Laplace transform of hypergeometric functions and Zeta distributions in the Dunkl setting

Dominik Brennecken

Paderborn University, Germany

October 19th, 2021

Laplace Transform in the Dunkl Setting of Type A

Laplace Transform in the Dunkl Setting of Type A

Dunkl-Laplace Transform

$$\mathcal{L}_k f(z) := \int_{\mathbb{R}_+^n} E_k^A(-x, z) f(x) \omega_k^A(x) \, \mathrm{d}x,$$

- $E_k^A(x, y)$= Dunkl kernel of root system $A_{n-1} = \{e_i - e_j \mid i \neq j\} \subseteq \mathbb{R}^n$ with multiplicity $k \geq 0$
- $\omega_k^A(x) = \prod_{i \neq j} |x_i - x_j|^k$ a weight function

Laplace Transform in the Dunkl Setting of Type A

Dunkl-Laplace Transform

$$\mathcal{L}_k f(z) := \int_{\mathbb{R}_+^n} E_k^A(-x, z) f(x) \omega_k^A(x) \, \mathrm{d}x,$$

- $E_k^A(x, y)$= Dunkl kernel of root system $A_{n-1} = \{e_i - e_j \mid i \neq j\} \subseteq \mathbb{R}^n$ with multiplicity $k \geq 0$
- $\omega_k^A(x) = \prod_{i \neq j} |x_i - x_j|^k$ a weight function
- used by Baker & Forrester and earlier in a symmetrized version by I.G. Macdonald with kernel $e(x, y)$ (the type A Bessel function). The analytic properties were studied later ($\rightarrow$ Rösler)

Laplace Transform in the Dunkl Setting of Type A

Dunkl-Laplace Transform

$$\mathcal{L}_k f(z) := \int_{\mathbb{R}_+^n} E_k^A(-x, z) f(x) \omega_k^A(x) \, \mathrm{d}x,$$

- $E_k^A(x, y)$= Dunkl kernel of root system $A_{n-1} = \{e_i - e_j \mid i \neq j\} \subseteq \mathbb{R}^n$ with multiplicity $k \geq 0$
- $\omega_k^A(x) = \prod_{i \neq j} |x_i - x_j|^k$ a weight function
- used by Baker & Forrester and earlier in a symmetrized version by I.G. Macdonald with kernel $e(x, y)$ (the type A Bessel function). The analytic properties were studied later ($\rightarrow$ Rösler)
- only for root systems of Type A (good decay of E_k^A and connection to Laplace transform on symmetric cones)

Motivation

Identity on symmetric cones

Important Laplace transform identity on a symmetric cone $\Omega = G/K$ of dimension m and rank n :

$$\int_\Omega e^{-\langle x,z\rangle}\Delta_{\mathbf{s}}(x)\det(x)^{-\frac{m}{n}}\,\mathrm{d}x = \Gamma_\Omega(\mathbf{s})\Delta_{\mathbf{s}}(z^{-1}),.$$

where $\Delta_{\mathbf{s}},\ \mathbf{s}\in\mathbb{C}^n$ generalizes the power function x^s on $]0,\infty[$ and Γ_Ω the Gindikin gamma function of Ω.

Motivation

Identity on symmetric cones

Important Laplace transform identity on a symmetric cone $\Omega = G/K$ of dimension m and rank n :

$$\int_\Omega e^{-\langle x,z\rangle}\Delta_{\mathbf{s}}(x)\det(x)^{-\frac{m}{n}}\,\mathrm{d}x = \Gamma_\Omega(\mathbf{s})\Delta_{\mathbf{s}}(z^{-1}),.$$

where $\Delta_{\mathbf{s}},\ \mathbf{s}\in\mathbb{C}^n$ generalizes the power function x^s on $]0,\infty[$ and Γ_Ω the Gindikin gamma function of Ω.

- the spherical functions $\varphi_{\mathbf{s}} = \int_K \Delta_{\mathbf{s}+\rho}\,\mathrm{d}k$ are expressed in type A Heckman-Opdam hypergeometric functions

Motivation

Identity on symmetric cones

Important Laplace transform identity on a symmetric cone $\Omega = G/K$ of dimension m and rank n :

$$\int_\Omega e^{-\langle x,z\rangle} \Delta_{\mathbf{s}}(x) \det(x)^{-\frac{m}{n}} \, \mathrm{d}x = \Gamma_\Omega(\mathbf{s}) \Delta_{\mathbf{s}}(z^{-1}), .$$

where $\Delta_{\mathbf{s}}, \; \mathbf{s} \in \mathbb{C}^n$ generalizes the power function x^s on $]0, \infty[$ and Γ_Ω the Gindikin gamma function of Ω.

- the spherical functions $\varphi_{\mathbf{s}} = \int_K \Delta_{\mathbf{s}+\rho} \, \mathrm{d}k$ are expressed in type A Heckman-Opdam hypergeometric functions
- expectation: existence of generalization in the Dunkl setting.

Laplace transform of Cherednik kernel and hypergeometric function

- $G_k(\lambda, z), F_k(\lambda, z)$ the Cherednik kernel and Heckman-Opdam hypergeometric function corresponding to $A_{n-1} \subseteq \mathbb{R}_0^n = \{\sum x_i = 0\}$

Laplace transform of Cherednik kernel and hypergeometric function

- $G_k(\lambda, z), F_k(\lambda, z)$ the Cherednik kernel and Heckman-Opdam hypergeometric function corresponding to $A_{n-1} \subseteq \mathbb{R}^n_0 = \{\sum x_i = 0\}$
- $\mathcal{G}_k(\lambda, z) := \prod_{i=1}^n z_i^{\lambda_i/n} \cdot G_k(\pi(\lambda), \pi(\log(z)))$ with $\lambda \in \mathbb{C}^n, \mathrm{Re}(z) > 0$

Laplace transform of Cherednik kernel and hypergeometric function

- $G_k(\lambda, z), F_k(\lambda, z)$ the Cherednik kernel and Heckman-Opdam hypergeometric function corresponding to $A_{n-1} \subseteq \mathbb{R}_0^n = \{\sum x_i = 0\}$
- $\mathcal{G}_k(\lambda, z) := \prod_{i=1}^n z_i^{\lambda_i/n} \cdot G_k(\pi(\lambda), \pi(\log(z)))$ with $\lambda \in \mathbb{C}^n, \mathrm{Re}(z) > 0$ is unique common eigenfunction of the Cherednik operators $\mathscr{D}_j(k) = x_j T_j^A(k) + k(1-n) + k \sum_{i>j} s_{ij}$ with $\mathscr{D}_j \mathcal{G}_k(\lambda, \cdot) = \lambda_j \mathcal{G}_k(\lambda, \cdot)$.

Laplace transform of Cherednik kernel and hypergeometric function

- $G_k(\lambda, z), F_k(\lambda, z)$ the Cherednik kernel and Heckman-Opdam hypergeometric function corresponding to $A_{n-1} \subseteq \mathbb{R}_0^n = \{\sum x_i = 0\}$
- $\mathcal{G}_k(\lambda, z) := \prod_{i=1}^n z_i^{\lambda_i/n} \cdot G_k(\pi(\lambda), \pi(\log(z)))$ with $\lambda \in \mathbb{C}^n, \mathrm{Re}(z) > 0$ is unique common eigenfunction of the Cherednik operators $\mathscr{D}_j(k) = x_j T_j^A(k) + k(1-n) + k \sum_{i>j} s_{ij}$ with $\mathscr{D}_j \mathcal{G}_k(\lambda, \cdot) = \lambda_j \mathcal{G}_k(\lambda, \cdot)$.
  Similar $\mathcal{F}_k$ (particular choice of $k \Rightarrow$ sph. fct. of symmetric cones).

Laplace transform of Cherednik kernel and hypergeometric function

- $G_k(\lambda, z), F_k(\lambda, z)$ the Cherednik kernel and Heckman-Opdam hypergeometric function corresponding to $A_{n-1} \subseteq \mathbb{R}_0^n = \{\sum x_i = 0\}$
- $\mathcal{G}_k(\lambda, z) := \prod_{i=1}^n z_i^{\lambda_i/n} \cdot G_k(\pi(\lambda), \pi(\log(z)))$ with $\lambda \in \mathbb{C}^n, \mathrm{Re}(z) > 0$ is unique common eigenfunction of the Cherednik operators $\mathscr{D}_j(k) = x_j T_j^A(k) + k(1-n) + k \sum_{i>j} s_{ij}$ with $\mathscr{D}_j \mathcal{G}_k(\lambda, \cdot) = \lambda_j \mathcal{G}_k(\lambda, \cdot)$.

  Similar $\mathcal{F}_k$ (particular choice of $k \Rightarrow$ sph. fct. of symmetric cones).

Theorem

For $\Delta(x) := x_1 \cdots x_n$ and $\mu_0 = k(n-1)$:

$$\mathcal{L}_k(\mathcal{G}_k(\lambda, \cdot)\Delta^{-\mu_0})(z) = \Gamma_n(\lambda + \rho)\mathcal{G}_k(\lambda, \tfrac{1}{z}),$$

Laplace transform of Cherednik kernel and hypergeometric function

- $G_k(\lambda, z), F_k(\lambda, z)$ the Cherednik kernel and Heckman-Opdam hypergeometric function corresponding to $A_{n-1} \subseteq \mathbb{R}_0^n = \{\sum x_i = 0\}$
- $\mathcal{G}_k(\lambda, z) := \prod_{i=1}^n z_i^{\lambda_i/n} \cdot G_k(\pi(\lambda), \pi(\log(z)))$ with $\lambda \in \mathbb{C}^n, \mathrm{Re}(z) > 0$ is unique common eigenfunction of the Cherednik operators $\mathscr{D}_j(k) = x_j T_j^A(k) + k(1-n) + k \sum_{i>j} s_{ij}$ with $\mathscr{D}_j \mathcal{G}_k(\lambda, \cdot) = \lambda_j \mathcal{G}_k(\lambda, \cdot)$.

  Similar $\mathcal{F}_k$ (particular choice of $k \Rightarrow$ sph. fct. of symmetric cones).

Theorem

For $\Delta(x) := x_1 \cdots x_n$ and $\mu_0 = k(n-1)$:

$$\mathcal{L}_k(\mathcal{G}_k(\lambda, \cdot)\Delta^{-\mu_0})(z) = \Gamma_n(\lambda + \rho)\mathcal{G}_k(\lambda, \tfrac{1}{z}),$$

where $\rho = -\frac{k}{2}(n+1, \ldots, -n+1)$ and $\Gamma_n(z) = c \prod_{j=1}^n \Gamma(z_j - k(j-1))$.

Laplace transform of Cherednik kernel and hypergeometric function

- $G_k(\lambda, z), F_k(\lambda, z)$ the Cherednik kernel and Heckman-Opdam hypergeometric function corresponding to $A_{n-1} \subseteq \mathbb{R}_0^n = \{\sum x_i = 0\}$
- $\mathcal{G}_k(\lambda, z) := \prod_{i=1}^n z_i^{\lambda_i/n} \cdot G_k(\pi(\lambda), \pi(\log(z)))$ with $\lambda \in \mathbb{C}^n, \mathrm{Re}(z) > 0$ is unique common eigenfunction of the Cherednik operators $\mathscr{D}_j(k) = x_j T_j^A(k) + k(1-n) + k \sum_{i>j} s_{ij}$ with $\mathscr{D}_j \mathcal{G}_k(\lambda, \cdot) = \lambda_j \mathcal{G}_k(\lambda, \cdot)$.
  Similar $\mathcal{F}_k$ (particular choice of $k \Rightarrow$ sph. fct. of symmetric cones).

Theorem

For $\Delta(x) := x_1 \cdots x_n$ and $\mu_0 = k(n-1)$:

$$\mathcal{L}_k(\mathcal{G}_k(\lambda, \cdot)\Delta^{-\mu_0})(z) = \Gamma_n(\lambda + \rho)\mathcal{G}_k(\lambda, \tfrac{1}{z}),$$

where $\rho = -\frac{k}{2}(n+1, \ldots, -n+1)$ and $\Gamma_n(z) = c \prod_{j=1}^n \Gamma(z_j - k(j-1))$.

Symmetrization over $\mathcal{S}_n \Rightarrow$ same for $\mathcal{F}_k$, generalizing Laplace transform formula for spherical functions.

The Proof

- *first step*: for a particular choice of λ, $\mathcal{G}_k(\lambda, \cdot)$ becomes a scalar multiplice of a non-symmetric Jack polynomial E_η of index $\frac{1}{k}$.

The Proof

- *first step*: for a particular choice of λ, $\mathcal{G}_k(\lambda, \cdot)$ becomes a scalar multiplice of a non-symmetric Jack polynomial E_η of index $\frac{1}{k}$.
- well-known: $(E_\eta)_{\eta \in \mathbb{N}_0^n}$ have a recurrence via $\mathcal{S}_n$ and the raising operator $\Phi f(x) = x_n f(x_n, x_1, \ldots, x_{n-1})$.

The Proof

- *first step*: for a particular choice of λ, $\mathcal{G}_k(\lambda,\cdot)$ becomes a scalar multiplice of a non-symmetric Jack polynomial E_η of index $\frac{1}{k}$.
- well-known: $(E_\eta)_{\eta\in\mathbb{N}_0^n}$ have a recurrence via $\mathcal{S}_n$ and the raising operator $\Phi f(x) = x_n f(x_n, x_1, \ldots, x_{n-1})$.
- inductive proof shows for $\mu \in \mathbb{C}$ and $\mathrm{Re}(\mu) > \mu_0$:

$$\begin{aligned}\mathcal{L}_k(E_\eta \cdot \Delta^{\mu-\mu_0})(z) &= \int_{\mathbb{R}_+^n} E_k^A(-x,z)E_\eta(x)\Delta(x)^{\mu-\mu_0}\omega_k^A(x)\,\mathrm{d}x \\ &= \Gamma_n(\eta_+ + \underline{\mu})E_\eta(\tfrac{1}{z})\Delta(z)^{-\mu}\end{aligned}$$

with the unique partition $\eta_+ \in \mathcal{S}_n\eta$ and $\underline{\mu} = (\mu,\ldots,\mu)$.

The Proof

- *first step*: for a particular choice of λ, $\mathcal{G}_k(\lambda, \cdot)$ becomes a scalar multiplice of a non-symmetric Jack polynomial E_η of index $\frac{1}{k}$.
- well-known: $(E_\eta)_{\eta \in \mathbb{N}_0^n}$ have a recurrence via $\mathcal{S}_n$ and the raising operator $\Phi f(x) = x_n f(x_n, x_1, \ldots, x_{n-1})$.
- inductive proof shows for $\mu \in \mathbb{C}$ and $\mathrm{Re}(\mu) > \mu_0$:

$$\mathcal{L}_k(E_\eta \cdot \Delta^{\mu-\mu_0})(z) = \int_{\mathbb{R}_+^n} E_k^A(-x, z) E_\eta(x) \Delta(x)^{\mu-\mu_0} \omega_k^A(x) \, \mathrm{d}x$$
$$= \Gamma_n(\eta_+ + \underline{\mu}) E_\eta(\tfrac{1}{z}) \Delta(z)^{-\mu}$$

  with the unique partition $\eta_+ \in \mathcal{S}_n \eta$ and $\underline{\mu} = (\mu, \ldots, \mu)$.
- the same holds for symmetric Jack polynomials, which was a conjecture of I.G. Macdonald

The Proof

- *first step*: for a particular choice of λ, $\mathcal{G}_k(\lambda,\cdot)$ becomes a scalar multiplice of a non-symmetric Jack polynomial E_η of index $\frac{1}{k}$.
- well-known: $(E_\eta)_{\eta\in\mathbb{N}_0^n}$ have a recurrence via $\mathcal{S}_n$ and the raising operator $\Phi f(x) = x_n f(x_n, x_1, \ldots, x_{n-1})$.
- inductive proof shows for $\mu \in \mathbb{C}$ and $\mathrm{Re}(\mu) > \mu_0$:

$$\begin{aligned}\mathcal{L}_k(E_\eta \cdot \Delta^{\mu-\mu_0})(z) &= \int_{\mathbb{R}_+^n} E_k^A(-x,z)E_\eta(x)\Delta(x)^{\mu-\mu_0}\omega_k^A(x)\,\mathrm{d}x \\ &= \Gamma_n(\eta_+ + \underline{\mu})E_\eta(\tfrac{1}{z})\Delta(z)^{-\mu}\end{aligned}$$

  with the unique partition $\eta_+ \in \mathcal{S}_n\eta$ and $\underline{\mu} = (\mu,\ldots,\mu)$.
- the same holds for symmetric Jack polynomials, which was a conjecture of I.G. Macdonald
- *final step*: analytic continuation via Carlson's Theorem

Laplace Transform of Hypergeometric series

- Choose renormalized version of non-symmetric and symmetric Jack polynomials $(L_\eta^k)_{\eta\in\mathbb{N}_0^n}$ and $(C_\lambda^k)_{\lambda\in\Lambda_+^n}$ of index $\frac{1}{k}$ s.th. $\sum_{|\eta|=m} L_\eta^k(x) = \sum_{|\lambda|=m} C_\lambda^k(x) = (x_1 + \ldots + x_n)^m$.

Laplace Transform of Hypergeometric series

- Choose renormalized version of non-symmetric and symmetric Jack polynomials $(L_\eta^k)_{\eta\in\mathbb{N}_0^n}$ and $(C_\lambda^k)_{\lambda\in\Lambda_+^n}$ of index $\frac{1}{k}$ s.th. $\sum_{|\eta|=m} L_\eta^k(x) = \sum_{|\lambda|=m} C_\lambda^k(x) = (x_1 + \ldots + x_n)^m$.
- hypergeometric series: ${}_pK_q^k(\genfrac{}{}{0pt}{}{\mu}{\nu}; w, z) := \sum_{\eta\in\mathbb{N}_0^n} \frac{[\mu_1]_{\eta_+}\cdots[\mu_p]_{\eta_+}}{[\nu_1]_{\eta_+}\cdots[\nu_q]_{\eta_+}} \frac{L_\eta^k(w)L_\eta^k(z)}{L_\eta^k(\underline{1})}$ and ${}_pF_q^k$ via the C_λ^k.

Laplace Transform of Hypergeometric series

- Choose renormalized version of non-symmetric and symmetric Jack polynomials $(L_\eta^k)_{\eta\in\mathbb{N}_0^n}$ and $(C_\lambda^k)_{\lambda\in\Lambda_+^n}$ of index $\frac{1}{k}$ s.th. $\sum_{|\eta|=m} L_\eta^k(x) = \sum_{|\lambda|=m} C_\lambda^k(x) = (x_1 + \ldots + x_n)^m$.
- hypergeometric series: ${}_pK_q^k(\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}; w, z) := \sum_{\eta\in\mathbb{N}_0^n} \frac{[\mu_1]_{\eta_+}\cdots[\mu_p]_{\eta_+}}{[\nu_1]_{\eta_+}\cdots[\nu_q]_{\eta_+}} \frac{L_\eta^k(w)L_\eta^k(z)}{L_\eta^k(\underline{1})}$ and ${}_pF_q^k$ via the C_λ^k.
  generalized Pochhammer symbol $[\mu]_\eta = \frac{\Gamma_n(\underline{\mu}+\eta)}{\Gamma_n(\underline{\mu})}$.

Laplace Transform of Hypergeometric series

- Choose renormalized version of non-symmetric and symmetric Jack polynomials $(L_\eta^k)_{\eta\in\mathbb{N}_0^n}$ and $(C_\lambda^k)_{\lambda\in\Lambda_+^n}$ of index $\frac{1}{k}$ s.th. $\sum_{|\eta|=m} L_\eta^k(x) = \sum_{|\lambda|=m} C_\lambda^k(x) = (x_1+\ldots+x_n)^m$.
- hypergeometric series: ${}_pK_q^k\left(\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}; w, z\right) := \sum_{\eta\in\mathbb{N}_0^n} \frac{[\mu_1]_{\eta_+}\cdots[\mu_p]_{\eta_+}}{[\nu_1]_{\eta_+}\cdots[\nu_q]_{\eta_+}} \frac{L_\eta^k(w)L_\eta^k(z)}{L_\eta^k(\underline{1})}$ and ${}_pF_q^k$ via the C_λ^k.
  generalized Pochhammer symbol $[\mu]_\eta = \frac{\Gamma_n(\underline{\mu}+\eta)}{\Gamma_n(\underline{\mu})}$.

Laplace Transform

For $w, z \in \mathbb{C}^n$ with small $\mathrm{Re}(z) > 0$ and $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > \mu_0$

$$\mathcal{L}_k\left({}_pK_q^k\left(\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}; w, \cdot\right)\cdot \Delta^{\mu'-\mu_0}\right)(z) = \Gamma_n(\mu')\, {}_{p+1}K_q^k\left(\begin{smallmatrix}\mu',\mu\\ \nu\end{smallmatrix}; w, \tfrac{1}{z}\right)\Delta(z)^{-\mu'}.$$

The same holds for ${}_pF_q$.

Laplace Transform of Hypergeometric series

- Choose renormalized version of non-symmetric and symmetric Jack polynomials $(L_\eta^k)_{\eta\in\mathbb{N}_0^n}$ and $(C_\lambda^k)_{\lambda\in\Lambda_+^n}$ of index $\frac{1}{k}$ s.th. $\sum_{|\eta|=m} L_\eta^k(x) = \sum_{|\lambda|=m} C_\lambda^k(x) = (x_1 + \ldots + x_n)^m$.
- hypergeometric series: ${}_pK_q^k(\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}; w, z) := \sum_{\eta\in\mathbb{N}_0^n} \frac{[\mu_1]_{\eta_+}\cdots[\mu_p]_{\eta_+}}{[\nu_1]_{\eta_+}\cdots[\nu_q]_{\eta_+}} \frac{L_\eta^k(w)L_\eta^k(z)}{L_\eta^k(\underline{1})}$ and ${}_pF_q^k$ via the C_λ^k.
  generalized Pochhammer symbol $[\mu]_\eta = \frac{\Gamma_n(\underline{\mu}+\eta)}{\Gamma_n(\underline{\mu})}$.

Laplace Transform

For $w, z \in \mathbb{C}^n$ with small $\mathrm{Re}(z) > 0$ and $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > \mu_0$

$$\mathcal{L}_k({}_pK_q^k(\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}; w, \cdot) \cdot \Delta^{\mu'-\mu_0})(z) = \Gamma_n(\mu')\,{}_{p+1}K_q^k(\begin{smallmatrix}\mu',\mu\\ \nu\end{smallmatrix}; w, \tfrac{1}{z})\Delta(z)^{-\mu'}.$$

The same holds for ${}_pF_q$.

This includes results for the Laplace transform of the Dunkl kernel $E_k^A = {}_0K_0^k$ and the Bessel function $J_k^A = {}_0F_0^k$

Application: Zeta distributions in the type B Dunkl setting

Similiar to the case of symmetric cones ($\rightarrow$ Faraut-Koranyi, Rubin, Herz...)

Application: Zeta distributions in the type B Dunkl setting

Similiar to the case of symmetric cones ($\rightarrow$ Faraut-Koranyi, Rubin, Herz...)

- Let E_κ^B be the type $B_n = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{e_i\}$ Dunkl kernel for the multiplicity $\kappa = (k, k')$.

Application: Zeta distributions in the type B Dunkl setting

Similiar to the case of symmetric cones ($\to$ Faraut-Koranyi, Rubin, Herz...)

- Let E_κ^B be the type $B_n = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{e_i\}$ Dunkl kernel for the multiplicity $\kappa = (k, k')$.
- $\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E_\kappa^B(x, \tau y) = {}_0K_1^k(\nu; \frac{x^2}{2}, \frac{y^2}{2}), \quad \nu = k' + k(n-1) + \frac{1}{2}.$

Application: Zeta distributions in the type B Dunkl setting

Similiar to the case of symmetric cones ($\to$ Faraut-Koranyi, Rubin, Herz...)

- Let E_κ^B be the type $B_n = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{e_i\}$ Dunkl kernel for the multiplicity $\kappa = (k, k')$.
- $\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E_\kappa^B(x, \tau y) = {}_0K_1^k(\nu; \frac{x^2}{2}, \frac{y^2}{2}), \quad \nu = k' + k(n-1) + \frac{1}{2}.$
- $\mathcal{Z}(f; \beta) := \int_{\mathbb{R}^n} f(x) \Delta(x^2)^\beta \omega_\kappa^B(x) \, \mathrm{d}x$, Zeta distribution: $\zeta_\alpha := \frac{\mathcal{Z}(\cdot, \alpha - \nu)}{\Gamma_n(\underline{\alpha})}$

Application: Zeta distributions in the type B Dunkl setting

Similiar to the case of symmetric cones ($\to$ Faraut-Koranyi, Rubin, Herz...)

- Let E_κ^B be the type $B_n = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{e_i\}$ Dunkl kernel for the multiplicity $\kappa = (k, k')$.
- $\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E_\kappa^B(x, \tau y) = {}_0K_1^k(\nu; \frac{x^2}{2}, \frac{y^2}{2}), \quad \nu = k' + k(n-1) + \frac{1}{2}.$
- $\mathcal{Z}(f;\beta) := \int_{\mathbb{R}^n} f(x)\Delta(x^2)^\beta \omega_\kappa^B(x)\,\mathrm{d}x$, Zeta distribution: $\zeta_\alpha := \frac{\mathcal{Z}(\cdot, \alpha-\nu)}{\Gamma_n(\underline{\alpha})}$

Theorem

1. Bernstein identity: $\Delta(T^B)^2 \zeta_\alpha = 4^n b(\alpha - \nu)\zeta_{\alpha-1}$ with a polynomial b and $\Delta(T^B)^2$ is the type B Dunkl operator associated to the polynomial Δ^2

Application: Zeta distributions in the type B Dunkl setting

Similiar to the case of symmetric cones ($\to$ Faraut-Koranyi, Rubin, Herz...)

- Let E_κ^B be the type $B_n = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{e_i\}$ Dunkl kernel for the multiplicity $\kappa = (k, k')$.
- $\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E_\kappa^B(x, \tau y) = {}_0K_1^k(\nu; \frac{x^2}{2}, \frac{y^2}{2}), \quad \nu = k' + k(n-1) + \frac{1}{2}.$
- $\mathcal{Z}(f; \beta) := \int_{\mathbb{R}^n} f(x) \Delta(x^2)^\beta \omega_\kappa^B(x)\, \mathrm{d}x$, Zeta distribution: $\zeta_\alpha := \frac{\mathcal{Z}(\cdot, \alpha - \nu)}{\Gamma_n(\underline{\alpha})}$

Theorem

1. Bernstein identity: $\Delta(T^B)^2 \zeta_\alpha = 4^n b(\alpha - \nu) \zeta_{\alpha - 1}$ with a polynomial b and $\Delta(T^B)^2$ is the type B Dunkl operator associated to the polynomial Δ^2
2. $\alpha \mapsto \zeta_\alpha$ extends to an entire map $\mathbb{C} \to \mathscr{S}'(\mathbb{R}^n)$.

Application: Zeta distributions in the type B Dunkl setting

Similiar to the case of symmetric cones ($\rightarrow$ Faraut-Koranyi, Rubin, Herz...)

- Let E_κ^B be the type $B_n = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{e_i\}$ Dunkl kernel for the multiplicity $\kappa = (k, k')$.
- $\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E_\kappa^B(x, \tau y) = {}_0K_1^k(\nu; \frac{x^2}{2}, \frac{y^2}{2}), \quad \nu = k' + k(n-1) + \frac{1}{2}.$
- $\mathcal{Z}(f; \beta) := \int_{\mathbb{R}^n} f(x) \Delta(x^2)^\beta \omega_\kappa^B(x) \, \mathrm{d}x$, Zeta distribution: $\zeta_\alpha := \frac{\mathcal{Z}(\cdot, \alpha - \nu)}{\Gamma_n(\underline{\alpha})}$

Theorem

1. Bernstein identity: $\Delta(T^B)^2 \zeta_\alpha = 4^n b(\alpha - \nu) \zeta_{\alpha - 1}$ with a polynomial b and $\Delta(T^B)^2$ is the type B Dunkl operator associated to the polynomial Δ^2
2. $\alpha \mapsto \zeta_\alpha$ extends to an entire map $\mathbb{C} \rightarrow \mathscr{S}'(\mathbb{R}^n)$.
3. $\zeta_\alpha = 2^{n(\alpha - \nu)} \mathcal{F}_\kappa^B \zeta_{\nu - \alpha}$, where $\mathcal{F}_\kappa^B$ is the type B Dunkl transform

Application: Zeta distributions in the type B Dunkl setting

Similiar to the case of symmetric cones ($\to$ Faraut-Koranyi, Rubin, Herz...)

- Let E_κ^B be the type $B_n = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{e_i\}$ Dunkl kernel for the multiplicity $\kappa = (k, k')$.
- $\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E_\kappa^B(x, \tau y) = {}_0K_1^k(\nu; \frac{x^2}{2}, \frac{y^2}{2}), \quad \nu = k' + k(n-1) + \frac{1}{2}.$
- $\mathcal{Z}(f; \beta) := \int_{\mathbb{R}^n} f(x) \Delta(x^2)^\beta \omega_\kappa^B(x)\, \mathrm{d}x$, Zeta distribution: $\zeta_\alpha := \frac{\mathcal{Z}(\cdot, \alpha - \nu)}{\Gamma_n(\underline{\alpha})}$

Theorem

1. Bernstein identity: $\Delta(T^B)^2 \zeta_\alpha = 4^n b(\alpha - \nu) \zeta_{\alpha - 1}$ with a polynomial b and $\Delta(T^B)^2$ is the type B Dunkl operator associated to the polynomial Δ^2
2. $\alpha \mapsto \zeta_\alpha$ extends to an entire map $\mathbb{C} \to \mathscr{S}'(\mathbb{R}^n)$.
3. $\zeta_\alpha = 2^{n(\alpha - \nu)} \mathcal{F}_\kappa^B \zeta_{\nu - \alpha}$, where $\mathcal{F}_\kappa^B$ is the type B Dunkl transform
4. ζ_α positive measure iff α is in the Wallach set $W_k = \{0, k, \ldots, k(n-1)\} \cup]k(n-1), \infty[$.

Application: Zeta distributions in the type B Dunkl setting

Similiar to the case of symmetric cones ($\rightarrow$ Faraut-Koranyi, Rubin, Herz...)

- Let E_κ^B be the type $B_n = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{e_i\}$ Dunkl kernel for the multiplicity $\kappa = (k, k')$.
- $\frac{1}{2^n} \sum_{\tau \in \mathbb{Z}_2^n} E_\kappa^B(x, \tau y) = {}_0K_1^k(\nu; \frac{x^2}{2}, \frac{y^2}{2}), \qquad \nu = k' + k(n-1) + \frac{1}{2}.$
- $\mathcal{Z}(f; \beta) := \int_{\mathbb{R}^n} f(x) \Delta(x^2)^\beta \omega_\kappa^B(x)\, \mathrm{d}x$, Zeta distribution: $\zeta_\alpha := \frac{\mathcal{Z}(\cdot, \alpha - \nu)}{\Gamma_n(\underline{\alpha})}$

Theorem

1. Bernstein identity: $\Delta(T^B)^2 \zeta_\alpha = 4^n b(\alpha - \nu) \zeta_{\alpha - 1}$ with a polynomial b and $\Delta(T^B)^2$ is the type B Dunkl operator associated to the polynomial Δ^2
2. $\alpha \mapsto \zeta_\alpha$ extends to an entire map $\mathbb{C} \rightarrow \mathscr{S}'(\mathbb{R}^n)$.
3. $\zeta_\alpha = 2^{n(\alpha - \nu)} \mathcal{F}_\kappa^B \zeta_{\nu - \alpha}$, where $\mathcal{F}_\kappa^B$ is the type B Dunkl transform
4. ζ_α positive measure iff α is in the Wallach set $W_k = \{0, k, \dots, k(n-1)\} \cup]k(n-1), \infty[$.
5. connection to type A Riesz-distributions (Riesz Dist. $\rightarrow$ Rösler)

Thanks for your attention!