Domains of orthogonality for generalized Chebyshev polynomials as semi-algebraic sets[1]

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Modern Analysis Related to Root Systems with Applications
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Contribution

- R crystallog. root system
  Λ weight lattice
  $\mathcal{W}$ Weyl group
- $T_\alpha \in \mathbb{R}[\mathrm{X}]$, $\alpha \in \Lambda$
  Chebyshev polynomial
- $Z : \mathbb{T}^n \to \mathbb{R}^n$
  generalized cosine

Theorem[1]

Let $\alpha, \beta \in \Lambda$. Then $\alpha \in \mathcal{W} \cdot \beta \Leftrightarrow$

$$\int_{\mathrm{im}(Z)} T_\alpha(\mathrm{X})\, T_\beta(\mathrm{X})\, \delta(\mathrm{X})\, \mathrm{dX} \neq 0.$$

Main Result

$$z \in \mathrm{im}(Z) \Leftrightarrow H(z) \succeq 0 \quad \text{for} \quad \mathrm{R} = \mathrm{A}_{n-1}, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$$

with $H \in \mathrm{Sym}_n(\mathbb{R}[\mathrm{X}])$ yields n polynomial inequations for $\mathrm{im}(Z)$.

Motivation: $\min \sum_{\alpha} c_\alpha T_\alpha(\mathrm{X}) \quad \text{s.t.} \quad \mathrm{X} \in \mathrm{im}(Z)$

[1] Hoffman, Withers 1988: *Generalized Chebyshev Polynomials associated with affine Weyl Groups*, Thm 5.1

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Generalized Chebyshev polynomials of the first kind

Assume $\Lambda = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$.
Identify $\mathbb{Z}[\Lambda]$ with $\mathbb{Z}[\mathrm{x}^\pm]$ via $\mathrm{x}^{\omega_i} = \mathrm{x}_i$.

$$\mathbb{Z}[\mathrm{x}^\pm]^{\mathcal{W}} := \left\{ f \in \mathbb{Z}[\mathrm{x}^\pm] : \forall A \in \mathcal{W} : A \cdot f = f \right\}$$

Theorem[1]

For $\alpha \in \Lambda$ define the **orbit polynomial** $\Theta_\alpha := \sum_{A \in \mathcal{W}} \mathrm{x}^{A\cdot\alpha}$. Then

$$\mathbb{Z}[\mathrm{x}^\pm]^{\mathcal{W}} = \mathbb{Z}[\Theta_{\omega_1}, \ldots, \Theta_{\omega_n}].$$

Definition[2]

The unique $T_\alpha \in \mathbb{R}[\mathrm{X}]$ satisfying $T_\alpha(\Theta_{\omega_1}, \ldots, \Theta_{\omega_n}) = \Theta_\alpha$ is the **Chebyshev polynomial of the first kind** associated to $\alpha \in \Lambda$.

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$\mathbb{T}^n := \{\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n : |\zeta_i| = 1\}$ the complex torus.

Definition

Define $\phi : \mathbb{R}^n \to \mathbb{T}^n$ and $\Theta : \mathbb{T}^n \to \mathbb{R}^n$ by

$$u \overset{\phi}{\mapsto} (\exp(-2\pi \mathrm{i} \langle \omega_1, u\rangle), \ldots, \exp(-2\pi \mathrm{i} \langle \omega_n, u\rangle)) \quad \text{and}$$

$$\zeta \overset{\Theta}{\mapsto} (\Theta_{\omega_1}(\zeta), \ldots, \Theta_{\omega_n}(\zeta)).$$

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$Z := \Theta \circ \phi$ is called the **generalized cosine** of R.

The function Z is periodic and $\mathrm{im}(Z) = \mathrm{im}(\Theta)$.

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The generalized cosine of a root system

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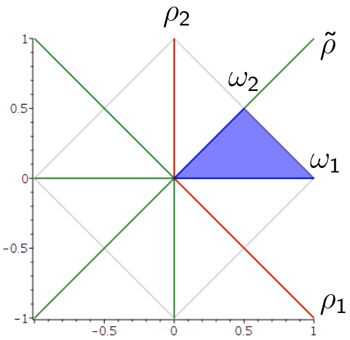
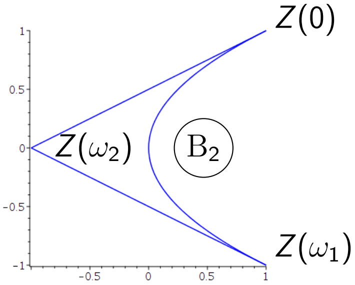
Example: The B_2 case I

Example

$Z : \mathbb{R}^2 \to \mathbb{R}^2, u \mapsto ((\cos(2\pi u_1) + \cos(2\pi u_2))/2, \cos(\pi u_1)\cos(\pi u_2))$



fundamental domain
of the affine Weyl group



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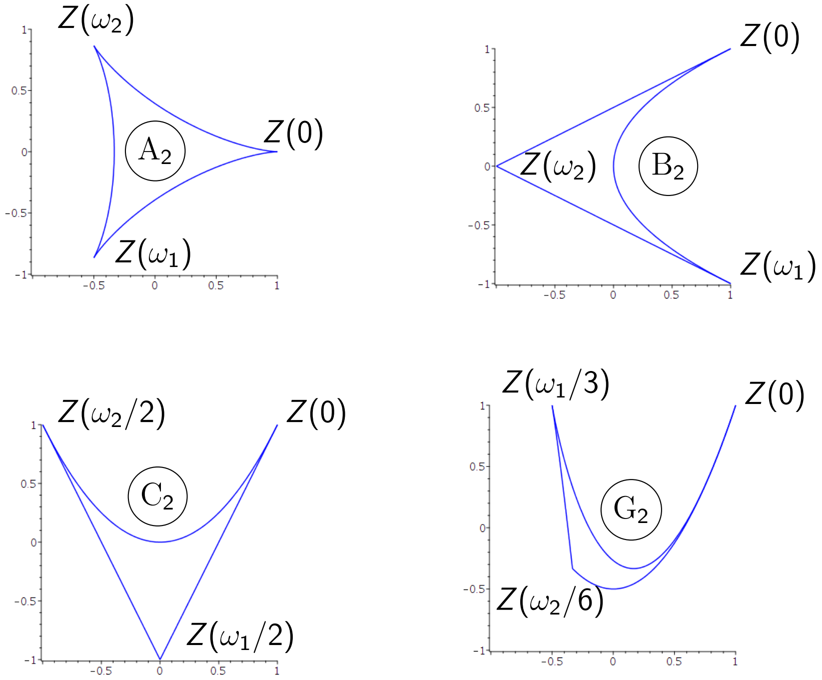
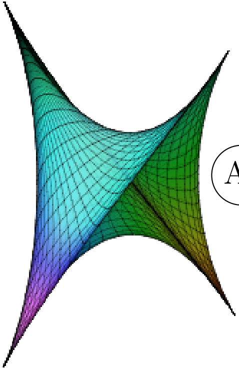
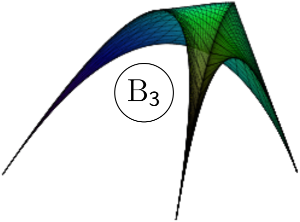
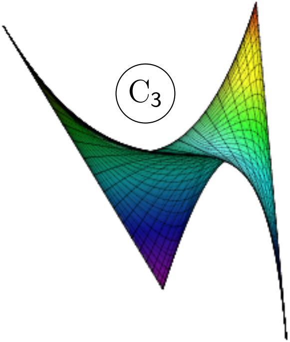

Z(ω2)
Z(0)
A2
Z(ω1)
Z(0)
Z(ω2)
B2
Z(ω1)
Z(ω2/2)
Z(0)
C2
Z(ω1/2)
Z(ω1/3)
Z(0)
G2
Z(ω2/6)



A_3





Relations to elementary symmetric polynomials σ_i

Assume $\mathcal{W} \cdot \mathrm{x}_1 = \{y_1, y_2, \ldots\}$ is the orbit of $\mathrm{x}_1 \rightsquigarrow \mathrm{y}(\mathrm{x}) \in [\mathrm{x}^{\pm}]^n$

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$\mathcal{W} = \mathfrak{S}_n$

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$\mathcal{W} = \mathfrak{S}_n \ltimes \{\pm 1\}^n$

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Main Result

Proposition

There exists a unique $f \in \mathbb{R}[\mathrm{X}][t]$ with

$$f(\Theta(\mathrm{x})) = \sum_{i=0}^{n} (-1)^i \sigma_i(\mathrm{y}(\mathrm{x}))\, t^{n-i} \in \mathbb{R}[t].$$

With a well-known result[1] from algebraic geometry, we obtain:

Theorem

Let $z \in \mathbb{R}^n$. Define the matrix $H \in \mathrm{Sym}_n(\mathbb{R}[\mathrm{X}])$:

$$\begin{aligned} H(\mathrm{X})_{ij} &= \mathrm{Trace}(4\,(C(\mathrm{X}))^{i+j-2} - (C(\mathrm{X}))^{i+j}) \quad \text{with} \\ C(\mathrm{X}) &= \mathrm{CompanionMatrix}(f(\mathrm{X})) \end{aligned}$$

Then $z \in \mathrm{im}(\Theta)$ iff $H(z) \succeq 0$.

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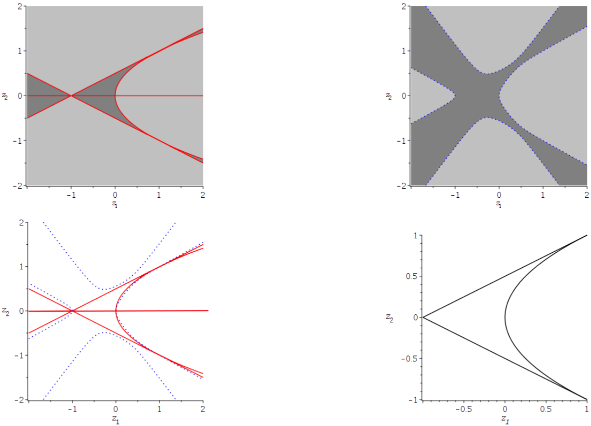
Example: The B_2 case II

$$H(X) = 16 \begin{pmatrix} -X_1^2 + 2X_2^2 - X_1 & -4X_1^3 + 12X_1\,X_2^2 - 6X_1^2 - 2X_1 \\ -4X_1^3 + 12X_1\,X_2^2 - 6X_1^2 - 2X_1 & -16X_1^4 + 64X_1^2\,X_2^2 - 32X_2^4 - 32X_1^3 + 32X_1\,X_2^2 - 20X_1^2 + 8X_2^2 - 4X_1 \end{pmatrix}$$



$\mathrm{Det}(H(z)) = 0$ (solid) $\mathrm{Trace}(H(z)) = 0$ (dots)

Thanks for your attention.

- Bourbaki: *Lie Groups and Lie Algebras ch IV, V, VI*, Elements of Mathematics, Springer 1981, www.springer.com
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Appendix A: Outline for the B_n case

Step ①

The orbit of x_1 under the Weyl group is

$$\mathcal{W} \cdot \mathrm{x}_1 = \{\underbrace{\mathrm{x}_1}_{=:\mathrm{y}_1}, \underbrace{\mathrm{x}_2\,\mathrm{x}_1^{-1}}_{=:\mathrm{y}_2}, \ldots, \underbrace{\mathrm{x}_{n-1}\,\mathrm{x}_{n-2}^{-1}}_{=:\mathrm{y}_{n-1}}, \underbrace{\mathrm{x}_n^2\,\mathrm{x}_{n-1}^{-1}}_{=:\mathrm{y}_n}\} \cup \{\ldots\}^{-1}.$$

Step ②

For $1 \leq i \leq n-1$:

$$\sigma_i(\mathrm{y}_1(\mathrm{x}) + \mathrm{y}_1(\mathrm{x})^{-1}, \ldots, \mathrm{y}_n(\mathrm{x}) + \mathrm{y}_n(\mathrm{x})^{-1}) = 2^i \binom{n}{i} \Theta_{\omega_i}(\mathrm{x}) \quad \text{and}$$

$$\sigma_n(\mathrm{y}_1(\mathrm{x}) + \mathrm{y}_1(\mathrm{x})^{-1}, \ldots, \mathrm{y}_n(\mathrm{x}) + \mathrm{y}_n(\mathrm{x})^{-1}) = 2^n \Theta_{2\,\omega_n}(\mathrm{x}).$$

Apply the recurrence formula $\boxed{\Theta_{2\omega_n} = 2^n \Theta_{\omega_n}^2 - \sum_{j=1}^{n-1} \binom{n}{j} \Theta_{\omega_j} - 1.}$

Appendix B: Characterizing roots of polynomials

Sturm Sylvester Theorem[1]

Consider $f \in \mathbb{R}[\mathrm{t}]$ of degree n and $A := \mathbb{R}[\mathrm{t}]/\langle f \rangle$.
For $q \in \mathbb{R}[\mathrm{t}]$ define $H_q : A \to \mathbb{R}$, $[p] \mapsto \mathrm{Trace}(m_{q\,p^2})$. Then:

$$\begin{aligned}\mathrm{Sign}\, H_q = \quad & \#\text{ real roots } r \text{ of } f \text{ with } q(r) > 0 \\ - & \#\text{ real roots } r \text{ of } f \text{ with } q(r) < 0\end{aligned}$$

Choose $q := 4 - \mathrm{t}^2$.

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