Sparse Interpolation in terms of the Multivariate Chebyshev Polynomials associated to a Weyl group

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Modern Analysis Related to Root Systems with Applications
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Joint work with **Michael Singer**, North Carolina State University,
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started in 2015 at the Fields Institute

1. Interpolation *vs* sparse interpolation
2. Generalized Chebyshev polynomials
3. Sparse interpolation in the monomial basis :
   from univariate to multivariate
4. Sparse interpolation in the Chebyshev basis (or with symmetry)

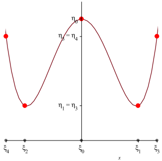
Interpolation

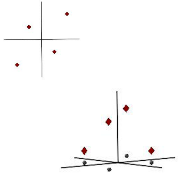
Input : r **number of nodes or** $r-1$ **the degree**

- $\xi_1, \ldots, \xi_r \subset \mathbb{K}$.
- $\eta_1, \eta_2, \ldots, \eta_r \in \mathbb{K}$.

Output : $p = p_r + p_{r-1}x + \ldots + p_1 x^{r-1}$ s.t. $p(\xi_1) = \eta_1, \ldots, p(\xi_r) = \eta_r$.

Symmetry :
$f(-x) = f(x)$
or
$f(-x) = -f(x)$





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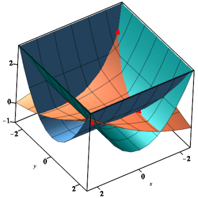

Sparse Interpolation

Input : r **number of terms**

- $f : \mathbb{K} \to \mathbb{K}$.

Output : $(d_1, a_1), \ldots (d_r, a_r) \in \mathbb{Z} \times \mathbb{K}$ s.t. $f(x) = a_1\, x^{d_1} + \ldots + a_r\, x^{d_r}$

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Alternative polynomial basis :

$$F(X) = a_1\, T_{d_1}(X) + \ldots + a_r\, T_{d_r}(X)$$

Symmetry : $f(x^{-1}) = f(x)$ *ie.*, $f(x) = F(x + x^{-1})$

$$f(x) = a_1(x^{d_1} + x^{-d_1}) + \ldots + a_r(x^{d_r} + x^{-d_r})$$

Multivariate sparse interpolation

Notation: $x^{\beta} = x_1^{\beta_1} \ldots x_n^{\beta_n}$

$$f(x) = \sum_{i=1}^{r} a_i\, x^{\beta_i}$$ with unknowns : $\beta_i \in \mathbb{Z}^n, a_i \in \mathbb{K}$

- With symmetry : $f(x^A) = f(x)$, for $A \in \mathcal{A}$
- W.r.t alternative bases of polynomials : generalized Chebyshev

Univariate Chebyshev polynomials (of the first kind)

$\tilde{T}_k(cos(\theta)) = \cos(k\,\theta)$

i.e., $\tilde{T}_k\left(\frac{e^{\mathrm{i}\theta}+e^{-\mathrm{i}\theta}}{2}\right) = \frac{e^{\mathrm{i}k\theta}+e^{-\mathrm{i}k\theta}}{2}$

or $\tilde{T}\left(\frac{x+x^{-1}}{2}\right) = \frac{x^d+x^{-d}}{2}$

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or $\tilde{T}\left(\frac{x+x^{-1}}{2}\right) = \frac{x^d+x^{-d}}{2}$

$$T_d(x + x^{-1}) = x^d + x^{-d}$$

$\mathcal{A}_1 = \{1, -1\} \subset \mathrm{GL}_1(\mathbb{Z})$

$$\begin{array}{ccc} \mathcal{A}_1 \times \mathbb{K}[x, x^{-1}] & \rightarrow & \mathbb{K}[x, x^{-1}] \\ (\epsilon, x^d) & \mapsto & x^{\epsilon d} \end{array}$$

$\mathbb{K}[x, x^{-1}]^{\mathcal{A}} = \langle x^d + x^{-d} \rangle_{\mathbb{K}} = \mathbb{K}[x + x^{-1}]$

$\mathcal{A}$ a finite subgoup of $\mathrm{GL}_n(\mathbb{Z})$

Linear action of $\mathcal{A}$ on $\mathbb{K}[x, x^{-1}] = \mathbb{K}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$

$$\begin{array}{ccc} \mathcal{A} \times \mathbb{K}[x, x^{-1}] & \to & \mathbb{K}[x, x^{-1}] \\ (A, x^{\beta}) & \mapsto & x^{A\beta} \end{array} \qquad \text{i.e. } (A \cdot f)(x) = f\left(x^A\right)$$

$\mathcal{A}_2 \sim \mathfrak{S}_3$

$$\underset{x^a y^b}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{\frac{y^a}{x^{a+b}}}{\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}}, \underset{\frac{x^b}{y^{a+b}}}{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}, \underset{\frac{y^{a+b}}{x^a}}{\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}}, \underset{\frac{1}{x^b y^a}}{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}, \underset{\frac{x^{a+b}}{y^b}}{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}$$

The orbit polynomials

$\Theta_\beta = \sum_{B \in \mathcal{A}} x^{B\beta}$ are invariants that span $\mathbb{K}[x, x^{-1}]^{\mathcal{A}}$

$$\Theta_{[a,b]}(x, y) = x^a y^b + \frac{y^a}{x^{a+b}} + \frac{x^b}{y^{a+b}} + \frac{y^{a+b}}{x^a} + \frac{1}{x^b y^a} + \frac{x^{a+b}}{y^b}$$

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The orbit polynomials

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Wanted :

$$\mathbb{K}[x, x^{-1}]^{\mathcal{A}} = \mathbb{K}[\Theta_{\omega_1}, \ldots, \Theta_{\omega_n}]$$

where

$\omega_1 = (1, \ldots, 0), \ldots, \omega_n = (0, \ldots, 0, 1)$

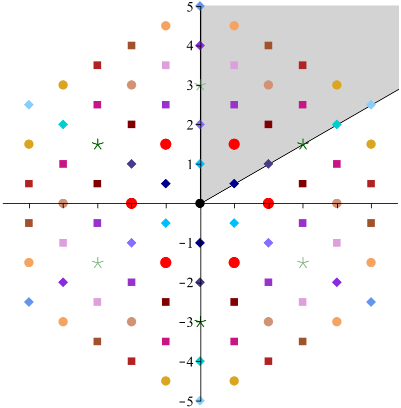
$$T_\alpha \in \mathbb{K}[X_1, \ldots X_n]$$

s.t.

$$\Theta_\alpha = T_\alpha(\Theta_{\omega_1}, \ldots, \Theta_{\omega_n})$$

Chebyshev polynomials associated to the Weyl group $\mathcal{A}$

$\mathcal{A}$ generated by reflections associated to a *crystalographic* *root system* R



Reflection w.r.t. $\rho \in \mathrm{R}$:

$$s_\rho(\omega) = \omega - 2\frac{\langle \omega, \rho \rangle}{\langle \rho, \rho \rangle}\rho$$

$\rho, \mu \in \mathrm{R} \Rightarrow$

- $s_\rho(\mu) \in \mathrm{R}$
- $2\frac{\langle \mu, \rho \rangle}{\langle \rho, \rho \rangle} \in \mathbb{Z}$.

Weight lattice :

$$\Omega = \left\{ \omega \in \mathbb{R}^n \mid 2\frac{\langle \rho, \omega \rangle}{\langle \rho, \rho \rangle} \in \mathbb{Z}\ \forall \rho \in \mathrm{R} \right\}$$

$$s_\rho(\Omega) = \Omega$$

Chebyshev polynomials associated to the Weyl group $\mathcal{A}$

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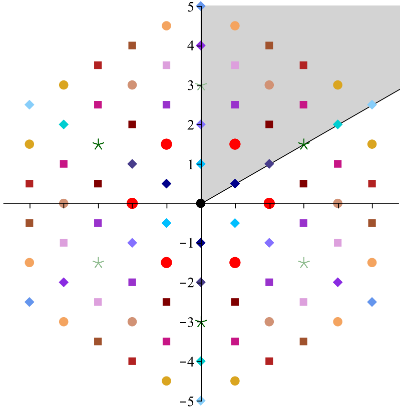
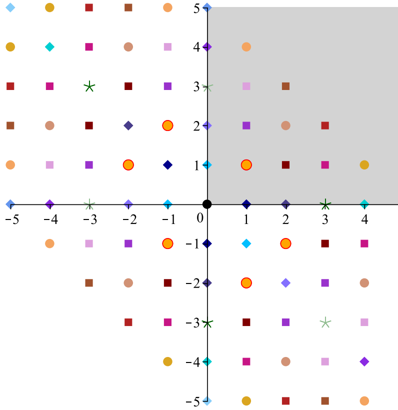




Weight lattice :

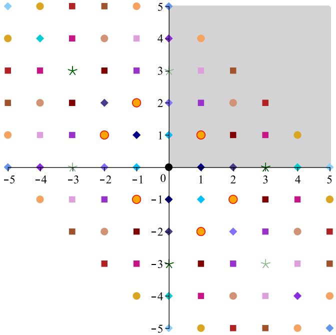
$$\Omega = \left\{ \omega \in \mathbb{R}^n \mid 2\frac{\langle \rho, \omega \rangle}{\langle \rho, \rho \rangle} \in \mathbb{Z} \; \forall \rho \in \mathrm{R} \right\}$$

$$\Omega = \mathbb{Z}^n$$

Chebyshev polynomials associated to the Weyl group $\mathcal{A}$

$\mathcal{A}$ generated by reflections associated to a *crystalographic* *root system* R

$$\mathcal{A} \subset \mathrm{GL}_n(\mathbb{Z})$$



Invariant scalar product : $\langle \beta, \alpha \rangle = \beta^{\mathsf{T}} S \alpha$

$A^{\mathsf{T}} S A = S$ for all $A \in \mathcal{A}$

Orbit polynomials : $\Theta_\beta = \sum_{A \in \mathcal{A}} x^{A\beta}$

Thm [Bourbaki]

$\mathbb{K}[x, x^{-1}]^{\mathcal{A}} = \mathbb{K}[\Theta_{\omega_1}, \ldots, \Theta_{\omega_n}] \qquad \omega_1 = (1, \ldots, 0), \ldots, \omega_n = (0, \ldots, 0, 1)$

There exists $T_\alpha \in \mathbb{K}[X_1, \ldots X_n]$ such that $\Theta_\alpha = T_\alpha(\Theta_{\omega_1}, \ldots, \Theta_{\omega_n})$.

$$\tilde{T}_k \, \tilde{T}_l = \tilde{T}_{k+l} + \tilde{T}_{|k-l|}$$

Multiplication

$$\Theta_\alpha \, \Theta_\beta = \sum_{A \in \mathcal{A}} \Theta_{\alpha + A\beta} = \sum_{\gamma \prec \alpha+\beta} c_\gamma \Theta_\gamma \qquad (c_{\alpha+\beta} \neq 0)$$

$\rightsquigarrow$ Recurrences on T_α

$$T_k = 2x^k + a_{k-2}x^{k-2} + a_{k-4}x^{k-4} + \dots$$

Property

$$T_\alpha = \sum_{\beta \prec \alpha} t_\beta \, x^\beta$$

$$\beta \prec \alpha \Leftrightarrow \alpha - \beta = m_1 \, \rho_1 - \dots - m_n \, \rho_n$$

$\mathcal{A}_2$ Chebyshev polynomials

$$T_{00} = 8$$

$$T_{10} = X, T_{01} = Y$$

$$T_{20} = \frac{1}{2}X^2 - 2Y,\ T_{11} = \frac{1}{4}YX - 3,\ T_{02} = \ldots$$

$$T_{30} = \frac{1}{4}X^3 - \frac{3}{2}XY + 6,\ \ T_{21} = \frac{1}{8}X^2Y - \frac{1}{2}Y^2 - \frac{1}{2}X,\ \ T_{12} = \ldots$$

The references from which I learned about Chebyshev polynomials :

H. Munthe-Kaas, M. Nome, and B. Ryland. Through the kaleidoscope: symmetries, groups and Chebyshev approximations from a computational point of view. In FoCM Budapest 2011, LMS Lecture Note Series 403.

M. Hoffman and W. Withers. Generalized Chebyshev polynomials associated with affine Weyl groups. Trans. Amer. Math. Soc., 308(1), 1988.

Chebyshev polynomials of the second kind

Classical univariate case

$$\tilde{U}_d(\cos\theta) = \frac{\sin(d+1)\theta}{\sin\theta}$$

$$\text{or } U_d(x+x^{-1}) = \frac{x^{d+1} - x^{d+1}}{x - x^{-1}} = x^d + x^{d-1} + \ldots + x + 1$$

The *signed orbit polynomials* : $\Upsilon_\alpha(x) = \sum_{A\in\mathcal{A}} \det(A)\, x^{A\alpha}$

$$A \cdot \Upsilon_\alpha(x) = \Upsilon_\alpha(x^A) = \det A\, \Upsilon_\alpha(x)$$

Weyl character formula :

$$U_\beta(\Theta_{\omega_1}, \ldots, \Theta_{\omega_n}) = \frac{\Upsilon_{\delta+\beta}}{\Upsilon_\delta} \in \mathbb{K}[x, x^{-1}]^{\mathcal{A}} \qquad \delta = (1, \ldots, 1)$$

Sparse interpolation : univariate monomials or exponentials

[Ben Or & Tiwari 1988]

Sparse Interpolation

$$f(x) = \sum_{i=1}^{r} a_i\, x^{d_i}$$

Unknowns : $d_i \in \mathbb{N}$, $a_i \in \mathbb{K}$

Choose to evaluate:

$$f(1),\ f(\xi), \ldots,\ f(\xi^{2r-1})$$

$$f(\xi^k) = \sum_{i=1}^{r} a_i\, \left(\xi^k\right)^{d_i} = \sum_{i=1}^{r} a_i\, \left(\xi^{d_i}\right)^k$$

Recover $\xi^{d_1}, \ldots, \xi^{d_r}$

[Prony 1795]

Exponential sums

$$f(z) = \sum_{i=1}^{r} a_i e^{\lambda_i z}$$

Unknowns : $\lambda_i \in \mathbb{C}$, $a_i \in \mathbb{C}$

Choose to evaluate:

$$f(0), \ldots,\ f(2r-1)$$

$$f(k) = \sum_{i=1}^{r} a_i e^{\lambda_i\, k} = \sum_{i=1}^{r} a_i\, \left(e^{\lambda_i}\right)^k$$

Recover $e^{\lambda_1}, \ldots, e^{\lambda_r}$

Hankel machinery in the univariate case

$$\mu_k = \sum_{j=1}^{r} a_j\, {\zeta_j}^k, \quad k = 0, \dots, 2r-1$$

- The nodes $\zeta_1, \dots, \zeta_r$ are the generalized eigenvalues of (H_1, H_x)

$$H_1 = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{r-1} \\ \mu_1 & & & \\ \vdots & & \iddots & \vdots \\ & \iddots & & \\ \mu_{r-1} & & \cdots & \mu_{2r-2} \end{pmatrix}, \quad H_x = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_r \\ \mu_2 & & & \\ \vdots & & \iddots & \vdots \\ & \iddots & & \\ \mu_r & & \cdots & \mu_{2r-1} \end{pmatrix}.$$

- The coefficients $a_1, \dots, a_r$ satisfy

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_r \\ \vdots & \vdots & & \vdots \\ {\xi_1}^{r-1} & {\xi_2}^{r-1} & \cdots & {\xi_r}^{r-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{r-1} \end{pmatrix}.$$

Multivariate sparse interpolation

Sparse Interpolation

$$f(x) = \sum_{i=1}^{r} a_i \, x^{\beta_i}$$

$$x^{\beta} = x_1^{\beta_1} \ldots x_n^{\beta_n}$$

Unknowns : $\beta_i \in \mathbb{N}^n$, $a_i \in \mathbb{F}_p$

[Kaltofen, Lee, Arnold, . . .]

Reduce to the univariate case

$$f\left(\xi_1^k, \ldots, \xi_n^k\right), \; k = 0 \ldots 2r-1$$

$\xi_1^{\beta_{i,1}} \ldots \xi_n^{\beta_{i,n}}$, $1 \leq i \leq r$

Exponential sums

$$f(z) = \sum_{i=1}^{r} a_i \, e^{\langle \lambda_i, z \rangle}$$

$$\langle \lambda_i, z \rangle = \lambda_1 z_1 + \ldots + \lambda_n z_n$$

Unknowns : $\lambda_i \in \mathbb{C}^n$, $a_i \in \mathbb{C}$

[Potts, Kunis, Mourrain, Sauer]

Evaluate

$$f(\alpha), \quad \alpha \in \Gamma \subset \mathcal{C}_r^n + \overline{\mathcal{C}_r^n}$$

$\left[e^{\lambda_{i,1}}, \ldots, e^{\lambda_{i,n}}\right]$, $1 \leq i \leq r$

Hankel Machinery : the mutivariate case

$$\mu_\alpha = \sum_{j=1}^{r} a_j \, \zeta_j{}^\alpha \qquad \zeta_1, \ldots, \zeta_r \in \mathbb{C}^n \qquad \alpha \in \mathbb{N}^n$$

- $\zeta_1, \ldots, \zeta_r$ are deduced from the generalized eigenvalues/vectors of

$$H_1^\Gamma \text{ and } \ell_1 \, H_{x_1}^\Gamma + \ldots + \ell_n \, H_{x_n}^\Gamma \qquad \text{where } H_{x^\gamma}^\Gamma = [\mu_{\alpha+\beta+\gamma}]_{\alpha,\beta\in\Gamma}$$

- $(a_1, \ldots, a_r)$ is the solution of a Vandermonde-like linear system

Hankel Machinery : the mutivariate case

$$\mu_\alpha = \sum_{j=1}^{r} a_j \, \zeta_j^{\,\alpha} \qquad \zeta_1, \ldots, \zeta_r \in \mathbb{C}^n \qquad \alpha \in \mathbb{N}^n$$

- Find Γ indexing a nonsingular $r \times r$ submatrix in $H_1^{\mathcal{C}_r^n} = [\mu_{\alpha+\beta}]_{\alpha,\beta \in \mathcal{C}_r^n}$ where $\mathcal{C}_r^n = \{\alpha \in \mathbb{N}^n \mid \prod_{i=1}^r (\alpha_i + 1) \leq r\}$ [Sauer 2018]

  $|\,\mathcal{C}_r^n\,| \leq r \log^{n-1}(r)$ [Lubich]

- $\zeta_1, \ldots, \zeta_r$ are deduced from the generalized eigenvalues/vectors of

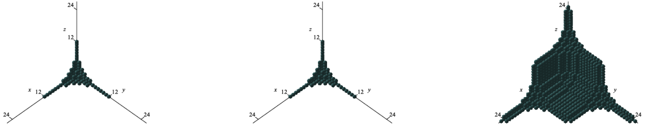
  $$H_1^\Gamma \text{ and } \ell_1 \, H_{x_1}^\Gamma + \ldots + \ell_n \, H_{x_n}^\Gamma \qquad \text{where } H_{x^\gamma}^\Gamma = [\mu_{\alpha+\beta+\gamma}]_{\alpha,\beta\in\Gamma}$$

- $(a_1, \ldots, a_r)$ is the solution of a Vandermonde-like linear system

Multivariate sparse interpolation : Algorithm Outline

$$f(x) = \sum_{i=1}^{r} a_i \, x^{\beta_i} \in \mathbb{K}[x, x^{-1}]$$

- $\xi \in \mathbb{Q}$ $\qquad\qquad \xi^{\alpha^T} = (\xi^{\alpha_1}, \ldots, \xi^{\alpha_n})$
- From the evaluations $f\left(\xi^{\alpha^T}\right)$, $\alpha \in \Gamma \subset \mathcal{C}_r^n + \overline{\mathcal{C}_r^n}$



⋆⋆ **Hankel machinery** ⋆⋆

Retrieve $\xi^{\beta_i^T} = \left[\xi^{\beta_{i,1}}, \ldots, \xi^{\beta_{i,n}}\right]$ for each $1 \leq i \leq r$

- Take logarithms and output $\beta_1, \ldots, \beta_r \in \mathbb{N}^n$

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Univariate : alternative polynomial basis & symmetry

$$\tilde{T}_d(cos(\theta)) = \cos(d\,\theta)$$

$$F(x + x^{-1}) = f(x)$$

Alternative basis in $\mathbb{K}[X]$

$$F(X) = \sum_{i=1}^{r} a_i\, \tilde{T}_{d_i}(X)$$

Symmetry in $\mathbb{K}[x, x^{-1}]$

$$f(x) = \sum_{i=1}^{r} a_i \left(x^{d_i} + x^{-d_i}\right)$$

$$\tilde{T}_d(\tilde{T}_k(X)) = \tilde{T}_k(\tilde{T}_d(X))$$

Invariance : $f(x^{-1}) = f(x)$

Univariate : alternative polynomial basis & symmetry

$$\tilde{T}_d(\cos(\theta)) = \cos(d\,\theta)$$

$$F(x + x^{-1}) = f(x)$$

Alternative basis in $\mathbb{K}[X]$

$$F(X) = \sum_{i=1}^{r} a_i\, \tilde{T}_{d_i}(X)$$

Symmetry in $\mathbb{K}[x, x^{-1}]$

$$f(x) = \sum_{i=1}^{r} a_i \left(x^{d_i} + x^{-d_i}\right)$$

[Arnold Kaltofen 2015]

Evaluate [Lakshman, Sauders 1995]

$$F\left(\tilde{T}_0(\tilde{\xi})\right), \dots, F\left(\tilde{T}_{2r-1}(\tilde{\xi})\right)$$

Evaluate

$$f\left(\xi^0\right), \dots, f\left(\xi^{2r-1}\right)$$

Recover $T_{d_1}(\tilde{\xi}),\ \dots,\ T_{d_r}(\tilde{\xi})$

Recover $(\xi^{d_1} + \xi^{-d_1}), \dots, (\xi^{d_r} + \xi^{-d_r})$

Inverse problem

$$\zeta = \xi^d + \xi^{-d} \qquad \xi > 1 \;\Rightarrow\; \xi^d = \lfloor \zeta \rfloor \;\Rightarrow\; d = \frac{\log \lfloor \zeta \rfloor}{\log \xi}$$

Sparse interpolation in terms of Multivariate Chebyshev

In : $F : \mathbb{Q}^n \to \mathbb{Q}$ where $F(X) = \sum_{i=1}^{r} a_i T_{\beta_i}(X) \in \mathbb{K}[X]$

Out: $\beta_1, \ldots, \beta_r$

$$f(x) := F\left(\Theta_{\omega_1}(x), \ldots, \Theta_{\omega_n}(x)\right) \in \mathbb{K}[x, x^{-1}]$$

$$\Theta_\beta(x) = \sum_{A \in \mathcal{A}} x^{A\beta}$$

Multivariate sparse interpolation with symmetry

In : $f : \mathbb{Q}^n \to \mathbb{Q}$ where $f(x) = \sum_{i=1}^{r} a_i \Theta_{\beta_i}(x) \in \mathbb{K}[x, x^{-1}]$

Out: $\beta_1, \ldots, \beta_r$

Commutativity [H. & Singer]

$$\xi \in \mathbb{K}^* \qquad \gamma^T = [\gamma_1, \ldots, \gamma_n] \in \mathbb{N}^n \qquad \xi^{\gamma^T} = [\xi^{\gamma_1}, \ldots, \xi^{\gamma_n}] \in (\mathbb{K}^*)^n$$

$$\Theta_\alpha\left(\xi^{\beta^T S}\right) = \Theta_\beta\left(\xi^{\alpha^T S}\right).$$

Inverse problem [H. & Singer]

For $\xi > (\frac{3}{2}|\mathcal{A}|)^2$ $\beta \in \mathbb{N}^n$ and $\mu \in (\mathbb{N}_{>0})^n$

$$\langle \mu, \beta \rangle \leq \log_\xi\left(\Theta_\beta\left(\xi^{\mu^T S}\right)\right) < \langle \mu, \beta \rangle + \frac{1}{2}$$

$$\Theta_\beta\left(\xi^{\mu^T S}\right) = \sum_{A \in \mathcal{A}} \xi^{\mu^T S A \beta} = \sum_{A \in \mathcal{A}} \xi^{\langle \mu, A\beta \rangle} = \xi^{\langle \mu, \beta \rangle}\left(1 + \sum_{A \neq 1, A \in \mathcal{A}} \xi^{\langle \mu, A\beta - \beta \rangle}\right)$$

From $\langle \mu_1, \beta \rangle, \ldots, \langle \mu_n, \beta \rangle$ for $\mu_1, \ldots, \mu_n \in (\mathbb{N}_{>0})^n$ recover β

Algorithm outline - with symmetry

$$f(x) = \sum_{i=1}^{r} a_i \, \Theta_{\beta_i}(x) \in \mathbb{K}[x, x^{-1}]$$

- $\xi \geq \left(\frac{3}{2}\,|\mathcal{A}|\right)^2$ $\qquad \xi^{\alpha^T} = (\xi^{\alpha_1}, \ldots, \xi^{\alpha_n})$
- From the evaluations $f\left(\xi^{\alpha^T S}\right)$, $\alpha \in \Gamma \subset \mathcal{C}_r^n + \mathcal{A}\,\mathcal{C}_r^n + \mathcal{A}\,\mathcal{C}_2^n$

  ⋆⋆ **Invariant Hankel machinery** ⋆⋆

  Retrieve $\vartheta_i = \left[\Theta_{\omega_1}\left(\xi^{\beta_i^T S}\right), \ldots, \Theta_{\omega_n}\left(\xi^{\beta_i^T S}\right)\right]$ for each $1 \leq i \leq r$

- Retrieve $\beta_i \in \mathbb{N}^n$ from $\left[\Theta_{\beta_i}(\xi^{\mu_1^T S}), \ldots, \Theta_{\beta_i}(\xi^{\mu_n^T S})\right]$

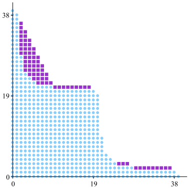
  where $\mu_1, \ldots, \mu_n \in (\mathbb{N}_{>0})^n$ linearly independent

  and

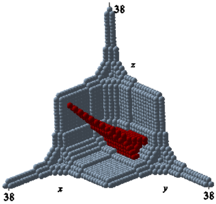
$$\Theta_{\beta_i}(\xi^{\mu^T S}) = \Theta_{\mu}(\xi^{\beta_i^T S}) = T_{\mu}(\vartheta_i)$$

Comparative Computational Cost

$\mathcal{B}_2$



$\mathcal{A}_3$



Size of the matrices

Deal with the matrix $\left[\sum_{A\in\mathcal{A}} f(\xi^{\alpha+A\beta})\right]_{\alpha,\beta\in\mathcal{C}^n_r}$

instead of $\left[f(\xi^{\alpha+\beta})\right]_{\alpha,\beta\in\mathcal{C}^n_{|\mathcal{A}|r}}$

Number of evaluations

Bounded by

$$|\,\mathcal{C}^n_r + \mathcal{A}\,\mathcal{C}^n_r + \mathcal{A}\,\mathcal{C}^n_2\,| \leq 2|\mathcal{A}|^2\, r^2\, \log^{2n-2}(r)$$

as opposed to

$$\left|\mathcal{C}^n_{|\mathcal{A}|r} + \mathcal{C}^n_{|\mathcal{A}|r} + \mathcal{C}^n_2\right| \leq (n+1)\,|\mathcal{A}|^2\, r^2\, \log^{2n-2}(|\mathcal{A}|r)$$

Sparse interpolation in terms of Chebyshev of the second kind

In : $F : \mathbb{Q}^n \to \mathbb{Q}$ where $F(X) = \sum_{i=1}^{r} a_i U_{\beta_i}(X) \in \mathbb{K}[X]$

Out: $\beta_1, \ldots, \beta_r$

$$f(x) := \Upsilon_\delta(x)\, F\left(\Theta_{\omega_1}(x), \ldots, \Theta_{\omega_n}(x)\right) \in \mathbb{K}[x, x^{-1}]$$

where $\Upsilon_\alpha(x) = \sum_{A \in \mathcal{A}} \det(A)\, x^{A\alpha}$

and $U_\beta(\Theta_{\omega_1}, \ldots, \Theta_{\omega_n}) = \dfrac{\Upsilon_{\delta+\beta}}{\Upsilon_\delta}$ *(Weyl Character formula)*:

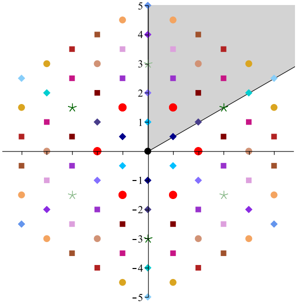
Multivariate sparse interpolation with antisymmetry

In : $f : \mathbb{Q}^n \to \mathbb{Q}$ where $f(x) = \sum_{i=1}^{r} a_i\, \Upsilon_{\delta+\beta_i}(x) \in \mathbb{K}[x, x^{-1}]$

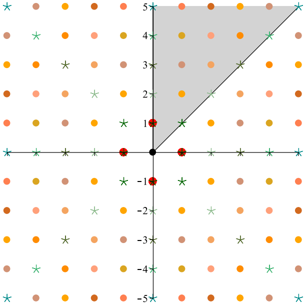
Out: $\beta_1, \ldots, \beta_r$

Thanks

$\mathcal{A}_2$



$\mathcal{B}_2$



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Sparse Interpolation in Terms of Multivariate Chebyshev Polynomials.
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