Addition of matrices at high and low temperatures

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General question

Self-adjoint $N \times N$ matrices A and B: $\begin{cases} \text{known eigenvalues and} \\ \text{unknown eigenvectors.} \end{cases}$

Q1: Eigenvalues of $M \times M$ **corner** A_M?

$$\left(\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ \hline A_{41} & A_{42} & A_{43} & A_{44} \end{array}\right)$$

Q2: Eigenvalues of **sum** $C = A + B$? (today)

Q3: Eigenvalues of **product** $C = AB$?

Parameters in play:

- Dimension N
- $\beta = 1/2/4$ for real/complex/quaternion matrices

Deterministic point of view

Task: characterize all possible eigenvalues.

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Theorem. (Weyl, Horn, Klyachko, Knutson-Tao) $C = A + B$ case: One equality Trace $C =$ Trace $A +$ Trace B and a **set of inequalities**

E.g., $\lambda^A_{\max} + \lambda^B_{\max} \ge \lambda^C_{\max}$ and many more!

Probabilistic point of view

Random self-adjoint matrices invariant under (unitary) conjugations.

Eigenvectors are uniformly distributed conditionally on eigenvalues.
(=no information about eigenvectors)

Q: You are given eigenvalues of independent A and B.
What are eigenvalues of $C = A + B$?

They are **random and high-dimensional** (last slide for support)

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They are **random and high-dimensional** (last slide for support)

Task: Investigate **asymptotic questions**.
Possible regimes: $N \to \infty$ $\beta \to \infty$ $\beta \to 0$

Toy example: Gaussian β ensemble

$N \times N$ matrix X with i.i.d. real/complex/quaternion Gaussian random variables normalized so that their real parts are $\mathcal{N}(0, \frac{2}{\beta})$.

$$M = \frac{X + X^*}{2} = \begin{pmatrix} M_{11} & M_{12} & \dots \\ M_{21} & M_{22} & \\ \vdots & & \ddots \end{pmatrix}$$

$\beta = 1, 2, 4$ is the **dimension** of the base (skew-) field.

Fixed point (up to rescaling) of addition and cutting corners.

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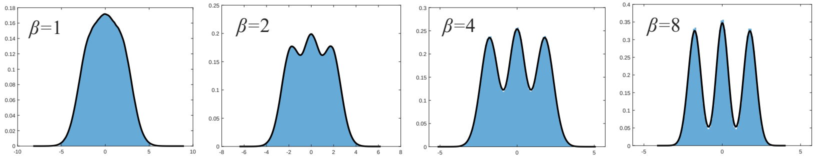
The density of **eigenvalues** $x_1 < x_2 < \cdots < x_N$:

$$\sim \prod_{1 \le i < j \le N} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{\beta}{4}(x_i)^2\right).$$

Meaningful for any **inverse temperature** $\beta > 0$!

GβE as $\beta \to \infty$: concentration

$$\prod_{1 \le i < j \le N} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{\beta}{4}(x_i)^2\right).$$



Theorem. $(x_1, \ldots, x_N)$ converges as $\beta \to \infty$ to N roots $(h_1, \ldots, h_N)$ of **Hermite polynomial $H_N(x)$** (of weight $\exp(-\frac{x^2}{2})$).

$N = 1$: x
$N = 2$: $x^2 - 1$
$N = 3$: $x^3 - 3x$
$N = 4$: $x^4 - 6x^2 + 3$

Equivalent property of **classical orthogonal polynomials** (Hermite, Laguerre, Jacobi) known already to Stieltjes (≈ 1885).

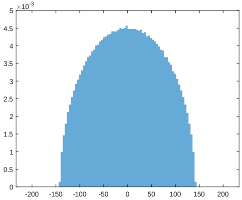
$N \to \infty$: Between Gaussian and semicircle

GβE at inverse temperature β: random $x_1 < \cdots < x_N$ of density

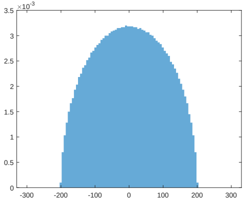
$$\sim \prod_{i<j} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2}\right).$$

Eigenvalues of self-adjoint matrices with i.i.d. Gaussian entries:

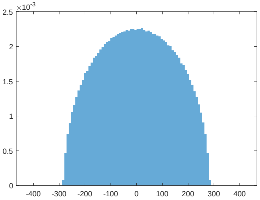
real $\beta = 1$



complex $\beta = 2$



quaternion $\beta = 4$

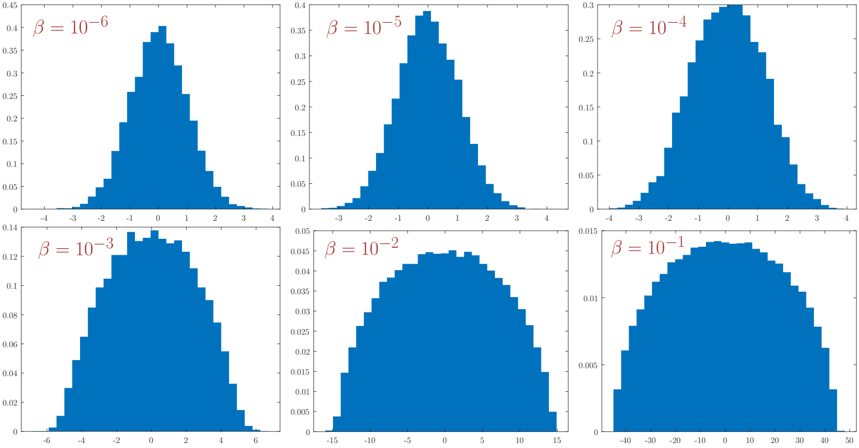


$N = 10000$ histogram of eigenvalues. Support $[-\sqrt{2\beta N}, \sqrt{2\beta N}]$

Wigner semicircle law of density $\displaystyle \frac{1}{2\pi R^2}\sqrt{4R^2 - x^2}$

$N \to \infty$: Between Gaussian and semicircle

$$\prod_{i<j} (x_j - x_i)^{\beta} \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2}\right), \qquad N = 10000$$



Small β leads to Gauss $\to$ semicircle crossover.

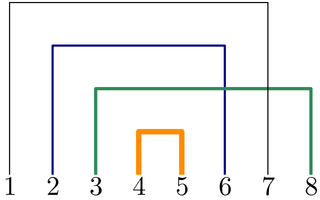
$N \to \infty$: Between Gaussian and semicircle

$$\prod_{i<j}(x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2}\right)$$

Theorem. [Benaych-Georges, Cuenca, G., 2021] Suppose $\beta N \to 2\gamma$

Then $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} = \mu_\gamma$, weakly, in probability.

$$\int_{\mathbb{R}} x^k \mu_\gamma(dx) = \sum_{\text{perfect matchings of } \{1,\dots,k\}} (\gamma+1)^{\text{roof(matching)}}.$$



- Heights $\downarrow$ by position of left leg.
- roof(matching)$= \#\{$roofs **not** intersected by legs$\}$.

$\longleftarrow$ roof(matching)=3

From no weight at $\gamma = 0$ to **non-crossing matchings at $\gamma = \infty$**.

$N \to \infty$: Between Gaussian and semicircle

$$\prod_{i<j}(x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2}\right)$$

Theorem. [Benaych-Georges, Cuenca, G., 2021] Suppose $\beta N \to 2\gamma$

Then $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1} \delta_{x_i} = \mu_\gamma,$ weakly, in probability.

$$\int_{\mathbb{R}} x^k \mu_\gamma(dx) = \sum_{\text{perfect matchings of } \{1,\dots,k\}} (\gamma+1)^{\text{roof(matching)}}.$$

Earlier results: limit with no simple formula for the moments

- [Allez, Bouchaud, Majumdar, Vivo, 2012] Gamma $\to$ Marchenko-Pastur
- [Duy, Shirai, 2015] integrals for density
- [Benaych-Georges, Péché, 2015] moments$= \sum$ of integrals over paths

Earlier results: Moments with no connection to GβE:

- [Drake, 2012] orthogonality of associated Hermite polynomials
- [Bozejko, Dołega, Ejsmont, Gal, 2021] in asymptotic rep. theory

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$$\frac{X + X^*}{2} = \begin{pmatrix} M_{11} & M_{12} & \cdots \\ M_{21} & M_{22} & \\ \vdots & & \ddots \end{pmatrix}$$

$\beta = 1, 2, 4$
dimension of the base field.

Conclusions:

- There is a meaningful extension of eigenvalues to $\beta > 0$

$$\sim \prod_{1 \le i < j \le N} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{\beta}{4}(x_i)^2\right).$$

- There are rich limits as:
  1. $\beta \to \infty$
  2. $N \to \infty$
  3. $\beta N \to 2\gamma$

Addition of matrices as $\beta \to \infty$ (**low temperature**)

Theorem. (G.–Marcus) $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ — eigenvalues of A, B, C. Take **Haar–random** orthogonal/unitary/symplectic U and V. Then

$\lim_{\beta\to\infty}$ $\boxed{C = UAU^* + VBV^*}$ is given by

$$\prod_{i=1}^{N}(z - c_i) = \frac{1}{N!} \sum_{\sigma \in S(N)} \prod_{i=1}^{N}(z - a_i - b_{\sigma(i)})$$

And the identity is true in **expectation** for each $\beta > 0$.

Finite free convolutions of polynomials.

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Finite free convolutions of polynomials.

How exactly do we add matrices at general $\beta > 0$?

Addition of general β random matrices

Theorem. Random $N \times N$ self-adjoint independent matrices A, B. The law of the sum $C = A + B$ is uniquely determined by

$$\boxed{\mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(CZ)\right) = \mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(AZ)\right) \cdot \mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(BZ)\right)},$$

which should be valid for each self-adjoint Z.

Definition. A: deterministic eigenvalues $(a_1, \dots, a_N)$ and uniformly random eigenvectors (invariant under $A \mapsto UAU^*$).
Then law of $\mathrm{Trace}(AZ)$ depends only on eigenvalues $(z_i)_{i=1}^N$ of Z and we define the **multivariate Bessel function** through

$$\boxed{B_{a_1,\dots,a_N}(\mathbf{i}z_1, \dots, \mathbf{i}z_N;\, \beta/2) = \mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(AZ)\right)}$$

Reformulation. For eigenvalues, addition $c = a \boxplus_\beta b$ is fixed by

$$\mathbb{E} B_{c_1,\dots,c_N}(z_1, \dots, z_N;\, \beta/2) = B_{a_1,\dots,a_N}(z_1, \dots, z_N;\, \beta/2) \cdot B_{b_1,\dots,b_N}(z_1, \dots, z_N;\, \beta/2)$$

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Extension to general $\beta > 0$ through eigenfunctions of (symmetric) Dunkl operators

$$D_i := \frac{\partial}{\partial z_i} + \frac{\beta}{2} \sum_{j: j \neq i} \frac{1}{z_i - z_j} \circ (1 - s_{i,j})$$

$$\sum_{i=1}^{N} (D_i)^k B_{a_1,\ldots,a_N}(z_1, \ldots, z_N;\ \tfrac{\beta}{2}) = \sum_{i=1}^{N} (a_i)^k B_{a_1,\ldots,a_N}(z_1, \ldots, z_N;\ \tfrac{\beta}{2})$$

MATRIX lectures: My homepage → Teaching for materials.

Addition of general β random matrices

Definition. Given deterministic eigenvalues $(a_i)_{i=1}^N$ and $(b_i)_{i=1}^N$ we define (random) eigenvalues $(c_i)_{i=1}^N$ of the sum of independent β-matrices with uniformly random eigenvectors through

$$\mathbb{E} B_{c_1,\dots,c_N}(z_1,\dots,z_N;\, \beta/2) = B_{a_1,\dots,a_N}(z_1,\dots,z_N;\, \beta/2) \cdot B_{b_1,\dots,b_N}(z_1,\dots,z_N;\, \beta/2)$$

- $c = a \boxplus_\beta b$ at $\beta = 1, 2, 4$ is the same old addition of matrices.
- At general $\beta > 0$ one needs to show the existence of **probability measure** defining $(c_i)_{i=1}^N$.
- It is well-defined as a generalized function (distribution), but being a measure is a known open problem.

[$\approx$ need positivity of structure constants of multiplication for **Macdonald polynomials**]

Addition of matrices as $\beta N \to 2\gamma$ (**high temperature**)

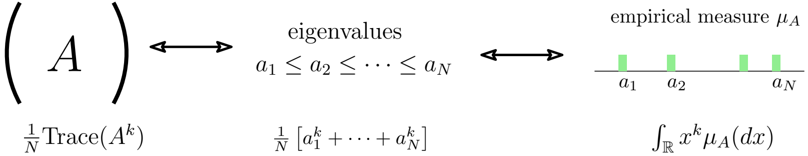
Theorem. [Benaych-Georges, Cuenca, G., 2021] Suppose for all k

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}(a_i)^k = m_k(A), \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}(b_i)^k = m_k(B).$$

Define $c = a \boxplus_\beta b$. Then there exists a limit as $\beta N \to 2\gamma > 0$:

$$\lim_{\substack{N\to\infty \\ \beta N \to 2\gamma}} \frac{1}{N}\sum_{i=1}^{N}(c_i)^k = m_k^\gamma(A+B)$$

γ–**convolution:** $\quad \big(\{m_k(A)\}, \{m_k(B)\}\big) \mapsto \{m_k^\gamma(A+B)\}$



γ–convolution

A binary operation $\big(\{m_k(A)\}, \{m_k(B)\}\big) \mapsto \{m_k^\gamma(A+B)\}$

1. **Conventional convolution** at $\gamma = 0$:
   - Independent random variables ξ and η.
   - $\mathbb{E}\xi^k = m_k$ and $E\eta^k = \tilde{m}_k$.
   - Then $\mathbb{E}(\xi+\eta)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} m_\ell \tilde{m}_{k-\ell}$
2. **Free convolution** at $\gamma = \infty$.
   - Independent $N \times N$ **Hermitian random matrices** A and B.
   - (Unitary) conjugation–invariant laws $A \mapsto UAU^*$, $B \mapsto UBU^*$.
   - Assume $\lim_{N\to\infty} \frac{1}{N}\mathrm{Trace}(A^k) = m_k(A)$ and
     $\lim_{N\to\infty} \frac{1}{N}\mathrm{Trace}(B^k) = m_k(B)$.
   - Then $\lim_{N\to\infty} \frac{1}{N}\mathrm{Trace}(A+B)^k =$ (expression in $m_\ell(A)$, $m_\ell(B)$)

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   - Then $\lim_{N\to\infty} \frac{1}{N}\mathrm{Trace}(A+B)^k =$ (expression in $m_\ell(A)$, $m_\ell(B)$)

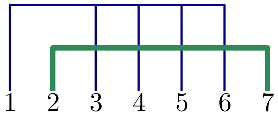
In terms of **γ–cumulants**: $\boxed{\kappa_\ell^{(\gamma)}(A) + \kappa_\ell^{(\gamma)}(B) = \kappa_\ell^{(\gamma)}(A+B)}$

γ-cumulants $\kappa_\ell^{(\gamma)}$ are defined recursively

$$m_k := \sum_{\pi = B_1 \sqcup \cdots \sqcup B_h \in \mathscr{P}(k)} \prod_{i=1}^{h} \left[\kappa_{|B_i|}^{(\gamma)} \cdot (\gamma + p(i) + 1)_{|B_i| - p(i) - 1} \cdot (1)_{p(i)} \right]$$

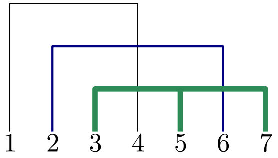
$$(x)_n = x(x+1)\cdots(x+n-1) \qquad \mathscr{P}(k) = \left\{\text{set partitions of } \{1, 2, \ldots, k\}\right\}$$

$$p(i) = \#\{\text{roofs of } B_i \text{ which intersect legs}\}$$



$|B_1| = 5$, $p(1) = 0$, $|B_2| = 2$, $p(2) = 1$

$(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)\kappa_5^{(\gamma)}\kappa_2^{(\gamma)}$



$|B_1| = 2$, $p(1) = 0$, $|B_2| = 2$, $p(2) = 1$

$|B_3| = 3$, $p(3) = 2$

$2(\gamma+1)\kappa_2^{(\gamma)}\kappa_2^{(\gamma)}\kappa_3^{(\gamma)}$

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1. **Conventional cumulants** at $\gamma = 0$.
   The weight does not depend on $p(i)$.
2. **Free cumulants** at $\gamma = \infty$.
   Only **non-crossing** partitions remain.

Another point of view on γ-convolution

Following [Faraut–Fourati-14], [Mergny–Potters-21] we define:

$$\boxed{\text{Probability measure } \nu} \xrightarrow{MK_\gamma} \boxed{\text{Probability measure } \mu}$$

$$\exp\left(-\gamma\int_{\mathbb{R}} \log(z-u)\nu(du)\right) \quad = \quad \int_{\mathbb{R}} \frac{1}{(z-t)^\gamma}\mu(dt)$$

Claim. The diagram is commutative:

$$\begin{array}{ccc}
(\nu_1, \nu_2) & \xrightarrow{MK_\gamma} & (\mu_1, \mu_2) \\
\downarrow & & \downarrow \\
\gamma\text{–convolution} & & \text{Conventional convolution} \\
\downarrow & & \downarrow \\
\nu & \xrightarrow{MK_\gamma} & \mu
\end{array}$$

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- $\gamma = 1$: relative of the **Markov-Krein transform** popularized by Kerov. Bijection of Markov and Hausdorff moment problems.
- The image of MK_γ is **unknown.** [Even at $\gamma = 1$.]

A mysterious duality

Low temperature. In dimension $N = D$, $c = \lim_{\beta\to\infty} a \boxplus_\beta b$ is:

$$\prod_{i=1}^{D}(z - c_i) = \frac{1}{D!} \sum_{\sigma\in S(D)} \prod_{i=1}^{D}(z - a_i - b_{\sigma(i)})$$

High temperature. $\lim_{\substack{N\to\infty \\ \beta N\to 2\gamma}} a \boxplus_\beta b$ is γ–convolution:

$$\kappa_\ell^{(\gamma)}(A) + \kappa_\ell^{(\gamma)}(B) = \kappa_\ell^{(\gamma)}(A + B)$$

- Operations are analytic continuations of each other.
- Formulas coincide under $\boxed{\gamma = -D}$.

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The end.