Stochastic self-duality from quantum algebra representations

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Modern Analysis Related to Root Systems with Applications
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Joint work with G.Carinci, C.Franceschini, C.Giardinà, F.Redig

Outline

Stochastic duality

$X_i = \{\eta_i(t) \mid t \geq 0\}$, $i = 1, 2$, Markov process with state space Ω_i.

Definition: The processes X_1 and X_2 are in duality if there exists a nonconstant function $D : \Omega_1 \times \Omega_2 \to \mathbb{C}$ (duality function) such that

$$\mathbb{E}_1\Big[D\big(\eta_1(t), \eta_2\big)\Big] = \mathbb{E}_2\Big[D\big(\eta_1, \eta_2(t)\big)\Big],$$

for all $t \geq 0$ and all starting configurations $\eta_1 \in \Omega_1$, $\eta_2 \in \Omega_2$.

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$L_i : F(\Omega_i) \to F(\Omega_i)$ generator for process X_i, i.e. $\exp(tL_i)$ is the semigroup corresponding to X_i.

Theorem: X_1 and X_2 are in duality with duality function D, iff

$$\big[L_1 D(\,\cdot\,, \eta_2)\big](\eta_1) = \big[L_2 D(\eta_1,\ \cdot\,)\big](\eta_2), \qquad (\eta_1, \eta_2) \in \Omega_1 \times \Omega_2.$$

Reversible measure and self-duality

Definition: A probability measure w on Ω is a reversible measure if

$$L(x,y)w(x) = L(y,x)w(y), \qquad \text{for all } x,y \in \Omega,$$

where $Lf(x) = \sum_{y \in \Omega} L(x,y) f(y)$.

Equivalent: L is self-adjoint on $\ell^2(\Omega, w)$

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Theorem: A process with a reversible measure is self-dual with duality function

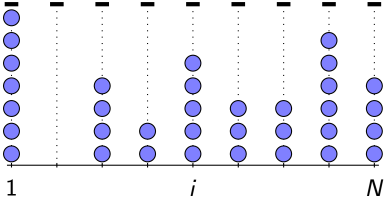
$$d(x,y) = \frac{\delta_{x,y}}{w(x)}.$$

The (generalized) symmetric exclusion process SEP($2j$)

$j \in \frac{1}{2}\mathbb{N}$

State space: $\Omega = \{0, 1, \ldots, 2j\}^N$

x_i: number of particles on site i



Process generator

$$L^{\text{SEP}} f(x) = \sum_{1=1}^{N-1} x_i(2j - x_{i+1}) \left[f(x^{i,i+1}) - f(x)\right] + x_{i+1}(2j - x_i) \left[f(x^{i+1,i}) - f(x)\right]$$

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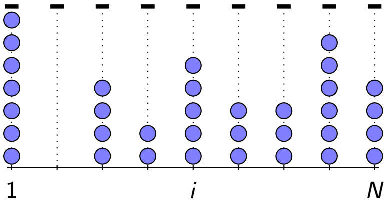

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Family of reversible measures

$$\prod_{i=1}^{N} w_p(x_i), \qquad w_p(x_i) = \binom{2j}{x_i} \left(\frac{p}{1-p} \right)^{x_i} (1-p)^{2j}, \qquad 0 < p < 1$$

Duality function: $D_p^{\text{ch}}(x, y) = \prod_{i=1}^{N} \frac{\delta_{x_i, y_i}}{w_p(x_i)}$

New self-dualities from old ones

Theorem:
If D is a self-duality function and X is a symmetry for the generator L, i.e. $[L, X] = 0$, then XD is again a self-duality function.

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Goal: find useful symmetries.

The Lie algebra $\mathfrak{su}(2)$

Generators: H, E, F with commutation relations

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

$*$**-structure:** $H^* = H, \quad E^* = F, \quad F^* = E.$

The Casimir element $\Omega = \frac{1}{2}H^2 + EF + FE \in \mathcal{U}(\mathfrak{su}(2))$ satisfies

- $[\Omega, X] = 0$ for all $X \in \mathfrak{su}(2)$
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Representation:
For $j \in \frac{1}{2}\mathbb{N}$, $0 < p < 1$, a $*$-representation on $\ell^2(w_p)$ is given by

$$\begin{aligned}
[Hf](x) &= (j-x)f(x),\\
[Ef](x) &= \sqrt{(1-p)/p}\, x\, f(x-1),\\
[Ff](x) &= \sqrt{p/(1-p)}\,(2j-x)f(x+1).
\end{aligned}$$

Relation with symmetric exclusion process

Coproduct: $\Delta : \mathcal{U}(\mathfrak{su}(2)) \to \mathcal{U}(\mathfrak{su}(2)) \otimes \mathcal{U}(\mathfrak{su}(2))$

$$\Delta(X) = 1 \otimes X + X \otimes 1, \qquad X \in \mathfrak{su}(2)$$

Coproduct of the Casimir:

$$\Delta(\Omega) = 1 \otimes \Omega + \Omega \otimes 1 + H \otimes H + 2F \otimes E + 2E \otimes F.$$

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Relation with generator L^{SEP}

Recall:

$$L^{\text{SEP}} f(x) = \sum_{1=1}^{N-1} x_i(2j - x_{i+1}) \left[f(x^{i,i+1}) - f(x) \right] + x_{i+1}(2j - x_i) \left[f(x^{i+1,i}) - f(x) \right]$$

Let

$$Y = \frac{1}{2}\Big(1 \otimes \Omega + \Omega \otimes 1 - \Delta(\Omega)\Big),$$

then

$$L^{\text{SEP}} = \sum_{1=1}^{N-1} (Y_{i,i+1} - 2j^2)$$

Self-duality functions for SEP($2j$) from symmetries

Three types of (factorized) duality functions:

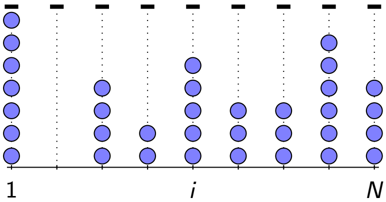
1. $$D_p^{\mathrm{ch}}(x,y) = \prod_{i=1}^{N} \binom{2j}{x_i}^{-1} \left(\frac{p}{1-p}\right)^{-x_i} \delta_{x_i,y_i}$$
2. $$D_p^{\mathrm{tr}}(x,y) = \prod_{i=1}^{N} \frac{x_i!(2j-y_i)!}{(x_i-y_i)!(2j)!} \left(\frac{1-p}{p}\right)^{y_i} \mathbb{1}_{y_i\le x_i}$$
3. $$D_p^{\mathrm{or}}(x,y) = \prod_{i=1}^{N} {}_2F_1\left(\begin{matrix}-x_i,-y_i\\ -2j\end{matrix};\frac{1}{p}\right) = \prod_{i=1}^{N} K_y(x;p,2j)$$

Theorem:

- $D_p^{\mathrm{tr}}(x,y) = C\left[\exp(E)D_p^{\mathrm{ch}}(\,\cdot\,,y)\right](x)$
- $D_p^{\mathrm{or}}(x,y) = C\left[S_{\alpha(p),\beta(p)}D_p^{\mathrm{ch}}(\,\cdot\,,y)\right](x)$ with $S_{\alpha,\beta} = \exp(\beta(E-F))\exp(\mathrm{i}\alpha H)$

The generalized asymmetric exclusion process ASEP(q, $2j$)

Introduced by G. Carinci, C. Giardinà, F. Redig, T. Sasamoto (2016)



$j \in \frac{1}{2}\mathbb{N}$
State space: $\Omega = \{0, 1, \ldots, 2j\}^N$
x_i: number of particles on site i

Process generator:

$$L^{\text{ASEP}} f(x) = \sum_{i=1}^{N-1} q^{1-4j} \frac{(1-q^{2x_i})(1-q^{4j-2x_{i+1}})}{(1-q^2)^2} \left[f(x^{i,i+1}) - f(x)\right]$$
$$+ q^{4j-1} \frac{(1-q^{-2x_{i+1}})(1-q^{-4j+2x_i})}{(1-q^{-2})^2} \left[f(x^{i+1,i}) - f(x)\right]$$

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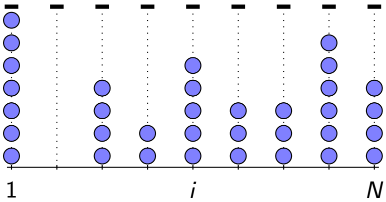

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Family of reversible measures: $w_\alpha(x) = \prod_{i=1}^{n} \binom{2j}{x_i}_q q^{-4jix_i} \alpha^{x_i}, \qquad \alpha > 0$

Duality function: $D_\alpha^{\mathrm{ch}}(x, y) = \dfrac{\delta_{x,y}}{w_\alpha(x)}$

The quantum algebra $\mathcal{U}_q(\mathfrak{su}(2))$

Generators: K, K^{-1}, E and F
Relations:

$$KK^{-1} = 1 = K^{-1}K,$$
$$KE = qEK, \quad KF = q^{-1}FK,$$
$$EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}}.$$

$*$-structure: $K^* = K, \quad E^* = F, \quad F^* = E$
Coproduct:

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = K \otimes E + E \otimes K^{-1},$$
$$\Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \quad \Delta(F) = K \otimes F + F \otimes K^{-1}.$$

The quantum algebra $\mathcal{U}_q(\mathfrak{su}(2))$

Representation:

$$[Kf](x) = q^{x-j} f(x)$$

$$[Ef](x) = \frac{q^{\frac{1}{2}-j}}{1-q^2}\sqrt{(1-q^{2x})(1-q^{4j-2x+2})}\, f(x-1)$$

$$[Ff](x) = \frac{q^{\frac{1}{2}-j}}{1-q^2}\sqrt{(1-q^{2x+2})(1-q^{4j-2x})}\, f(x+1)$$

Casimir: $\Omega = \dfrac{q^{-1}K^2 + qK^{-2} - 2}{(q^{-1}-q)^2} + EF$ satisfies

- $[\Omega, X] = 0$ for all $X \in \mathcal{U}_q(\mathfrak{su}(2))$
- $\Omega^* = \Omega$.

Process generator:

$$L^{\mathrm{ASEP}} = G(x)^{-1}\left(\sum_{i=1}^{N-1} \Delta(\Omega)_{i,i+1} - \frac{(q^{2j}-q^{-2j})(q^{2j+1}-q^{-2j-1})}{(q-q^{-1})^2}\right) G(x).$$

Self-duality functions from symmetries

q-Exponential functions:

$$E_{q^2}(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} x^n}{(q^2; q^2)_n} = (-x; q^2)_\infty, \quad e_{q^2}(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q^2; q^2)_n} = \frac{1}{(x; q^2)_\infty}$$

Symmetries:

$$S_\alpha = e_{q^2}\left(\sqrt{\alpha}(1-q^2)\Delta^{N-1}(KE)\right)$$
$$\widehat{S}_\alpha = E_{q^2}\left(\sqrt{\alpha} q^{-j(1+2N)}(1-q^2)\Delta^{N-1}(K^{-1}E)\right).$$

Self-duality functions:

- $D_\alpha^{\mathrm{tr}}(x,y) = [G^{-1} S_\alpha G\, D_\alpha^{\mathrm{ch}}(\,\cdot\,, y)](x) = \prod_{i=1}^{N} \frac{\binom{x_i}{y_i}_q}{\binom{2j}{x_i}_q} q^{x_i(2N_{i-1}^-(y)+y_i)+4jiy_i} \alpha^{y_i} \mathbb{1}_{y_i \le x_i}$
- $\widehat{D}_\alpha^{\mathrm{tr}}(x,y) = [G^{-1} \widehat{S}_\alpha G\, D_\alpha^{\mathrm{ch}}(\,\cdot\,, y)](x) = \prod_{i=1}^{N} \frac{\binom{x_i}{y_i}_q}{\binom{2j}{x_i}_q} q^{-y_i(2N_{i-1}^-(x)+x_i)-4jiy_i} \alpha^{y_i} \mathbb{1}_{y_i \le x_i}$

Biorthogonal self-duality functions from scalar products

Scalar product: $\langle f, g\rangle_\alpha = \sum_{x\in\Omega} f(x)\overline{g(x)}w_\alpha(x)$.

Theorem: Let d_1 and d_2 be self-duality functions. Then

$$D(x,y) = \langle d_1(x,\,\cdot\,), d_2(y,\,\cdot\,)\rangle_\alpha$$

is also a self-duality function.

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is also a self-duality function.

Theorem: Let d_1, d_2, $\widetilde{d}_1$ and $\widetilde{d}_2$ be self-duality functions satisfying

$$\langle d_1(x,\,\cdot\,), d_2(\,\cdot\,,y)\rangle_\alpha = \frac{\delta_{x,y}}{w_\beta(x)}, \qquad \langle \widetilde{d}_2(x,\,\cdot\,), \widetilde{d}_1(\,\cdot\,,y)\rangle_\beta = \frac{\delta_{x,y}}{w_\alpha(x)}.$$

Then the functions

$$D(x,y) = \langle \widetilde{d}_1(x,\,\cdot\,), d_1(y,\,\cdot\,)\rangle_\alpha, \qquad \widetilde{D}(x,y) = \langle \widetilde{d}_2(\,\cdot\,,x), d_2(\,\cdot\,,y)\rangle_\alpha$$

satisfy the biorthogonality relations

$$\langle D(\,\cdot\,,y), \widetilde{D}(\,\cdot\,,y')\rangle_\beta = \frac{\delta_{y,y'}}{w_\beta(y)}.$$

Biorthogonal self-duality functions for ASEP$(q, 2j)$)

Biorthogonality for triangular self-dualities: $\langle D^{\text{tr}}_{1/\alpha q}(x, \cdot\,), \widehat{D}^{\text{tr}}_{-q/\beta}(\,\cdot\,, y)\rangle_{-\alpha} = \dfrac{\delta_{x,y}}{w_\beta(x)}$

New biorthogonal self-duality functions:

Use theorem with

$$d_1 = D^{\text{tr}}_{1/\alpha q}, \quad \widetilde{d}_1 = \widehat{D}^{\text{tr}}_{q/\alpha}, \quad d_2 = \widehat{D}^{\text{tr}}_{-q/\alpha}, \quad \widetilde{d}_2 = D^{\text{tr}}_{-1/\alpha q},$$

then

$$D_\alpha(x, y) = \prod_{i=1}^{N} {}_2\varphi_1 \left(\begin{matrix} q^{-2x_i}, q^{-2y_i} \\ q^{-4j} \end{matrix} ; q^2, \frac{1}{\alpha} q^{1+2y_i+2N^+_{i+1}(y)-2N^-_{i-1}(x)+2j(2i-1)} \right)$$

$$\widetilde{D}_\alpha(x, y) = c(x)\widetilde{c}(y) D_\alpha(x, y)$$

satisfy

$$\langle D_\alpha(\,\cdot\,, y), \widetilde{D}_\alpha(\,\cdot\,, y')\rangle_\alpha = \frac{\delta_{x,y}}{w_\alpha(y)}.$$

Remark: D_α is a nested product of quantum q-Krawtchouk polynomials

$$K_n(q^{-x}; p, c; q) = {}_2\varphi_1 \left(\begin{matrix} q^{-x}, q^{-n} \\ q^{-c} \end{matrix} ; q; pq^{n+1} \right)$$

Orthogonal self-duality functions

Theorem: The function

$$D_\alpha^{\text{or}}(x,y) = \prod_{i=1}^{N} K_{y_i}(q^{-2x_i}; p_{i,\alpha}(x,y), 2j; q^2)$$

is a self-duality functions for ASEP$(q,2j)$, satisfying

$$\langle D_\alpha^{\text{or}}(\,\cdot\,,y), D_\alpha^{\text{or}}(\,\cdot\,,y')\rangle_{\omega_\alpha} = \frac{\delta_{y,y'}}{\omega_\alpha(y)}$$

where $\omega_\alpha(x) = w_\alpha(x)c(x)$ with $c(x)$ a function of the total number of particles.

From scalar products to symmetries

In matrix notation:

- $D^{\mathrm{or}} = \widehat{D}^{\mathrm{tr}} G^2 (D^{\mathrm{tr}})^T$ with $G = \left(\sqrt{w(x)}\delta_{x,y}\right)$
- $D^{\mathrm{tr}} = G^{-1} S G^{-1}$ with $S = e_{q^2}\left(\sqrt{\alpha}(1-q^2)\Delta^{N-1}(KE)\right)$
- $\widehat{D}^{\mathrm{tr}} = G^{-1} \widehat{S} G^{-1}$ with $\widehat{S} = E_{q^2}\left(\sqrt{\alpha} q^{-j(1+2N)}(1-q^2)\Delta^{N-1}(K^{-1}E)\right)$

Then

$$D^{\mathrm{or}} = G^{-1} \widehat{S} S^* G^{-1},$$

or equivalently

$$D^{\mathrm{or}}(x,y) = [G^{-1} \widehat{S} S^* G \, D^{\mathrm{ch}}(\,\cdot\,, y)](x).$$

Thanks for your attention!

- G.Carinci, C.Franceschini, C.Giardinà, W.Groenevelt, F.Redig, *Orthogonal Dualities of Markov Processes and Unitary Symmetries*, SIGMA 15, 2019, No.053.
- G.Carinci, C.Franceschini, W.Groenevelt, *q-Orthogonal dualities for asymmetric particle systems*, Electron. J. Probab. 26 (2021), No.108.