Nonsymmetric Jack and Macdonald Superpolynomials

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Superpolynomials and the Symmetric Group

Start with $2N$ variables x_i (bosonic, $x_ix_j = x_jx_i$), θ_i (fermionic $\theta_i\theta_j = -\theta_j\theta_i$), $x_i\theta_j = \theta_jx_i$, and the symmetric group $\mathcal{S}_N$ (permutations of $\{1, 2, \ldots, N\}$. The transpositions are denoted by (i, j) and $s_i := (i, i+1)$. The braid relations are $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ and $s_is_j = s_js_i$ for $|i-j| \geq 2$ (and $s_i^2 = 1$).

For $w \in \mathcal{S}_N$ let $wp(x;\theta) = p\left(x_{w(1)}, \ldots, x_{w(N)}; \theta_{w(1)}, \theta_{w(2)}, \ldots, \theta_{w(N)}\right)$.

Example: $s_1\theta_1\theta_2 = -\theta_1\theta_2$; $s_2\theta_1\theta_3 = \theta_1\theta_2$.

Basis elements for polynomials in $\{\theta_i\}$ are defined by

$$\phi_E := \theta_{i_1} \cdots \theta_{i_m},\ E = \{i_1, i_2, \cdots, i_m\}, 1 \leq i_1 < i_2 < \cdots < i_m \leq N.$$

Denote $\mathrm{span}\{\phi_E : \#E = m\}$ by $\mathcal{P}_m$. The span is over some extension field of $\mathbb{Q}$ (typically $\mathbb{Q}(\kappa)$ or $\mathbb{Q}(q, t)$), where κ is a generic parameter, transcendental or real $\neq m/n$ with $2 \leq n \leq N$, $m \in \mathbb{Z}$, $m/n \notin \mathbb{Z}$)

The bosonic variables are spanned by $x^\alpha = \prod_{i=1}^{N} x_i^{\alpha_i}, \alpha \in \mathbb{N}_0^N$. Define

$s\mathcal{P}_m := \mathrm{span}\left\{x^\alpha\phi_E : \alpha \in \mathbb{N}_0^N, \#E = m\right\}$.

Dunkl operators

The *Dunkl* and *Cherednik-Dunkl* operators are $(1 \leq i \leq N, p \in s\mathcal{P}_m)$

$$\mathcal{D}_i p(x;\theta) := \frac{\partial p(x;\theta)}{\partial x_i} + \kappa \sum_{j \neq i} \frac{p(x;\theta(i,j)) - p(x(i,j);\theta(i,j))}{x_i - x_j},$$

$$\mathcal{U}_i p(x;\theta) := \mathcal{D}_i (x_i p(x;\theta)) - \kappa \sum_{j=1}^{i-1} p(x(i,j);\theta(i,j)).$$

The same commutation relations as for the scalar case hold, that is,

$$\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i,\ \mathcal{U}_i \mathcal{U}_j = \mathcal{U}_j \mathcal{U}_i,\ 1 \leq i,j \leq N$$
$$w\mathcal{D}_i = \mathcal{D}_{w(i)} w, \forall w \in \mathcal{S}_N;\ s_j \mathcal{U}_i = \mathcal{U}_i s_j,\ j \neq i-1, i;$$
$$s_i \mathcal{U}_i s_i = \mathcal{U}_{i+1} + \kappa s_i,\ \mathcal{U}_i s_i = s_i \mathcal{U}_{i+1} + \kappa,\ \mathcal{U}_{i+1} s_i = s_i \mathcal{U}_i - \kappa.$$

Simultaneous eigenfunctions of $\{\mathcal{U}_i : 1 \leq i \leq N\}$ are called nonsymmetric Jack polynomials. These are a special case of the generalized Jack polynomials constructed by S. Griffeth for the family $G(n,p,N)$ of complex reflection groups, *TAMS* 2010.

Submodules of $\mathcal{P}_m$

To determine the irreducible $\mathcal{S}_N$-modules define ∂_i by $\partial_i \theta_i \phi_E = \phi_E$ and $\partial_i \phi_E = 0$ for $i \notin E$ (example $\partial_2 \theta_1 \theta_2 \theta_3 = -\theta_1 \theta_3$) and $D := \sum_{i=1}^{N} \partial_i$, then $D^2 = 0$. Also define

$$Mp(\theta) = \sum_{i=1}^{N} \theta_i p(\theta)$$

(example: $N = 4$, $M(\theta_1 \theta_3) = -\theta_1 \theta_2 \theta_3 + \theta_1 \theta_2 \theta_4$). Then $M^2 = 0$, M and D commute with the group action and $MD + DM = N$. Denote

$$\begin{aligned} \mathcal{P}_{m,0} &= \ker D \cap \mathcal{P}_m, s\mathcal{P}_{m,0} = \ker D \cap s\mathcal{P}_m \\ \mathcal{P}_{m,1} &= \ker M \cap \mathcal{P}_m, s\mathcal{P}_{m,1} = \ker D \cap s\mathcal{P}_m. \end{aligned}$$

The relations for M, D imply $s\mathcal{P}_m = s\mathcal{P}_{m,0} \oplus s\mathcal{P}_{m,1}$.(See C. D. *Ramanujan* J. 2021)

Young Tableaux

Representations of $\mathcal{S}_N$ are indexed by partitions of N. Given a partition λ ($\lambda \in \mathbb{N}_0^N, \lambda_1 \geq \lambda_2 \geq \ldots$ and $|\lambda| = N$) there is a Ferrers diagram : boxes at (i,j) with $1 \leq i \leq \ell(\lambda) = \max\{j : \lambda_j > 0\}$ and $1 \leq j \leq \lambda_i$. The module is spanned by reverse standard Young tableaux (abbr. RSYT) - the numbers $1, \ldots, N$ are inserted into the Ferrers diagram so that the entries in each row and in each column are decreasing. If $i, i+1$ are in the same row, resp. column of RSYT Y then $s_i Y = Y$, resp. $-Y$.
If k is in cell (i,j) of RSYT Y then the *content* $c(k,Y) = j - i$; the *content vector* $[c(k,Y)]_{k=1}^N$ determines Y uniquely. The *Jucys-Murphy* elements $\omega_i = \sum_{j=i+1}^N (i,j)$ satisfy $\omega_i Y = c(i,Y)\, Y$ for each i. So if one finds a simultaneous eigenfunction of $\{\omega_i\}$ then the eigenvalues determine the label (partition) of an irreducible. representation.

Isotypes of the submodules

Illustration: let
$N = 5, m = 2, E = \{3, 4, 5\}, \tau_E := D(\theta_3\theta_4\theta_5) = \theta_4\theta_5 - \theta_3\theta_5 + \theta_3\theta_4$
then τ_E is an eigenfunction of each ω_i with eigenvalues $[2, 1, -2, -1, 0]$ $(i = 1, 2, \ldots, 5)$ - this is the content vector of

$$\begin{bmatrix} 5 & 2 & 1 \\ 4 & & \\ 3 & & \end{bmatrix}$$

and indeed the rep. is $(3, 1, 1)$ (degree 6). (called a *hook tableau*).
In general let $\tau_{E_0} := D\phi_{E_0}$ where $E_0 = \{N - m, N - m + 1, \ldots, N\}$, the eigenvalues $[N - m - 1, \cdots, 1, -m, 1 - m, \cdots, -1, 0]$ correspond to the content vector of

$$\begin{bmatrix} N & N-m-1 & \cdots & \cdots & \cdots & 1 \\ \diagdown & N-1 & \cdots & \cdots & N-m & \end{bmatrix}$$

(displayed with column 1 folded under row 1 to save space). Thus the isotype of $\mathcal{P}_{m,0}$ is $(N - m, 1^m)$. The analysis of $\mathcal{P}_{m,1}$ is very similar and is omitted in this talk; the isotype is $(N - m + 1, 1^{m-1})$.

Construction of basis

For a subset E define $\mathrm{inv}E = \#\left\{(i,j) \in E \times E^C : i < j\right\}$ so $\mathrm{inv}E_0 = 0$ and $\mathrm{inv}E_1 = m(N-m-1)$ where $E_1 := \{1,2,\ldots,m,N\}$ (maximum). Let Y_E denote the RSYT with the entries of column 1 consisting of E, and $c(i,E) := c(i,Y_E)$. Let $\mathcal{E}_0 := \{E : \#E = m+1, N \in E\}$; for each $E \in \mathcal{E}_0$ there is a polynomial $\tau_E \in \mathcal{P}_{m,0}$ such that $\omega_i \tau_E = c(i,E)\,\tau_E$ and

$$\tau_E = D\phi_E + \sum_{\mathrm{inv}F < \mathrm{inv}E} a_{F,E} D\phi_F$$

constructed by induction on $\mathrm{inv}E$. Suppose $(i,i+1) \in E^C \times (E\backslash\{N\})$ then $\mathrm{inv}(s_i E) = \mathrm{inv}E + 1$ and

$$\tau_{s_i E} = s_i \tau_E + \frac{1}{c(i+1,E) - c(i,E)} \tau_E .$$

If $\{i,i+1\} \subset E$ then $s_i \tau_E = -\tau_E$ and if $\{i,i+1\} \subset E^C \cup \{N\}$ then $s_i \tau_E = \tau_E$.

Example

$N = 5, m = 2$

$$\begin{aligned}
\tau_{\{3,4,5\}} &= \theta_3\theta_4 - \theta_3\theta_5 + \theta_4\theta_5 \\
\tau_{\{2,4,5\}} &= \theta_2\theta_4 - \theta_2\theta_5 - \frac{1}{3}\theta_3\theta_4 + \frac{1}{3}\theta_3\theta_5 + \frac{2}{3}\theta_4\theta_5 \\
\tau_{\{1,4,5\}} &= \frac{1}{4}\left(4\theta_1 - \theta_2 - \theta_3\right)\left(\theta_4 - \theta_5\right) + \frac{1}{2}\theta_4\theta_5 \\
\tau_{\{1,2,5\}} &= \theta_1\theta_2 - \frac{1}{3}\left(\theta_1 - \theta_2\right)\left(\theta_3 + \theta_4 + \theta_5\right)
\end{aligned}$$

Observe the actions: $s_1\tau_{\{3,4,5\}} = \tau_{\{3,4,5\}}$, $s_2\tau_{\{1,4,5\}} = \tau_{\{1,4,5\}}$ and $s_4\tau_{\{1,4,5\}} = -\tau_{\{1,4,5\}}$. Note dim $(3,1,1) = \binom{4}{2}$ and $\sum_{E\in\mathcal{E}_0} q^{\mathrm{inv}E} = \left[{4 \atop 2}\right]_q = 1 + q + 2q^2 + q^3 + q^4$ (in general this sum equals $\left[{N-1 \atop m}\right]_q$).

Eigenfunctions

For each $\alpha \in \mathbb{N}_0^N, E \in \mathcal{E}_0$ there is a $\{\mathcal{U}_i\}$-simultaneous eigenfunction $J_{\alpha,E} \in s\mathcal{P}_{m,0}$ .The expression uses a partial order on $\mathbb{N}_0^N$

$$\alpha \prec \beta \iff \sum_{j=1}^{i} \alpha_j \leq \sum_{j=1}^{i} \beta_j,\ 1 \leq i \leq N,\ \alpha \neq \beta,$$

$$\alpha \lhd \beta \iff (|\alpha| = |\beta|) \wedge \left[\left(\alpha^+ \prec \beta^+\right) \vee \left(\alpha^+ = \beta^+ \wedge \alpha \prec \beta\right)\right].$$

and the *rank* function

$$r_\alpha\left(i\right) := \#\left\{1 \leq j \leq i : \alpha_j \geq \alpha_i\right\} + \#\left\{j > i : \alpha_j > \alpha_i\right\};$$

thus $r_\alpha \in \mathcal{S}_N$, and $r_\alpha = I$ if and only if $\alpha \in \mathbb{N}_0^{N,+}$ $(\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N)$.
Example: $\alpha = (4,5,0,4)$, $r_\alpha = [2,1,4,3]$, and $r_\alpha\alpha = (5,4,4,0) = \alpha^+$
Then

$$J_{\alpha,E}\left(x;\theta\right) = x^\alpha\left(r_\alpha^{-1}\tau_E\right) + \sum_{\alpha \rhd \beta} x^\beta v_{\alpha,\beta,T}\left(\kappa;\theta\right),$$

where $v_{\alpha,\beta,T}\left(\kappa;\theta\right) \in \mathcal{P}_{m,0}$. The coefficients of the polynomials $v_{\alpha,\beta,T}\left(\kappa;\theta\right)$ are rational functions of κ. Note $r_\alpha^{-1}\tau_E\left(\theta\right) = \tau_E\left(\theta r_\alpha^{-1}\right)$.

Spectral vector

- These polynomials satisfy

$$\mathcal{U}_i J_{\alpha,E} = \zeta_{\alpha,E}(i) J_{\alpha,E},$$
$$\zeta_{\alpha,E}(i) := \alpha_i + 1 + \kappa c(r_\alpha(i), E), \ 1 \leq i \leq N.$$

Then $\left[\zeta_{\alpha,E}(i)\right]_{i=1}^{N}$ is called the spectral vector of α, E (note α is clearly determined by $\zeta_{\alpha,E}$ and then r_α recovers the content vector of E).

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- *Example* $N = 4, m = 2, \ \alpha = (0,1,1,0), \ E = \{2,3,4\} \in \mathcal{E}_0$, $[c(j,E)]_{j=1}^{4} = [1,-2,-1,0]$ (thus $r_\alpha = [3,1,2,4]$, $\zeta_{\alpha,E} = [1-\kappa, 2+\kappa, 2-2\kappa, 1]$)

$$J_{\alpha,E} = \left(x_2x_3 - \frac{\kappa x_2 x_4}{1-2\kappa}\right)(-\theta_1\theta_3 + \theta_1\theta_4 - \theta_3\theta_4)$$
$$+\frac{\kappa x_3 x_4}{(1-2\kappa)(1+\kappa)}\left\{\begin{array}{c}(1-\kappa)\theta_1\theta_2 - (1-2\kappa)(\theta_1\theta_3 - \theta_2\theta_3)\\ -\kappa(\theta_1\theta_4 - \theta_2\theta_4)\end{array}\right\}.$$

Inner Product and Norms

Require the properties: $\langle f, g\rangle = \langle s_i f, s_i g\rangle = \langle g, f\rangle$; $\langle x_i f, g\rangle = \langle f, \mathcal{D}_i g\rangle$, if $\deg f \neq \deg g$ then $\langle f, g\rangle = 0$. Write $\|f\|^2 = \langle f, f\rangle$ without claiming positivity.
At the lowest x-degree, declare the ϕ_E to be an orthonormal set so that $\langle \tau_{E0}, \tau_{E_0}\rangle = m+1$, and for $E \in \mathcal{E}_0$

$$\|\tau_E\|^2 = (m+1) \prod_{\substack{1\leq i<j<N \\ (i,j)\in E\times E^C}} \left(1 - \frac{1}{\left(c(i,E) - c(j,E)\right)^2}\right)$$

(follows from $\langle \tau_E, \tau_{s_i E}\rangle = 0$, the $\{\omega_i\}$-eigenfunction property). Note if $(i,j) \in E \times E^C$ and $i<j$ then $c(i,E) - c(j,E) \leq -2$.

Norms II

From the adjacency relation for $\alpha_i < \alpha_{i+1}$

$$J_{s_i\alpha,E} = \left(s_i - \frac{\kappa}{\zeta_{\alpha,E}(i) - \zeta_{\alpha,E}(i+1)} \right) J_{\alpha,E}$$

and the orthogonality $\langle J_{\alpha,E}, J_{s_i\alpha,E} \rangle = 0$ we relate $\| J_{\alpha,E} \|^2$ to $\| J_{\alpha^+,E} \|^2$.
For $\alpha \in \mathbb{N}_0^N, z = 0, 1$ let

$$\mathcal{R}_z(\alpha, E) = \prod_{\substack{1 \le i < j \le N \\ \alpha_i < \alpha_j}} \left(1 + \frac{(-1)^z \kappa}{\alpha_j - \alpha_i + \kappa \left(c\left(r_\alpha(j), E\right) - c\left(r_\alpha(i), E\right) \right)} \right)$$

$\mathcal{R}$ is for "rearrangement" , let $\mathcal{R}(\alpha, E) = \mathcal{R}_0(\alpha, E)\, \mathcal{R}_1(\alpha, E)$ then

$$\| J_{\alpha,E} \|^2 = \mathcal{R}(\alpha, E)^{-1} \| J_{\alpha^+,E} \|^2 .$$

Norms III

- To change degrees we use the affine step $J_{\Phi\alpha,E} = x_N w_N^{-1} J_{\alpha,E}$ where $\Phi\alpha := (\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1 + 1)$ and $w_N = s_1 s_2 \cdots s_{N-1}$ (a cyclic shift). For $\lambda \in \mathbb{N}_0^{N,+}$, $E \in \mathcal{E}_0$ let

$$\begin{aligned}\mathcal{P}(\lambda, E) &= \prod_{i=1}^{N} (1 + \kappa c(i, E))_{\lambda_i} \\ &\times \prod_{1 \le i < j \le N} \prod_{\ell=1}^{\lambda_i - \lambda_j} \left(1 - \left(\frac{\kappa}{\ell + \kappa (c(i, E) - c(j, E))}\right)^2\right).\end{aligned}$$

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- Then

$$\|J_{\lambda,E}\|^2 = \|\tau_E\|^2 \mathcal{P}(\lambda, E)$$

The inner product is positive-definite for $-1/N < \kappa < 1/N$. The norm formula is a special case of Griffeth's results.

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- *Example*: $N = 4, m = 2, \ \alpha = (0, 1, 1, 0), \ E = \{2, 3, 4\}$

$$\|J_{\alpha,E}\|^2 = \frac{3(1 - 3\kappa)(1 + 2\kappa)(1 - \kappa)}{(1 + \kappa)(1 - 2\kappa)}.$$

The Hecke algebra

We keep the braid relations of $\{s_i\}$ but change the quadratic relation: introduce a parameter t (such that $t^n \neq 1$ for $2 \leq n \leq N$) then $\mathcal{H}_N(t)$ is the unital associative algebra generated by $T_1, \ldots, T_{N-1}$ satisfying $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $T_i T_j = T_j T_i$ for $|i-j| \geq 2$, and $(T_i - t)(T_i + 1) = 0$. Use an extension field $\mathbb{K}$ of $\mathbb{Q}(q,t)$. There is a linear (not multiplicative!) isomorphism between the group algebra $\mathbb{K}\mathcal{S}_N$ and $\mathcal{H}_N(t)$. Given $u \in \mathcal{S}_N$ there is a shortest expression $u = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ where $\ell = \#\{(i,j) : i < j, u(i) > u(j)\}$ and set $T(u) = T_{i_1} T_{i_2} \cdots T_{i_\ell}$ (well-defined because of the braid relations). The representation theory is very similar to that of $\mathcal{S}_N$ (partitions, RSYT etc.). First define an action of $\mathcal{H}_N(t)$ on $\mathcal{P}_m$ - the example suffices:

$$\begin{aligned} T_1 1 &= t, \ T_1\theta_1 = \theta_2, \\ T_1\theta_2 &= t\theta_1 + (t-1)\theta_2, \ T_1\theta_1\theta_2 = -\theta_1\theta_2, \end{aligned}$$

so that $T_1(\theta_1 + \theta_2) = t(\theta_1 + \theta_2)$. Example: $T_2\theta_1\theta_2\theta_4 = \theta_1\theta_3\theta_4$ and $T_2\theta_1\theta_3\theta_4 = t\theta_1\theta_2\theta_4 + (t-1)\theta_1\theta_3\theta_4$.

Submodules

The operators M, D are defined in this setting: M is the same while $D := \sum_{i=1}^{N} t^{i-1}\partial_i$. Then M, D commute with each T_i, $M^2 = 0$, $D^2 = 0$ and

$$MD + DM = [N]_t := \frac{1 - t^N}{1 - t}.$$

With the same notations $\mathcal{P}_{m,0} = \ker D \cap \mathcal{P}_m$, and $\mathcal{P}_{m,1} = \ker M \cap \mathcal{P}_m$, both are irreducible $\mathcal{H}_N(t)$-modules.

The analogous *Jucys-Murphy elements* are

$$\omega_i = t^{i-N} T_i T_{i+1} \cdots T_{N-1} T_{N-1} T_{N-2} \cdots T_i$$

for $i < N$ while $\omega_N = 1$. The Hecke algebra is represented on the span of RSYT's Y of a given shape and $\omega_i Y = t^{c(i,Y)} Y$. If $i, i+1$ are in the same row, resp. column of Y then $T_i Y = tY$ resp. $-Y$. (As before the eigenvalues of an $\{\omega_i\}$-eigenvector determine a partition of N.)

Start of the Basis Construction

- *Illustration*: let $N = 5, m = 2, E = \{3, 4, 5\}, \tau_E = D(\theta_3\theta_4\theta_5) = t^2\theta_4\theta_5 - t^3\theta_3\theta_5 + t^4\theta_3\theta_4$ then $\omega_i\tau_E = \left[t^2, t, t^{-2}, t^{-1}, 1\right]_i \tau_E$ $(i = 1, 2, \ldots, 5)$ - this is the t-exponential content vector of

$$\begin{bmatrix} 5 & 2 & 1 \\ \diagdown & 4 & 3 \end{bmatrix}.$$

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- In general let $\tau_{E_0} := D\phi_{E_0}$ where $E_0 = \{N - m, N - m + 1, \ldots, N\}$, the $\{\omega_i\}$-eigenvalues correspond to the t-exponential content vector of

$$\begin{bmatrix} N & N-m-1 & \cdots & \cdots & \cdots & 1 \\ \diagdown & N-1 & \cdots & \cdots & N-m & \end{bmatrix},$$

of isotype $(N - m, 1^m)$, degree $\binom{N-1}{m}$. Acting on τ_{E_0} with $\{T_i\}$ generates the submodule $\mathcal{P}_{m,0}$. (Details in C.D. *SIGMA* 21-054)

Basis

Let Y_E denote the RSYT with the entries of column 1 consisting of E, and define $c(i,E) := c(i,Y_E)$.

- Example: let $N = 8, m = 3, E = \{2,5,7,8\}$ then

$$Y_E = \begin{bmatrix} 8 & 6 & 4 & 3 & 1 \\ \diagdown & 7 & 5 & 2 & \end{bmatrix}$$

and $[c(i,E)]_{i=1}^{8} = [4,-3,3,2,-2,1,-1,0]$.

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- Let $\mathcal{E}_0 := \{E : \#E = m+1, N \in E\}$; for each $E \in \mathcal{E}_0$ there is a polynomial $\tau_E \in \mathcal{P}_{m,0}$ such that $\omega_i \tau_E = t^{c(i,E)} \tau_E$ $\forall i$ and

$$\tau_E = D\phi_E + \sum_{\mathrm{inv}F < \mathrm{inv}E} a_{F,E} D\phi_F$$

  constructed by induction on $\mathrm{inv}E$, starting with E_0. Suppose $(i, i+1) \in E^C \times (E \backslash \{N\})$ then $\mathrm{inv}(s_i E) = \mathrm{inv}E + 1$ and

$$\tau_{s_i E} = \left(T_i + \frac{(t-1)\, t^{c(i,E)}}{t^{c(i+1,E)} - t^{c(i,E)}} \right) \tau_E .$$

Tools and example

- Utility functions:
$$u_0(z) := \frac{t-z}{1-z},\; u_1(z) := \frac{1-tz}{1-z},\; u(z) := u_0(z)\, u_1(z)$$

Tools and example

- Utility functions:
$u_0(z) := \frac{t-z}{1-z},\ u_1(z) := \frac{1-tz}{1-z},\ u(z) := u_0(z)\,u_1(z)$
- example $N=5, m=2$ (recall $[n]_t = (1-t^n)/(1-t)$)

$$\tau_{\{3,4,5\}} = t^4\theta_3\theta_4 - t^3\theta_3\theta_5 + t^2\theta_4\theta_5$$

$$\tau_{\{2,4,5\}} = t^5\theta_2\theta_4 - t^4\theta_2\theta_5 + \frac{t^3}{[3]_t}\left(\theta_3\theta_5 - t\theta_3\theta_4 + [2]_t\,\theta_4\theta_5\right)$$

$$\tau_{\{1,4,5\}} = t^4\left(t\theta_1 - \frac{1}{[4]_t}(\theta_2+\theta_3)\right)(t\theta_4-\theta_5) + \frac{t^4\,[2]_t}{[4]_t}\theta_4\theta_5$$

$$\tau_{\{1,2,5\}} = t^7\theta_1\theta_2 - \frac{t^6}{[3]_t}(t\theta_1-\theta_2)(\theta_3+\theta_4+\theta_5).$$

(omitted $\tau_{\{2,3,5\}}, \tau_{\{1,3,5\}}$)

Inner Product

The motivation for the definition is to make T_i (and thus ω_i) into a self-adjoint operator so that the $\{\tau_E\}$ will be mutually orthogonal (tacit assumption: $t > 0$)

For $E, F \subset \{1, 2, \ldots, N\}$ define $\langle \phi_E, \phi_F \rangle = \delta_{E,F} t^{-\mathrm{inv}(E)}$ and extend the form to $\mathcal{P}$ by linearity. This satisfies $\langle T_i\phi_E, \phi_F \rangle = \langle \phi_E, T_i\phi_F \rangle$ for each i.

For a set F and $k = 0, 1$ let

$$\mathcal{C}_k(F) = \prod_{1\leq i<j<N, c(i,F)<0<c(j,F)} u_k\left(t^{c(i,F)-c(j,F)}\right)$$

Suppose $E \in \mathcal{E}_0$ then

$$\|\tau_E\|^2 = t^{2(N-m-1)} [m+1]_t \; \mathcal{C}_0(E)\,\mathcal{C}_1(E).$$

Action on Superpolynomials

Suppose $p \in s\mathcal{P}_m$ and $1 \leq i < N$ then set

$$\mathbf{T}_i p(x;\theta) := (1-t)\, x_{i+1} \frac{p(x;\theta) - p(xs_i;\theta)}{x_i - x_{i+1}} + T_i p(xs_i;\theta).$$

Note that T_i acts on the θ variables (according to the previous definition). Let $T^{(N)} := T_{N-1} T_{N-2} \cdots T_1$ (like a shift). Introduce another parameter q, then for $p \in s\mathcal{P}_m$ and $1 \leq i \leq N$ define

$$\mathbf{w} p(x;\theta) := T^{(N)} p(qx_N, x_1, x_2, \ldots, x_{N-1};\theta),$$
$$\xi_i p(x;\theta) := t^{i-N} \mathbf{T}_i \mathbf{T}_{i+1} \cdots \mathbf{T}_{N-1} \mathbf{w} \mathbf{T}_1^{-1} \mathbf{T}_2^{-1} \cdots \mathbf{T}_{i-1}^{-1} p(x;\theta).$$

The ξ_i are Cherednik (*IMRN* 1995) operators, (also Baker and Forrester, *IMRN* 1997). The ξ_i mutually commute. There is a basis of $s\mathcal{P}_m$ consisting of simultaneous eigenvectors of $\{\xi_i\}$ and these are the nonsymmetric Macdonald superpolynomials (henceforth abbreviated to "NSMP"). The $\mathcal{H}_N(t)$-module version is due to C.D. and Luque *SLC* 2012.

Macdonald Superpolynomials

- Suppose $p(\theta)$ is independent of x then $\mathbf{T}_i p = T_i p$ and

$$\begin{aligned}\xi_i p(\theta) &= t^{i-N} T_i T_{i+1} \cdots T_{N-1} \left(T_{N-1} \cdots T_2 T_1\right) T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1} p(\theta) \\ &= t^{i-N} T_i T_{i+1} \cdots T_{N-1} T_{N-1} \cdots T_i p(\theta) = \omega_i p(\theta),\end{aligned}$$

that is ξ_i agrees with ω_i on polynomials of x-degree 0. For $\alpha \in \mathbb{N}_0^N$ the rank is used in $R_\alpha := T(r_\alpha)^{-1}$ (if $r_\alpha = s_{i_1} s_{i_2} \ldots s_{i_k}$ then $R_\alpha = \left(T_{i_1} T_{i_2} \cdots T_{i_k}\right)^{-1}$)

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- Suppose $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{Y}_0$, then there exists a (ξ_i)-simultaneous eigenfunction NSMP

$$M_{\alpha,E}(x;\theta) = t^{e(\alpha^+)} q^{b(\alpha)} x^\alpha R_\alpha(\tau_E(\theta)) + \sum_{\beta \lhd \alpha} x^\beta v_{\alpha,\beta,E}(\theta; q, t)$$

where $v_{\alpha,\beta,E}(\theta; q, t) \in \mathcal{P}_{m,0}$ and whose coefficients are rational functions of q, t. Also $\xi_i M_{\alpha,E}(x;\theta) = \zeta_{\alpha,E}(i) M_{\alpha,E}(x;\theta)$ where $\zeta_{\alpha,E}(i) = q^{\alpha_i} t^{c(r_\alpha(i),E)}$ for $1 \le i \le N$. The exponents $b(\alpha) := \sum_{i=1}^N \binom{\alpha_i}{2}$ and $e(\alpha^+) := \sum_{i=1}^N \alpha_i^+ (N - i + c(i,E))$.

Yang-Baxter Graph Method

The nodes of the graph are labeled by (α, E) and directed edges join adjacent labels (idea of Lascoux).

- if $\alpha = (0, 0, \ldots, 0)$ then $M_{\alpha,E} = \tau_E$

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- if $\alpha_i = \alpha_{i+1}$ and $j = r_\alpha(i), (j, j+1) \in E^C \times E \backslash \{N\}$ then let $z = \zeta_{\alpha,E}(i+1)/\zeta_{\alpha,E}(i) = t^{c(j+1,E)-c(j,E)}$ and

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- If $\alpha_i = \alpha_{i+1}$ and $j = r_\alpha(i)$ then (1) $\{j, j+1\} \subset E$ implies $\mathbf{T}_i M_{\alpha.E} = -M_{\alpha,E}$ (2) $\{j, j+1\} \subset E^C \cup \{N\}$ implies $\mathbf{T}_i M_{\alpha.E} = t M_{\alpha,E}$

Affine step and an Example

- For any α let $\Phi\alpha = (\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1 + 1)$ then $M_{\Phi\alpha,E} = x_N \mathbf{w} M_{\alpha,E}$. The transformed spectral vector is $\zeta_{\Phi\alpha,E} = \left[\zeta_{\alpha,E}(2), \zeta_{\alpha,E}(3), \ldots, \zeta_{\alpha,E}(N), q\zeta_{\alpha,E}(1)\right]$. The proofs use commutation rules such as $\mathbf{w}\mathbf{T}_{i+1} = \mathbf{T}_i\mathbf{w}$, $\xi_N x_N \mathbf{w} = q x_N \mathbf{w} \xi_1$ and $\xi_i x_N \mathbf{w} = x_N \mathbf{w} \xi_{i+1}$ for $1 \leq i < N$.

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- *Example*: Let $N = 5$, $m = 2$, $E = \{3, 4, 5\}$ and $\alpha = (0, 0, 1, 0, 0)$ (thus $r_\alpha = [2, 3, 1, 4, 5]$)

$$M_{\alpha,E} = t^6 x_3 \left(t^3\theta_2\theta_4 - t^2\theta_2\theta_5 + \theta_4\theta_5\right) +$$

$$\frac{(t-1)\, t^9 q}{qt^3 - 1} \left\{x_4 \left(t^3\theta_2\theta_3 - t\theta_2\theta_5 + \theta_3\theta_5\right) - x_5 \left(t^2\theta_2\theta_3 - t\theta_2\theta_4 + \theta_3\theta_4\right)\right\}$$

The spectral vector is $\left[t, t^{-2}, qt^2, t^{-1}, 1\right]$ and $\mathbf{T}_4 M_{\alpha,E} = -M_{\alpha,E}$. Observe a typical pole at $q = t^{-3}$.

Inner product and D operators

We would like an analog of the Jack-type inner product in which the Jack polynomials are mutually orthogonal and which satisfies a degree-changing relation $\langle x_i f, g\rangle = \langle f, D_i g\rangle$. Baker and Forrester defined an analog of D_i: Suppose $f \in s\mathcal{P}_m$ then

$$\mathbf{D}_N f := \frac{1}{x_N}\left(f - \xi_N f\right), \ \mathbf{D}_i f := \frac{1}{t}\mathbf{T}_i \mathbf{D}_{i+1} \mathbf{T}_i f, \ i < N.$$

These operators map polynomials to those of lower x-degree: Suppose $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{E}_0$;

- if $\alpha_N = 0$ then $r_\alpha(N) = N, c(N, E) = 0$, $\xi_N M_{\alpha,E} = M_{\alpha,E}$ and $\left(1 - \xi_N\right) M_{\alpha,E} = 0$ so that $\mathbf{D}_N M_{\alpha,E} = 0$;

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- if $\alpha_N \geq 1$ then $\alpha = \Phi\beta$ with $|\beta| = |\alpha| - 1$ and $(1 - \xi_N) M_{\alpha,E} = \left(1 - \zeta_{\alpha,E}(N)\right) M_{\alpha,E}$ $= \left(1 - \zeta_{\alpha,E}(N)\right) x_N \mathbf{w} M_{\beta,E}$ thus $\mathbf{D}_N M_{\alpha,E} = \left(1 - \zeta_{\alpha,E}(N)\right) \mathbf{w} M_{\beta,E}$

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- The operators $\{\mathbf{D}_i\}$ mutually commute.

Axioms for the Inner Product

(1) $\langle \mathbf{T}_i f, g \rangle = \langle f, \mathbf{T}_i g \rangle, 1 \leq i < N$ (2) $\langle \xi_N f, g \rangle = \langle f, \xi_N g \rangle$
then $\xi_i = t^{-1} \mathbf{T}_i \xi_{i+1} \mathbf{T}_i$ implies $\langle \xi_i f, g \rangle = \langle f, \xi_i g \rangle$ for all i, implying the orthogonality of $\{M_{\alpha,E}\}$. (recall $u(z) = (t-z)(1-tz)/(1-z)^2$)

- Suppose $\alpha_i < \alpha_{i+1}$ then these axioms imply $\langle M_{\alpha,E}, M_{s_i\alpha,E} \rangle = 0$ and

$$\|M_{s_i\alpha,E}\|^2 = u\left(q^{\alpha_{i+1}-\alpha_i} t^{c(r_\alpha(i+1),E) - c(r_\alpha(i),E)}\right) \|M_{\alpha,E}\|^2 .$$

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- For $k = 0, 1$ let
$$\mathcal{R}_k(\alpha, E) := \prod\left\{u_k\left(q^{\alpha_j-\alpha_i} t^{c(r_\alpha(j),E)-c(r_\alpha(i),E)}\right) : i < j, \alpha_i < \alpha_j\right\}.$$
and $\mathcal{R}(\alpha, E) := \mathcal{R}_0(\alpha, E)\,\mathcal{R}_1(\alpha, E)$ then
$$\|M_{\alpha^+,E}\|^2 = \mathcal{R}(\alpha, E)\,\|M_{\alpha,E}\|^2.$$

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- Suppose $\alpha_i < \alpha_{i+1}$ then these axioms imply $\langle M_{\alpha,E}, M_{s_i\alpha,E}\rangle = 0$ and
$$\|M_{s_i\alpha,E}\|^2 = u\left(q^{\alpha_{i+1}-\alpha_i} t^{c(r_\alpha(i+1),E)-c(r_\alpha(i),E)}\right)\|M_{\alpha,E}\|^2 .$$
- For $k = 0, 1$ let
$$\mathcal{R}_k(\alpha, E) := \prod\left\{u_k\left(q^{\alpha_j-\alpha_i} t^{c(r_\alpha(j),E)-c(r_\alpha(i),E)}\right) : i < j, \alpha_i < \alpha_j\right\}.$$
and $\mathcal{R}(\alpha, E) := \mathcal{R}_0(\alpha, E)\mathcal{R}_1(\alpha, E)$ then
$$\|M_{\alpha^+,E}\|^2 = \mathcal{R}(\alpha, E)\|M_{\alpha,E}\|^2 .$$
- Axiom (3) is $\langle \mathbf{w}^{-1}\mathbf{D}_N f, g\rangle = (1-q)\langle f, x_N \mathbf{w} g\rangle$ (the Jack property does not work); the reason for the factor $(1-q)$ is to allow the limit $t \to 1$ when $q = t^{1/\kappa}$.The idea is to derive a formula using the axioms and then prove it works , C.D. *SLC* 2019.

Degree raising

Suppose $E \in \mathcal{E}_0, \alpha \in \mathbb{N}_0^N$ then

$$\left\| M_{\Phi\alpha,E} \right\|^2 = \frac{1 - q^{\alpha_1+1} t^{c(r_\alpha(1),E)}}{1-q} \left\| M_{\alpha,E} \right\|^2 .$$

Proof: set $g = M_{\alpha,E}$ and $f = M_{\Phi\alpha,E}$ then
$(1-q) \langle f, x_N \mathbf{w} g \rangle = (1-q) \left\| M_{\Phi\alpha,E} \right\|^2$, also

$$\begin{aligned} \mathbf{D}_N f &= \frac{1}{x_N} (1 - \xi_N) f = \frac{1}{x_N} \left(1 - \zeta_{\Phi\alpha,E}(N)\right) M_{\Phi\alpha,E} \\ &= \left(1 - \zeta_{\Phi\alpha,E}(N)\right) \mathbf{w} M_{\alpha,E}, \\ \left\langle \mathbf{w}^{-1} \mathbf{D}_N f, g \right\rangle &= \left(1 - \zeta_{\Phi\alpha,E}(N)\right) \langle M_{\alpha,E}, M_{\alpha,E} \rangle , \end{aligned}$$

thus $\left\| M_{\Phi\alpha,E} \right\|^2 = \dfrac{1 - \zeta_{\Phi\alpha,E}(N)}{1-q} \left\| M_{\alpha,E} \right\|^2$ and
$\zeta_{\Phi\alpha,E}(N) = q \zeta_{\alpha,E}(1) = q^{\alpha_1+1} t^{c(r_\alpha(1),E)}$.

Using edges of YB-graph for norm computation

Then we derive a hypothetical formula for $\|M_{\lambda,E}\|^2$ in terms of a lower degree value. Suppose $\lambda \in \mathbb{N}_0^{N,+}$, with $\lambda_k \geq 1$ and $\lambda_j = 0$ for $k < j \leq N$ then use the above formulas to express the norms, so compute the squared norm in terms of the previous value at each stage of
$(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k - 1, 0.., 0) \xrightarrow{T_*} (\lambda_k - 1, \lambda_1, \ldots, \lambda_{k-1}, 0, \ldots, 0)$
$\xrightarrow{\Phi} (\lambda_1, \ldots, \lambda_{k-1}, 0, \ldots, 0, \lambda_k) \xrightarrow{T_*} \lambda.$
We are led to the following formula. Note that it is required to prove that different paths to the same (α, E) produce the same value. The start is at the level $\|\tau_E\|^2$.

Formula for squared norms

The formulas are ($\lambda \in \mathbb{N}_0^{N,+}, \alpha, \beta \in \mathbb{N}_0^N$, $E \in \mathcal{E}_0$)

$$\begin{aligned} \left\langle M_{\alpha,E}, M_{\beta,F} \right\rangle &= 0, \ \ (\alpha, E) \neq (\beta, F) \\ \left\| M_{\alpha,E} \right\|^2 &= \mathcal{R}\left(\alpha, E\right)^{-1} \left\| M_{\alpha^+,E} \right\|^2, \end{aligned}$$

$$\begin{aligned} \left\| M_{\lambda.E} \right\|^2 &= t^{k(\lambda)} \left\| \tau_E \right\|^2 (1-q)^{-|\lambda|} \prod_{i=1}^{N} \left(q t^{c(i,E)}; q \right)_{\lambda_i} \\ &\times \prod_{1 \le i < j \le N} \frac{\left(q t^{c(i,E)-c(j,E)-1}; q \right)_{\lambda_i - \lambda_j} \left(q t^{c(i,E)-c(j,E)+1}; q \right)_{\lambda_i - \lambda_j}}{\left(q t^{c(i,E)-c(j,E)}; q \right)^2_{\lambda_i - \lambda_j}}. \end{aligned}$$

where $(a;q)_n = \prod_{i=1}^n \left(1 - aq^i\right)$, $k(\lambda) = \sum_{i=1}^N (N - 2i + 1)\lambda_i$. This form does satisfy the axioms. Furthermore $\left\| M_{\alpha,E} \right\|^2 > 0$ if $q > 0$ and $\min\left(q^{-1/N}, q^{1/N}\right) < t < \max\left(q^{-1/N}, q^{1/N}\right)$.

Evaluation in Special Cases

Let $F = \{1, 2, \cdots, m, N\}$, $\lambda \in \mathbb{N}_0^{N,+}$ with $\lambda_i = 0$ for $i > m$ and let $x^{(1)} = \left(1, t^{-1}, t^{-2}, \ldots, t^{1-N}\right)$. Then

$$M_{\lambda,F}\left(x^{(1)};\theta\right) = q^{\beta(\lambda)} t^{e_1(\lambda)} \frac{\left(qt^{-N};q,t^{-1}\right)_\lambda \left(qt^{-m};q,t^{-1}\right)_\lambda}{\left(qt^{1-N};q,t^{-1}\right)_\lambda h_{q,1/t}\left(qt^{-1};\lambda\right)} \tau_F(\theta),$$

where $((i,j) \in \lambda$ refers to the Ferrers diagram of $\lambda)$

$$\begin{aligned} (a;q,t)_\lambda &:= \prod_{i=1}^{N} \left(at^{1-i};q\right)_{\lambda_i} \\ h_{q,t}(a;\lambda) &:= \prod_{(i,j)\in\lambda} \left(1 - aq^{\mathrm{arm}(i,j;\lambda)} t^{\mathrm{leg}(i,j;\lambda)}\right) \end{aligned}$$

$\mathrm{arm}(i,j;\lambda) := \lambda_i - j$, $\mathrm{leg}(i,j;\lambda) := \#\{k : i < k \le \ell(\lambda), j \le \lambda_k\}$, and $\ell(\lambda) := \max\{i : \lambda_i \ge 1\}$. The exponents are $\beta(\lambda) := \sum_{i=1}^{m} \binom{\lambda_i}{2}$ and $e_1(\lambda) := \sum_{i=1}^{m} \lambda_i (N - m - i)$. (see C.D. *Symmetry* 2021, 13(5))

Symmetrization

Given a particular $M_{\alpha,E}$ what polynomials $M_{\beta,F}$ can be produced by a sequence of steps of the form $\mathbf{T}_i + b$? We describe the $\mathcal{H}_N(t)$-module generated by $M_{\alpha,E}$, this is based on the following:
For $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{E}_0$ let $\lfloor \alpha, E \rfloor$ denote the tableau obtained from Y_E by replacing i by α_i^+ for $1 \leq i \leq N$. Let
$\mathcal{M}(\alpha, E) := \operatorname{span}\{M_{\beta,F} : \lfloor \beta, F \rfloor = \lfloor \alpha, E \rfloor\}$. This is indeed the $\mathcal{H}_N(t)$-module generated by $M_{\alpha,E}$ (C.D. and Luque). Note
$\mathcal{M}(\alpha, E) = \mathcal{M}(\alpha^+, E)$, and $\lfloor \beta, F \rfloor = \lfloor \alpha, E \rfloor$ implies $\zeta_{\beta,F}$ is a permutation of $\zeta_{\alpha,E}$.
Example: let $N = 9, m = 4, E = \{2,3,6,8,9\}$, $\alpha = (3,5,6,2,2,1,4,4,6)$, $\alpha^+ = (6,6,5,4,4,3,2,2,1)$ and

$$Y_E = \begin{bmatrix} 9 & 7 & 5 & 4 & 1 \\ \diagdown & 8 & 6 & 3 & 2 \end{bmatrix}, \lfloor \alpha, E \rfloor = \begin{bmatrix} 1 & 2 & 4 & 4 & 6 \\ \diagdown & 2 & 3 & 5 & 6 \end{bmatrix}.$$

Is there a symmetric polynomial in $\mathcal{M}(\alpha, E)$, that is, $p(x;\theta)$ such that $\mathbf{T}_i p = tp$ for $1 \leq i < N$? (warning: not the same as $\mathcal{S}_N$-symmetry)

Column-strict Property

(Due to C.D. and Luque): if $\lfloor \alpha, E \rfloor$ is column-strict (the entries in column 1 are increasing) then there is a unique non-zero (up to scalar multiplication) symmetric $p \in \mathcal{M}(\alpha, E)$ otherwise there is none.
We use methods of Baker and Forrester (*Ann. Comb.* 1999) to analyze the symmetric p. Blondeau-Fournier, Desrosiers, Lapointe, and Mathieu (*J. Combin.* 2012) constructed Macdonald superpolynomials which are conceptually different from ours - however their definition of *superpartition* is relevant here: for fermionic degree m it is an N-tuple $(\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N)$ which satisfies $\Lambda_1 > \Lambda_2 > \cdots > \Lambda_m$ and $\Lambda_{m+1} \geq \Lambda_{m+2} \geq \cdots \geq \Lambda_N$. In the example the superpartition is $[6, 5, 3, 2; 6, 4, 4, 2, 1]$. In general for isotype $(N - m, 1^m)$ the numbers $(\Lambda_1, \ldots, \Lambda_m, \Lambda_N)$ are the entries in column 1 of $\lfloor \alpha, E \rfloor$ and thus $\Lambda_N < \Lambda_m)$. The number of tableaux of shape $(N - m, 1^m)$ with entries nondecreasing in row 1 and strictly increasing in column 1 with sum of entries $= n$ is the coefficient of q^n in
$q^{m(m+1)/2} \left\{ \left(1 - q^N\right) (q; q)_m (q; q)_{N-m-1} \right\}^{-1}$.

Expansion of the Symmetric Polynomial

$$p = \sum \left\{ A(\beta, F) M_{\beta,F} : \lfloor \beta, F \rfloor = \lfloor \alpha, E \rfloor \right\}, \mathbf{T}_i p = tp \ \forall i$$

find equations satisfied by the $A(\beta, F)$; not difficult because under the action of $\mathbf{T}_i$ the space $\mathcal{M}(\alpha, E)$ decomposes into a direct sum of two- and one-dimensional submodules (one-dim from $\mathbf{T}_i M_{\beta,F} = t M_{\beta,F}$)
Suppose (case 1) $\beta_i < \beta_{i+1}$ then the matrix of $\mathbf{T}_i$ acting on the span of $M_{\beta,E}, M_{s_i\beta,E}$ is (with $z = \zeta_{\beta,E}(i+1) / \zeta_{\beta,E}(i)$)

$$\begin{bmatrix} -\frac{t-1}{z-1} & \frac{(1-zt)(t-z)}{(1-z)^2} \\ 1 & \frac{z(t-1)}{z-1} \end{bmatrix}$$

then $\left[A(\beta, E), A(s_i\beta, E)\right]^T$ is an eigenvector with eigenvalue t when $A(\beta, E) = \frac{t-z}{1-z} A(s_i\beta, E)$.

Calculation of Coefficients

It is possible for different E to appear, we arrange by the inv-count. Suppose (case 2) $\beta_i = \beta_{i+1}, j = r_\beta(F)$ and $c(j,F) < 0 < c(j+1,F)$ then $\operatorname{inv}(s_j F) = \operatorname{inv}(F) - 1$ and (with $z = t^{c(j,F)-c(j+1,F)}$)

$$\mathbf{T}_i M_{\beta,s_j F} = -\frac{t-1}{z-1} M_{\beta,s_j F} + M_{\beta,F}$$

and the eigenvalue equation implies $A(\beta, s_j F) = \dfrac{t-z}{1-z} A(\beta,F)$. Among these E there are two extreme cases: the root E_R which minimizes the entries of Y_E in row 1 (and thus $\operatorname{inv}(E)$), and the sink E_S which maximizes these entries (depends on α implicitly). In the example

$$Y_E = \begin{bmatrix} 9 & 7 & 5 & 4 & 1 \\ \diagdown & 8 & 6 & 3 & 2 \end{bmatrix}, \lfloor \alpha, E \rfloor = \begin{bmatrix} 1 & 2 & 4 & 4 & 6 \\ \diagdown & 2 & 3 & 5 & 6 \end{bmatrix}.$$

$E_R = E$ and $E_S = \{1,3,6,7,9\}, \operatorname{inv}(E_R) = 7, \operatorname{inv}(E_S) = 9.$

The symmetric polynomial

Suppose $p = \sum \left\{ A(\beta, F) M_{\beta,F} : \lfloor \beta, F \rfloor = \lfloor \alpha, E \rfloor \right\}$ satisfies $\mathbf{T}_i p = tp$ then (case 1) $A(\beta, F) = \dfrac{\mathcal{R}_0(\beta, F)}{\mathcal{R}_0(s_i\beta, F)} A(s_i\beta, F) = \mathcal{R}_0(\beta, F) A\left(\beta^+, F\right)$ since $\mathcal{R}_0\left(\beta^+, F\right) = 1$ and (case 2) $A(\beta, s_j F) = \dfrac{\mathcal{C}_0(F)}{\mathcal{C}_0(s_j F)} A(\beta, F)$.

Set $A(\beta, F) = \dfrac{\mathcal{C}_0(E_S)}{\mathcal{C}_0(F)} \mathcal{R}_0(\beta, F)$ (and $\lambda = \beta^+ = \alpha^+$) then

$$p_{\lambda,E} = \sum_{\lfloor \beta, F \rfloor = \lfloor \lambda, E \rfloor} \frac{\mathcal{C}_0(E_S) \mathcal{R}_0(\beta, F)}{\mathcal{C}_0(F)} M_{\beta,F}$$

is the supersymmetric polynomial in $\mathcal{M}(\lambda, E)$, unique when the coefficient of M_{λ,E_S} is 1.

Examples of symmetric superpolynomials

- $N = 3, \lambda = (1,0,0), E = \{2,3\}$ then

$$\begin{aligned} p &= t^3 \left(\theta_2 + \theta_3 - t(t+1)\theta_1\right) x_1 + t^3 \left(t^3\theta_1 - t(t+1)\theta_2 + \theta_3\right) x_2 \\ &\quad + t^4 \left(t^2\theta_1 + t^2\theta_2 - (t+1)\theta_3\right) x_3 \end{aligned}$$

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- $N=4, \lambda=(2,1,0,0), E=\{1,2,4\}$, sample terms

$$\begin{aligned} & x_1^2x_2t^6q\left\{t^2\left(t+1\right)\theta_1\theta_2-\left(t\theta_1-\theta_2\right)\left(\theta_3+\theta_4\right)\right\} \\ & x_1x_2^2t^7q\left\{-t^2\left(t+1\right)\theta_1\theta_2+\left(t\theta_1-\theta_2\right)\left(\theta_3+\theta_4\right)\right\} \\ & x_1^2x_3t^6q\left\{\theta_1\left(-t^4\theta_2+t^2\left(t+1\right)\theta_3-t\theta_4\right)-\left(t-1\right)\theta_2\theta_4+\theta_3\theta_4\right\} \\ & x_1x_2x_3t^6q\left\{\begin{array}{c} t\left(t-1\right)\left(t^3\theta_1\theta_2+t\left(t+1\right)\theta_1\theta_3+\theta_2\theta_3\right)+ \\ \left(t-1\right)\left(t\theta_1+t\left(t-1\right)\theta_2-\theta_3\right)\theta_4 \end{array}\right\}. \end{aligned}$$

Symmetrization

Define $X_0 = 1, X_i = 1 + \mathbf{T}_i X_{i-1}$ for $i \geq 1$ and $S^{(N)} = X_1 X_2 \cdots X_{N-1}$ then for any $p \in s\mathcal{P}_m$

$$\mathbf{T}_i \left(S^{(N)} p \right)(x;\theta) = t \left(S^{(N)} p \right)(x;\theta),\ 1 \leq i < N,$$

and $\left(S^{(N)} \right)^2 = [N]_t ! S^{(N)}$ (idea of proof: replace $\mathbf{T}_i$ by s_i and show that one obtains the $\mathcal{S}_N$-symmetrization operator). In fact
$S^{(N)} = \sum_{u \in \mathcal{S}_N} \mathbf{T}(u)$, also $S^{(N)}$ is self-adjoint since $\mathbf{T}(u)^* = \mathbf{T}\left(u^{-1}\right)$
(e.g. $\langle \mathbf{T}_i \mathbf{T}_j f, g \rangle = \langle f, \mathbf{T}_j \mathbf{T}_i g \rangle$) and $\sum_{u \in \mathcal{S}_N} \mathbf{T}(u) = \sum_{u \in \mathcal{S}_N} \mathbf{T}\left(u^{-1}\right)$.
(Recall $[N]_t ! := \prod_{n=1}^{N} [n]_t$.)
From this it follows that if $\lfloor \alpha, F \rfloor = \lfloor \lambda, E \rfloor$ then $S^{(N)} M_{\alpha,F} = c p_{\lambda,E}$ for some constant c, because of the uniqueness of $p_{\lambda,E}$ in $\mathcal{M}(\lambda, E)$. This leads to the evaluation of $\left\| p_{\lambda,E} \right\|^2$, which does not use summation over all $\lfloor \alpha', F' \rfloor = \lfloor \lambda, E \rfloor$.

Evaluation of squared norm

$$\left\langle p_{\lambda,E}, S^{(N-1)} M_{\alpha,F} \right\rangle = c \left\langle p_{\lambda,E}, p_{\lambda,E} \right\rangle = \left\langle S^{(N-1)} p_{\lambda,E}, M_{\alpha,F} \right\rangle$$

$$= [N]_t! \left\langle p_{\lambda,E}, M_{\alpha,F} \right\rangle = [N]_t! \frac{\mathcal{C}_0 (E_S) \mathcal{R}_0 (\alpha, F)}{\mathcal{C}_0 (F)} \left\| M_{\alpha,F} \right\|^2 .$$

Let $\alpha = \lambda^-$, the nondecreasing rearrangement of λ, and $F = F = E_R$. For each $i \le \lambda_1$ let m_i be the multiplicity of i in row 1 of $\lfloor \lambda, E_S \rfloor$, that is $m_i = \# \{ j : \lfloor \lambda, E_S \rfloor [1, j] = i \}$. Then the coefficient of M_{λ,E_S} in $S^{(N)} M_{\lambda_-,E_R}$ is $\prod_{i=0}^{\lambda_1} [m_i]_t!$ (and the coefficient of M_{λ,E_S} in $p_{\lambda,E}$ is 1). Thus

$$\left\| p_{\lambda,E_S} \right\|^2 = \frac{[N]_t!}{\prod_{i \ge 0} [m_i]_t!} \frac{\mathcal{C}_0 (E_S) \mathcal{R}_0 \left(\lambda^-, E_R\right)}{\mathcal{C}_0 (E_R)} \left\| M_{\lambda^-,E_R} \right\|^2$$

$$= \frac{[N]_t!}{\prod_{i \ge 0} [m_i]_t!} \frac{\mathcal{C}_0 (E_S) \mathcal{R}_0 \left(\lambda^-, E_R\right)}{\mathcal{C}_0 (E_R) \mathcal{R} \left(\lambda^-, E_R\right)} \left\| M_{\lambda,E_R} \right\|^2 .$$

Conclusion!

- With some computation we obtain

$$\left\|p_{\lambda,E_S}\right\|^2 = t^{2(N-m-1)+k(\lambda)}\left[m+1\right]_t (1-q)^{-|\lambda|}\prod_{i=1}^{N}\left(qt^{c(i,E_S)};q\right)_{\lambda_i}$$

$$\times\prod_{1\le i<j\le N}\frac{\left(qt^{c(i,E_S)-c(j,E_S)-1};q\right)_{\lambda_i-\lambda_j}\left(qt^{c(i,E_S)-c(j,E_S)+1};q\right)_{\lambda_i-\lambda_j-1}}{\left(1-q^{\lambda_i-\lambda_j}t^{c(i,E_S)-c(j,E_S)}\right)\left(qt^{c(i,E_S)-c(j,E_S)};q\right)^2_{\lambda_i-\lambda_j-1}}$$

$$\times\frac{\left[N\right]_t!}{\prod_{i\ge 0}\left[m_i\right]_t!}\mathcal{C}_0\left(E_S\right)\mathcal{C}_1\left(E_R\right).$$

the last line involves only t. (recall $k(\lambda) := \sum_{i=1}^{N}(N-2i+1)\lambda_i$). Let $q = t^{1/\kappa}$ and let $t\to 1$ to obtain formulas for symmetric Jack superpolynomials.

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$$\times \frac{[N]_t!}{\prod_{i\ge 0} [m_i]_t!} \mathcal{C}_0 (E_S) \mathcal{C}_1 (E_R) .$$

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- Thank you.