z' \in \mathbb{T}^2,$$

with respect to the Lebesgue measure $dz = d\Re z \, d\Im z$.

- We define the reflection and shift of DPP on $\mathbb{T}^2$ as follows.

Definition 3.5 Consider a DPP $(\Xi, K, \lambda(dz))$ on $\mathbb{T}^2$, where we write $\Xi(\cdot) = \sum_j \delta_{Z_j}(\cdot)$.

(i) The inversion operator $\mathcal{R}$ is defined by

$$\mathcal{R}\Xi := \sum_j \delta_{-Z_j}, \quad \mathcal{R}K(z, z') := K(-z, -z'), \quad \mathcal{R}\lambda(dz) := \lambda(-dz).$$

We write $(\mathcal{R}\Xi, \mathcal{R}K, \mathcal{R}\lambda(dz))$ simply as $\mathcal{R}(\Xi, K, \lambda(dz))$.

(ii) For $u \in \mathbb{C}$, the shift operator $\mathcal{S}_u$ is defined by

$$\mathcal{S}_u\Xi := \sum_j \delta_{Z_j - u}, \quad \mathcal{S}_uK(z, z') := K(z + u, z' + u), \quad \mathcal{S}_u\lambda(dz) := \lambda(u + dz).$$

We write $(\mathcal{S}_u\Xi, \mathcal{S}_uK, \mathcal{S}_u\lambda(dz))$ simply as $\mathcal{S}(\Xi, K, \lambda(dz))$.

We can prove the following symmetry which characterizes the seven types of DPPs on $\mathbb{T}^2$.

Proposition 3.6 (i) For $\boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^\vee, \boldsymbol{C}_n, \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n, \boldsymbol{D}_n$, the reflection invariance is established: $\mathcal{R}(\Xi, K, \lambda(dz)) \overset{\text{(law)}}{=} (\Xi, K, \lambda(dz))$.

(ii) The following shift invariance are satisfied:

$$\mathcal{S}_{2\pi/n}(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz) \overset{\text{(law)}}{=} (\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz),$$

$$\mathcal{S}_{2\pi|\tau|i/n}(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz) \overset{\text{(law)}}{=} (\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz),$$

$$\mathcal{S}_\pi(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz) \overset{\text{(law)}}{=} (\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz), \quad \boldsymbol{R}_n = \boldsymbol{B}_n^\vee, \boldsymbol{C}_n, \boldsymbol{D}_n,$$

$$\mathcal{S}_{\pi|\tau|i}(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz) \overset{\text{(law)}}{=} (\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz), \quad \boldsymbol{R}_n = \boldsymbol{C}_n, \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n, \boldsymbol{D}_n.$$

(iii) The densities of points $\rho_{\mathbb{T}^2}^{\boldsymbol{R}_n}(z) = K_{\mathbb{T}^2}^{\boldsymbol{R}_n}(z, z), z \in \mathbb{T}^2$ with respect to the Lebesgue measure dz have the following zeros:

$$\rho_{\mathbb{T}^2}^{\boldsymbol{B}_n}(0) = 0,$$

$$\rho_{\mathbb{T}^2}^{\boldsymbol{B}_n^\vee}(0) = \rho_{\mathbb{T}^2}^{\boldsymbol{B}_n^\vee}(\pi) = 0,$$

$$\rho_{\mathbb{T}^2}^{\boldsymbol{R}_n}(0) = \rho_{\mathbb{T}^2}^{\boldsymbol{R}_n}(\pi|\tau|i) = 0, \quad \boldsymbol{R}_n = \boldsymbol{C}_n^\vee, \boldsymbol{BC}_n,$$

$$\rho_{\mathbb{T}^2}^{\boldsymbol{C}_n}(0) = \rho_{\mathbb{T}^2}^{\boldsymbol{C}_n}(\pi) = \rho_{\mathbb{T}^2}^{\boldsymbol{C}_n}(\pi|\tau|i) = 0.$$

3.3 Infinite Particle Limits

- We note that the periods $2\pi/n \in (0, \infty)$ and $2\pi|\tau|i/n \in i(0, \infty)$ of $(\Xi, K_{\mathbb{T}^2}^{A_{n-1}}, dz)$ shown by Proposition 3.6 (ii) become zeros as $n \to \infty$. Hence, as the $n \to \infty$ limit of $(\Xi, K_{\mathbb{T}^2}^{A_{n-1}}, dz)$, it is expected to obtain a translation-invariant system of infinite number of points on $\mathbb{C}$.

- Here we introduce three kinds of infinite DPPs on $\mathbb{C}$.

- Let the reference measure be the complex normal distribution,

$$\lambda_N(dz) := \frac{1}{\pi} e^{-|z|^2} dz.$$

- **Put**

$$\mathcal{K}^A_{\text{Ginibre}}(z, z') = e^{z\overline{z'}},$$

$$\mathcal{K}^C_{\text{Ginibre}}(z, z') = \sinh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} - e^{-z\overline{z'}}),$$

$$\mathcal{K}^D_{\text{Ginibre}}(z, z') = \cosh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} + e^{-z\overline{z'}}), \quad z, z' \in \mathbb{C}.$$

- **Then the Ginibre DPPs of type R are defined by $(\Xi, \mathcal{K}^R_{\text{Ginibre}}, \lambda_N(dz))$ for $R = A, C,$ and D, respectively.**

- **The Ginibre DPP of type A describes the eigenvalue distribution of the Gaussian random complex matrix in the bulk scaling limit [Gin65].** **The density of points is uniform with the Lebesgue measure dz on $\mathbb{C}$ and translation-invariant as**

$$\rho^A_{\text{Ginibre}}(x)dz = \mathcal{K}^A_{\text{Ginibre}}(z, z)\lambda_N(dz) = \frac{1}{\pi}dz, \quad z \in \mathbb{C}.$$

[Gin65] Ginibre, J.: Statistical ensembles of complex, quaternion, and real matrices, J. Math. Phys. <u>6</u> 440–449 (1965)
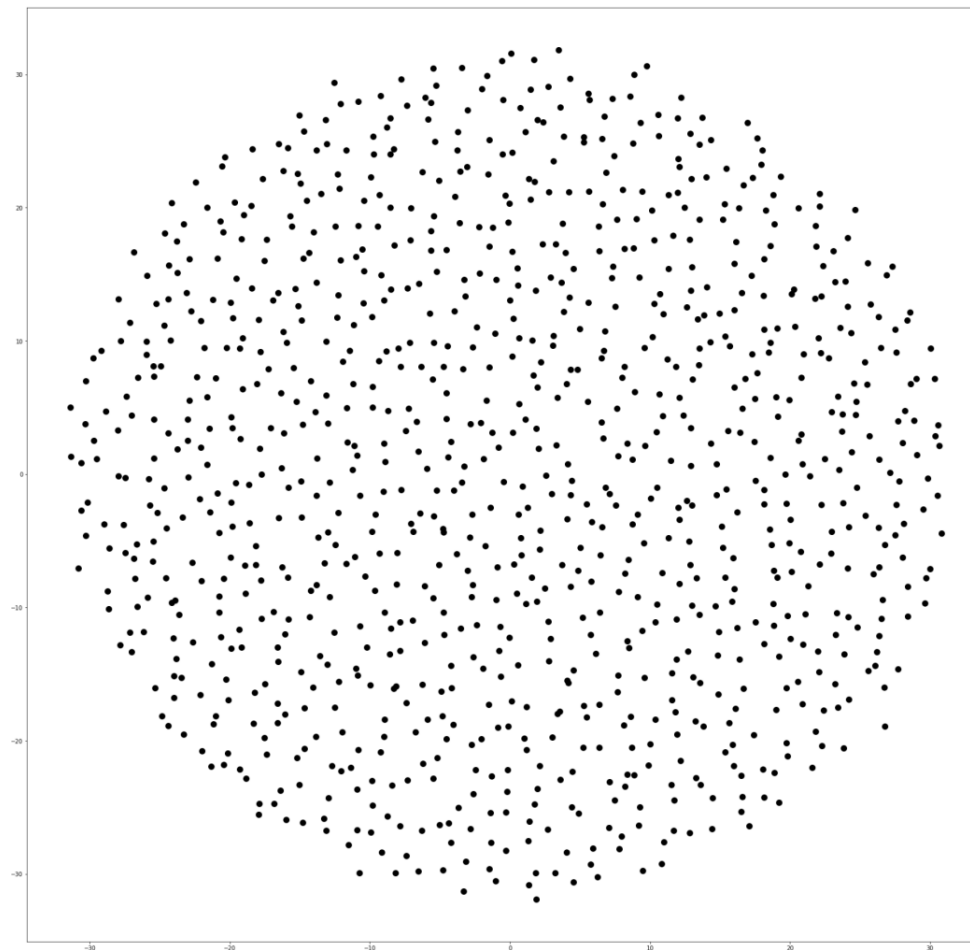
- **Put**

$$\mathcal{K}^A_{\text{Ginibre}}(z, z') = e^{z\overline{z'}},$$

$$\mathcal{K}^C_{\text{Ginibre}}(z, z') = \sinh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} - e^{-z\overline{z'}}),$$

$$\mathcal{K}^D_{\text{Ginibre}}(z, z') = \cosh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} + e^{-z\overline{z'}}), \quad z, z' \in \mathbb{C}.$$

- **Then the Ginibre DPPs of type R are defined by $(\Xi, \mathcal{K}^R_{\text{Ginibre}}, \lambda_{\text{N}}(dz))$ for $R = A, C,$ and D, respectively.**

- **The Ginibre DPP of type A describes the eigenvalue distribution of the Gaussian random complex matrix in the bulk scaling limit [Gin65]. The density of points is uniform with the Lebesgue measure dz on $\mathbb{C}$ and translation-invariant as**

$$\rho^A_{\text{Ginibre}}(x)dz = \mathcal{K}^A_{\text{Ginibre}}(z, z)\lambda_{\text{N}}(dz) = \frac{1}{\pi}dz, \quad z \in \mathbb{C}.$$

[Gin65] Ginibre, J.: Statistical ensembles of complex, quaternion, and real matrices, J. Math. Phys. **6** 440–449 (1965)

Finite approximation of Ginibre DPP of type A: eigenvalues of Gaussian random complex matrix

(Computer simulation by T. Matsui (Chuo U.))

- On the other hands, the Ginibre DPPs of types C and D with the correlation kernels,

$$\mathcal{K}^C_{\text{Ginibre}}(z, z') = \sinh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} - e^{-z\overline{z'}}),$$

$$\mathcal{K}^D_{\text{Ginibre}}(z, z') = \cosh(z\overline{z'}) = \frac{1}{2}(e^{z\overline{z'}} + e^{-z\overline{z'}}), \quad z, z' \in \mathbb{C},$$

are rotationally symmetric around the origin, but non-uniform on $\mathbb{C}$ and the translation-symmetry is broken. The density of points with the Lebesgue measure dz on $\mathbb{C}$ are given by

$$\rho^C_{\text{Ginibre}}(x)dx = \mathcal{K}^C_{\text{Ginibre}}(x, x)\lambda_{\text{N}}(dx) = \frac{1}{2\pi}(1 - e^{-2|z|^2})dz, \quad z \in \mathbb{C},$$

$$\rho^D_{\text{Ginibre}}(z)dz = \mathcal{K}^D_{\text{Ginibre}}(z, z)\lambda_{\text{N}}(dz) = \frac{1}{2\pi}(1 + e^{-2|z|^2})dz, \quad z \in \mathbb{C}.$$

They were first obtained by the following limit theorems.

- These three types of Ginibre DPPs on $\mathbb{C}$ are obtained by the **infinite particle limits** of our seven types of DPPs on $\mathbb{T}^2$.

Proposition 3.7 The following weak convergence is established,

$$
\frac{1}{2}\sqrt{\frac{n}{\pi|\tau|}} \circ \left(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{A}_{n-1}}, dz\right) \overset{n\to\infty}{\Longrightarrow} \left(\Xi, \mathcal{K}_{\text{Ginibre}}^{A}, \lambda_{\text{N}}(dz)\right),
$$

$$
\sqrt{\frac{n}{2\pi|\tau|}} \circ \left(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{R}_n}, dz\right) \overset{n\to\infty}{\Longrightarrow} \left(\Xi, \mathcal{K}_{\text{Ginibre}}^{C}, \lambda_{\text{N}}(dz)\right), \quad \boldsymbol{R}_n = \boldsymbol{B}_n, \boldsymbol{B}_n^{\vee}, \boldsymbol{C}_n, \boldsymbol{C}_n^{\vee}, \boldsymbol{BC}_n,
$$

$$
\sqrt{\frac{n}{2\pi|\tau|}} \circ \left(\Xi, K_{\mathbb{T}^2}^{\boldsymbol{D}_n}, dx\right) \overset{n\to\infty}{\Longrightarrow} \left(\Xi, \mathcal{K}_{\text{Ginibre}}^{D}, \lambda_{\text{N}}(dz)\right).
$$

- Remark that we have the **degeneracy** from the seven finite DPPs to three infinite DPPs.

- Here we show a sketch of proof for the first limit theorem by showing the convergence of the correlation kernels of type $\boldsymbol{A}_{n-1}$. Similar calculation proves the other two limit theorems.

A Sketch of Proof.

- By definition of $\Psi_j^{A_{n-1}}(z)$ with the fact $\lim\limits_{n\to\infty} \theta(\zeta;p^n) = 1 - \zeta$ for $p = e^{-2\pi|\tau|}$, we can evaluate

$$\Psi_j^{A_{n-1}}\left(2\sqrt{\frac{\pi|\tau|}{n}}z\right) \sim e^{-n\pi|\tau|/4}(2|\tau|)^{1/4}n^{1/4}e^{-y^2}$$

$$\times \exp\left[-\pi|\tau|\left(\frac{j-1-n/2}{\sqrt{n}}\right)^2 + 2i\sqrt{\pi|\tau|}z\frac{j-1-n/2}{\sqrt{n}}\right]$$

$$\times \left\{e^{-i\sqrt{n\pi|\tau|}z}\exp\left(\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right) + e^{i\sqrt{n\pi|\tau|}z}\exp\left(-\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right)\right\}$$

as $n \to \infty$.

- **Hence we have**

$$K_{\mathbb{T}^2}^{A_{n-1}}\left(2\sqrt{\frac{\pi|\tau|}{n}}z, 2\sqrt{\frac{\pi|\tau|}{n}}z'\right)\left(2\sqrt{\frac{\pi|\tau|}{n}}\right)^2$$

$$\sim e^{-n\pi|\tau|/2}(2|\tau|)^{1/2}e^{-(y^2+y'^2)}4\pi|\tau|\frac{1}{(2\pi)^2|\tau|}$$

$$\times\frac{1}{n^{1/2}}\sum_{j=0}^{n}\exp\left[-2\pi|\tau|\left(\frac{j-1-n/2}{\sqrt{n}}\right)^2+2i\sqrt{\pi|\tau|}\frac{j-1-n/2}{\sqrt{n}}(z-\overline{z'})\right]$$

$$\times\left[e^{-i\sqrt{n\pi|\tau|}z}\exp\left(\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right)+e^{i\sqrt{n\pi|\tau|}z}\exp\left(-\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right)\right]$$

$$\times\left[e^{i\sqrt{n\pi|\tau|}\overline{z'}}\exp\left(\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right)+e^{-i\sqrt{n\pi|\tau|}\overline{z'}}\exp\left(-\pi|\tau|\sqrt{n}\frac{j-1-n/2}{\sqrt{n}}\right)\right]$$

$$\sim e^{-n\pi|\tau|/2}\frac{(2|\tau|)^{1/2}}{\pi}\left\{e^{-i\sqrt{n\pi|\tau|}(z-\overline{z'})}I_+ + 2\cos(\sqrt{n\pi|\tau|}(z+\overline{z'}))I_0 + e^{i\sqrt{n\pi|\tau|}(z-\overline{z'})}I_-\right\},$$

as $n \to \infty$.

- **Here**

$$I_\pm := \int_{-\sqrt{n}/2}^{\sqrt{n}/2} e^{-2\pi|\tau|u^2 + 2i\sqrt{\pi|\tau|}(z-\overline{z'})u \pm 2\pi|\tau|\sqrt{n}u} du$$

$$= e^{n\pi|\tau|/2} e^{\pm i\sqrt{n\pi|\tau|}(z-\overline{z'})} e^{-(z-\overline{z'})^2/2} \int_{a_\pm - i(z-\overline{z'})/2\sqrt{\pi|\tau|}}^{b_\pm - i(z-\overline{z'})/2\sqrt{\pi|\tau|}} e^{-2\pi|\tau|v^2} dv$$

with $a_+ = -b_- = -\sqrt{n}$, $b_+ = a_- = 0$, and

$$I_0 := \int_{-\sqrt{n}/2}^{\sqrt{n}/2} e^{-2\pi|\tau|u^2 + 2i\sqrt{\pi|\tau|}(z-\overline{z'})u} du = e^{-(z-\overline{z'})^2/2} \int_{-\sqrt{n}/2 - i(z-\overline{z'})/2\sqrt{\pi|\tau|}}^{\sqrt{n}/2 - i(z-\overline{z'})/2\sqrt{\pi|\tau|}} e^{-2\pi|\tau|v^2} dv.$$

We see

$$I_\pm \sim e^{n\pi|\tau|/2} \frac{1}{2} \frac{1}{(2|\tau|)^{1/2}} e^{\pm i\sqrt{n\pi|\tau|}(z-\overline{z'})} e^{-(z-\overline{z'})^2/2},$$

$$I_0 \sim \frac{1}{(2|\tau|)^{1/2}} e^{-(z-\overline{z'})^2/2}, \quad n \to \infty.$$

- Therefore, we have

$$K_{\mathbb{T}^2}^{A_{n-1}}\left(2\sqrt{\frac{\pi|\tau|}{n}}z, 2\sqrt{\frac{\pi|\tau|}{n}}z'\right)\left(2\sqrt{\frac{\pi|\tau|}{n}}\right)^2 \to \frac{1}{\pi}e^{-(y^2+y'^2)-(z-\overline{z'})^2/2}.$$

- Since

$$\frac{1}{\pi}e^{-(y^2+y'^2)-(z-\overline{z'})^2/2} = \frac{e^{-ixy}}{e^{-ix'y'}}e^{z\overline{z'}}\frac{1}{\pi}e^{-(|z|^2+|z'|^2)/2}$$

$$= \frac{e^{-ixy}}{e^{-ix'y'}} \times \sqrt{\frac{1}{\pi}e^{-|z|^2}} \times e^{z\overline{z'}} \times \sqrt{\frac{1}{\pi}e^{-|z'|^2}}$$

with

$$\mathcal{K}_{\mathrm{Ginibre}}^{A}(z,z') = e^{z\overline{z'}}, \qquad \lambda_{\mathrm{N}}(dz) = \frac{1}{\pi}e^{-|z|^2}dz,$$

the **Gauge invariance of DPP** (Lemma 3.1) and the equivalence

$$(\Xi, K(x,y), g(x)\lambda(dx)) \stackrel{(\mathrm{law})}{=} (\Xi, \sqrt{g(x)}K(x,y)\sqrt{g(y)}, \lambda(dx)),$$

imply the assertion. ∎

Thank you very much
for your attention.

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