The algebraic Heun-Askey-Wilson operator, Leonard pairs and the algebraic Bethe ansatz

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→ *P.B-Tsujimoto-Vinet-Zhedanov, arXiv:1811.11407*
→ *P.B-R. Pimenta, arXiv:1909.02464*
→ *P.B.-R. Pimenta (in progress)*

Some motivations...

- The **spectral problem (eigenvalues, eigenstates)** for a class of Hamiltonians describing a 3-sites spin chain on a triangle with inhomogeneous couplings in a magnetic field of the general form $(a,b,c=x,y,z,\ i,j=1,2,3)$:

$$\mathcal{H}=\sum_a\sum_{i,j}J^a_{ij}\sigma^a_i\sigma^a_j+\sum_{a\neq b}\sum_{i,j}K^{ab}_{ij}\sigma^a_i\sigma^b_j+\sum_{\{a,b,c\}}\sum_{i,j,k}L^{abc}_{ijk}\sigma^a_i\sigma^b_j\sigma^c_k+\sum_i b^z_i\sigma^z_i$$

Example : (N.B. Special case $\kappa=\kappa^*, j_1=j_2=j_3=1/2$, result known [Sklyanin,'88])

$$\begin{aligned}\mathcal{H} \quad &= \quad \frac{(q-q^{-1})^2}{2}\Big(\kappa(S^x_1S^x_2+S^y_1S^y_2+\Delta S^z_1S^z_2)+\kappa^*(S^x_2S^x_3+S^y_2S^y_3+\Delta S^z_2S^z_3)\\ &\quad -\frac{(q-q^{-1})}{2}(\kappa^*S^z_3-\kappa S^z_1+(\kappa-\kappa^*)S^z_2)\Big).\end{aligned}$$

- **Toy model** for N sites spin chains (q-Onsager, higher rank Askey-Wilson algebras)
- **TQ-relation** [Cao et al.,'13] $\rightarrow$ Baxter Q-polynomials vs. special functions
- The **time and band limiting problem** [Slepian,Laudau,Pollack,80's],...,[Grünbaum-Vinet-Zhedanov,'17]
- The **spectral problem** for the reduced density matrix in **free-fermion entanglement** [Eisler-Peschel,'18],...,[Crampé-Nepomechie-Vinet,'19]
- Finding a **determinental representation** for q-Askey scheme orth. polynomials

Background [Wiegmann-Zabrodin,'95],[Grünbaum-Vinet-Zhedanov,'17]

- **Heun equation** (Fuchsian form).

$$\frac{d^2}{dx^2}\psi(x)+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\epsilon}{x-d}\right)\frac{d}{dx}\psi(x)+\frac{\alpha\beta x-q}{x(x-1)(x-d)}\psi(x)=0\;,\quad \alpha+\beta-\gamma-\delta+1=0\;.$$

- **The Heun operator** M. The Heun equation can be written $M\psi(x)=\lambda\psi(x)$

$$M=x(x-1)(x-d)\frac{d^2}{dx^2}+(\rho_2x^2+\rho_1x+\rho_0)\frac{d}{dx}+r_1x+r_0\,.$$

$\rightarrow$ ρ_i, r_i parameters in $(\alpha,\beta,\gamma,\epsilon,d,\lambda)$
$\rightarrow$ M is the most general 2nd order diff. op. / $M: n\rightarrow n+1$, $deg(Poly(x))=n$.

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In [GVZ,'17] **'tridiagonalization' approach** :

$$M = \pi(\kappa A + \kappa^* A^* + \tilde{\kappa}_+ AA^* + \tilde{\kappa}_- A^* A)$$

with $A \stackrel{\pi}{\rightarrow} x\,, \qquad A^* \stackrel{\pi}{\rightarrow} D_x = x(x-1)(x-d)\frac{d^2}{dx^2} + (\alpha+1-(\alpha+\beta+2)x)\frac{d}{dx}\,.$

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Bispectral problem related with $P_n^{\alpha,\beta}(x)$ Jacobi polynomial, $\theta^*_n = -n(n+\alpha+\beta+1)$

$$\pi(A^*)P_n^{\alpha,\beta}(x) = \theta^*_n P_n^{\alpha,\beta}(x) \ , \qquad \pi(A)P_n^{\alpha,\beta}(x) \quad = \quad P_{n+1}^{\alpha,\beta}(x) + b_n P_n^{\alpha,\beta}(x) + u_n P_{n-1}^{\alpha,\beta}(x)$$

Remarks [GVZ,'17] :

$\rightarrow$ Let A, A^* satisfy **Jacobi algebra** (J) then A, M satisfy **Heun-Jacobi algebra** (HJ)

Define the embedding $\pi : J \to \mathcal{D}$ (differential operators).

$\rightarrow$ $\pi(A), \pi(A^*)$ bispectral operators + $\pi(M)$ **Heun-Jacobi operator**

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Idea [GVZ,'17] : Let $\pi(A), \pi(A^*)$ bispectral operators generating q-Askey-scheme orthogonal polynomials $\Rightarrow$ Heun-Askey-Wilson algebra ? algebraic HAW operator ? Diagonalization,... ?

The subject of this talk is to present the following results :

Let $(\bar{\pi}, \bar{V})$ be an irreducible finite dimensional representation of the **Askey-Wilson algebra** with generators A, A^*. Denote $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*)$ the corresponding **Leonard pair** of Askey-Wilson type.

$\Rightarrow$ Solve the spectral problem for $\bar{\pi}(\mathsf{I}^{AW})$ where

$$\mathsf{I}^{AW} = \kappa\,\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\,[\mathsf{A}, \mathsf{A}^*]_q + \kappa_-\,[\mathsf{A}^*, \mathsf{A}]_q \qquad , \qquad [X, Y]_q = qXY - q^{-1}YX \ .$$

Remark : For a Leonard pair of Racah type ($q = 1$ case), see [Bernard et al.,'20].

$\Rightarrow$ Relate the q-Racah polynomials and scalar products of Bethe states

Remark : Towards determinental representations for q-Racah polynomials.

AW, HAW algebras, maps and Leonard pairs

In this talk, firstly we will be interested in two algebras (AW and HAW) and how they are closely related :

- The **Askey-Wilson algebra** (AW) with generators $\{A, A^*\}$
- The **Heun-Askey-Wilson algebra** (HAW) with generators $\{X, I\}$
- The map HAW $\rightarrow$ AW

Motivated by some problems in physics, we will be interested in the **diagonalization of** operators that are **images of HAW generators** $\{\mathbf{X}, \mathbf{I}\}$ in some representations. For the applications in physics, we will need to introduce the algebra $U_q(sl_2)$ and briefly recall the notion of Leonard pair.

- The **algebra** $U_q(sl_2)$
- **Examples** of maps : AW $\rightarrow U_q(sl_2)$ and AW $\rightarrow U_q(sl_2)^{\otimes 3}$
- AW algebra and **Leonard pairs**

$\mathbb{K}$ field of characteristic zero. $q \in \mathbb{K}$ not a root of unity. $[x,y]_q = qxy - q^{-1}yx$.

Definition : [Zhedanov,'91] Let $\rho, \omega, \eta, \eta^*$ be scalars in $\mathbb{K}$. The **Askey-Wilson algebra** (AW) is generated by A, A^* subject to the relations

$$\left[\mathsf{A}, \left[\mathsf{A}, \mathsf{A}^*\right]_q\right]_{q^{-1}} = \rho\,\mathsf{A}^* + \omega\,\mathsf{A} + \eta\ ,$$
$$\left[\mathsf{A}^*, \left[\mathsf{A}^*, \mathsf{A}\right]_q\right]_{q^{-1}} = \rho\,\mathsf{A} + \omega\,\mathsf{A}^* + \eta^*\ .$$

Definition : [B-Tsujimoto-Vinet-Zhedanov,'18] Let e_i, b_i be scalars in $\mathbb{K}$. The **Heun-Askey-Wilson algebra** (HAW) is generated by X, I subject to the relations

$$[\mathsf{X}, [\mathsf{X}, \mathsf{I}]_q]_{q^{-1}} = e_1\mathsf{X}^3 + b_1\mathsf{X}^2 + b_2\{\mathsf{X}, \mathsf{I}\} + b_3\mathsf{X} + b_4\mathsf{I} + b_5\ ,$$
$$[\mathsf{I}, [\mathsf{I}, \mathsf{X}]_q]_{q^{-1}} = e_2\mathsf{X}^3 + e_3\mathsf{X}\mathsf{I}\mathsf{X} + e_4\mathsf{X}^2 + b_1'\{\mathsf{X}, \mathsf{I}\} + b_2\mathsf{I}^2 + b_3'\mathsf{I} + b_6\mathsf{X} + b_7\ ,$$

$$e_3 = e_1(q^2 + q^{-2} + 1), \quad b_1' = b_1 + e_1b_2, \quad b_3' = b_3 + e_1b_4.$$

Special case : for $e_i = 0\ \forall i$, HAW algebra reduces to the AW algebra.

$$\mathsf{X} \mapsto \mathsf{A}\ ,\ \ \mathsf{I} \mapsto \mathsf{A}^* + a^*$$

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Proposition : [BTVZ,'18] Let $\kappa, \kappa^*, \kappa_\pm \in \mathbb{K}$. There exists $\phi : HAW \to AW$:

$$\mathsf{X} \mapsto \mathsf{A}\ , \qquad \mathsf{I} \mapsto \kappa\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+ \left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_- \left[\mathsf{A}^*, \mathsf{A}\right]_q\ .$$

In the context of physics, the **AW/HAW** algebras essentially arise in relation with the quantum algebra $U_q(sl_2)$. For applications (e.g. quantum Euler top, three spins interacting systems), the following material is needed :

Definition : The algebra $U_q(sl_2)$ is generated by $\{q^{\pm s_3}, S_\pm\}$ subject to the relations :

$$[s_3, S_\pm] = \pm S_\pm \ , \quad [S_+, S_-] = \frac{q^{2s_3} - q^{-2s_3}}{q - q^{-1}} \ .$$

Casimir element : $\quad C = (q-q^{-1})^2 S_- S_+ + q^{2s_3+1} + q^{-2s_3-1}$

Coproduct : $\Delta : U_q(sl_2) \to U_q(sl_2) \otimes U_q(sl_2)$:

$$\Delta(S_+) = S_+ \otimes 1 + q^{2s_3} \otimes S_+ \quad \Delta(S_-) = S_- \otimes q^{-2s_3} + 1 \otimes S_- \ , \quad \Delta(q^{s_3}) = q^{s_3} \otimes q^{s_3} \ .$$

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Example 1. [Granovskii-Zhedanov,'93] Let $k_\pm, \epsilon_\pm \in \mathbb{K}$. Map $AW \to U_q(sl_2)$:

$$\begin{aligned} \mathsf{A} \quad &\to \quad k_+ v q^{1/2} S_+ q^{s_3} + k_- v^{-1} q^{-1/2} S_- q^{s_3} + \epsilon_+ q^{2s_3} \ , \\ \mathsf{A}^* \quad &\to \quad k_+ v^{-1} q^{-1/2} S_+ q^{-s_3} + k_- v q^{1/2} S_- q^{-s_3} + \epsilon_- q^{-2s_3} \ . \end{aligned}$$

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Example 2. [Granovskii-Zhedanov,'93] Map (U)$AW \to U_q(sl_2) \otimes U_q(sl_2) \otimes U_q(sl_2)$:

$$\mathsf{A} \to \Delta(C) \otimes 1\ , \qquad \mathsf{A}^* \to 1 \otimes \Delta(C)\ .$$

In the context of physics, **eigenproblem for images of AW/HAW generators** $\{A, A^*\}/\{X, I\}$ arise. A classification of irreducible finite dimensional representations of AW algebra is needed.

Definition :[Terwilliger,'01]. Let $\bar{V}$ be a finite dimensional vector space. A **Leonard pair** A, A^* is such that :

(i) in the eigenbasis of A, then A^* acts as a tridiagonal matrix ;
(ii) in the eigenbasis of A^*, then A acts as a tridiagonal matrix.

Theorem :[Terwilliger-Vidunas,'04]. Let A, A^* denote a Leonard pair on $\bar{V}$. Then there exists a sequence of scalars $\{\beta, \gamma, \gamma^*, \omega, \eta, \eta^*\} \in \mathbb{K}$ such that

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In the following analysis, we consider families of **Leonard pairs** such that :

$\beta = q^2 + q^{-2}$, $\rho = \rho^*$ and $\gamma = \gamma^* = 0$ $\Rightarrow$ **Askey-Wilson type**

PROBLEM 1

Let $(\bar{\pi}, \bar{V})$ be an irreducible finite dimensional representation of the **Askey-Wilson algebra** with generators A, A^*. Denote $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*)$ the corresponding **Leonard pair of Askey-Wilson type**.

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Remark : Solution well-known for $\kappa = 1, \kappa^* = \kappa_\pm = 0$ (or $\kappa^* = 1, \kappa = \kappa_\pm = 0$)
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Strategy : **AW algebra** $\sim$ **Example of reflection algebra**

$\Rightarrow$ HAW operator $\bar{\pi}(\mathsf{I}^{AW}) \sim$ Transfer matrix

$\Rightarrow$ Apply **(Modified) algebraic Bethe Ansatz (ABA)**

- **Step 1 :** Use the reflection algebra presentation of AW algebra. **Relate the image of the HAW generator** I in AW to a generating function $t(u)$:

$$t(u) = (q^2u^2 - q^{-2}u^{-2})(u^2 - u^{-2}) \underbrace{\left(\kappa \, \mathsf{A} + \kappa^* \mathsf{A}^* + \kappa_+ \left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_- \left[\mathsf{A}^*, \mathsf{A}\right]_q\right)}_{\equiv \mathsf{I}^{AW}}$$

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• **Step 2 :** Let $(\bar{\pi}, \bar{V})$ such that $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*)$ is a Leonard pair. **Diagonalize the transfer matrix** $\bar{\pi}(t(u))$ using the framework of ABA $\Rightarrow$

$$\bar{\pi}\left(\mathsf{I}^{AW}\right) |\Psi^M_{a,\epsilon}(\bar{u})\rangle = \Lambda^M_{a,\epsilon} |\Psi^M_{a,\epsilon}(\bar{u})\rangle$$

is **diagonalized** for :
- $\rightarrow$ Special cases ('a'=sp) $\kappa^* = \kappa_\pm = 0$ or $\kappa = \kappa_\pm = 0$;
- $\rightarrow$ Diagonal case ('a'=d) $\kappa_\pm = 0,\ \kappa, \kappa^* \neq 0$;
- $\rightarrow$ Generic case ('a'=g) $\kappa, \kappa^*, \kappa_\pm \neq 0$.

The **eigenstates** are **Bethe states** of the general form $(a \in \{sp, d, g\}$:

$$|\Psi^M_{a,\pm}(\bar{u}, m_0)\rangle = \bar{\pi}(\mathscr{B}^\pm(u_1, m_0 + 2(M-1)) \cdots \mathscr{B}^\pm(u_M, m_0))|\Omega^\pm\rangle\,, \qquad \mathscr{B}^\pm(u) \in \mathbb{K}[u, u^{-1}] \otimes AW$$

The Bethe roots $\bar{u} = \{u_1, ..., u_M\}$ satisfy a system of **Bethe equations** :

$$P_a(U_i) = 0 \ \ i = 1, \ldots, dim(\bar{V}) \quad \text{where} \quad U_i = \frac{qu_i^2 + q^{-1}u_i^{-2}}{q + q^{-1}} \quad \text{with} \quad a \in \{sp, d, g\}.$$

Step 1-a : A reflection algebra presentation for the AW algebra

Recall the **AW algebra** with defining relations :

$$[\mathsf{A},[\mathsf{A},\mathsf{A}^*]_q]_{q^{-1}} = \rho\,\mathsf{A}^* + \omega\,\mathsf{A} + \eta\;, \qquad [\mathsf{A}^*,[\mathsf{A}^*,\mathsf{A}]_q]_{q^{-1}} = \rho\,\mathsf{A} + \omega\,\mathsf{A}^* + \eta^*\;.$$

Define the R-matrix and K-operator :

$$R(u) = \begin{pmatrix} uq-u^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & u-u^{-1} & q-q^{-1} & 0 \\ 0 & q-q^{-1} & u-u^{-1} & 0 \\ 0 & 0 & 0 & uq-u^{-1}q^{-1} \end{pmatrix}, \qquad K(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix},$$

$$\mathcal{A}(u) = (u^2-u^{-2})\left(qu\,\mathsf{A} - q^{-1}u^{-1}\mathsf{A}^*\right) - (q+q^{-1})\rho^{-1}\left(\eta u + \eta^* u^{-1}\right),$$

$$\mathcal{D}(u) = (u^2-u^{-2})\left(qu\,\mathsf{A}^* - q^{-1}u^{-1}\mathsf{A}\right) - (q+q^{-1})\rho^{-1}\left(\eta^* u + \eta u^{-1}\right),$$

$$\mathcal{B}(u) = \chi(u^2-u^{-2})\left(\rho^{-1}\left(\left[\mathsf{A}^*,\mathsf{A}\right]_q + \frac{\omega}{q-q^{-1}}\right) + \frac{qu^2+q^{-1}u^{-2}}{q^2-q^{-2}}\right),$$

$$\mathcal{C}(u) = \rho\chi^{-1}(u^2-u^{-2})\left(\rho^{-1}\left(\left[\mathsf{A},\mathsf{A}^*\right]_q + \frac{\omega}{q-q^{-1}}\right) + \frac{qu^2+q^{-1}u^{-2}}{q^2-q^{-2}}\right).$$

Proposition :(first example in [Zabrodin,'95]) [B,'04] The AW algebra admits a presentation in the form of the a **reflection algebra** :

$$R(u/v)\;(K(u)\otimes I\!I)\;R(uv)\;(I\!I\otimes K(v)) \;=\; (I\!I\otimes K(v))\;R(uv)\;(K(u)\otimes I\!I)\;R(u/v)\;.$$

Step 1-b : Generating function $t(u)$ and the HAW generator I^{AW}

Given a reflection algebra 'RKRK' with a K-operator $K(u)$ a **generating function for mutually commuting elements** is provided by ($\rightarrow$ transfer matrix) [Sklyanin,'88]
$t(u) = \text{tr}\left(K^+(u)K(u)\right)$ where

$$K^+(u) = \begin{pmatrix} qu\kappa + q^{-1}u^{-1}\kappa^* & \kappa_+(q^2u^2 - q^{-2}u^{-2}) \\ \kappa_-\rho(q^2u^2 - q^{-2}u^{-2}) & qu\kappa^* + q^{-1}u^{-1}\kappa \end{pmatrix},$$

satisfies the "dual" reflection equation given by [DeVega-Gonzales-Ruiz,'93]. Using the K-matrix in terms of A, A^* one gets :

$$t(u) = (q^2u^2 - q^{-2}u^{-2})(u^2 - u^{-2})\underbrace{\left(\kappa\,\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_-\left[\mathsf{A}^*, \mathsf{A}\right]_q\right)}_{=\text{ image of HAW generator : } \mathsf{I}^{AW}} + \underbrace{\mathcal{F}_0(u)}_{\text{scalar function}},$$

Diagonalization of $\bar{\pi}(t(u))$ using ABA $\Leftrightarrow$ Diagonalization of $\bar{\pi}(\mathsf{I}^{AW})$

Hint : combine three tools...

$\rightarrow$ Theory of Leonard pairs [Terwilliger et al.,'99],
$\rightarrow$ Gauge transformations for open systems [Cao et al.,'03],
$\rightarrow$ Modified algebraic Bethe ansatz [Belliard-Crampé et al.,'13-'15].

Step 2-a : Diagonalization of $t(u)$ using ABA (reference states)

• **Reference states.** In order to diagonalize $t(u) = \text{tr}\left(K^+(u)K(u)\right)$ using ABA, where

$$K^+(u) = \begin{pmatrix} qu\kappa + q^{-1}u^{-1}\kappa^* & \kappa_+(q^2u^2 - q^{-2}u^{-2}) \\ \kappa_-\rho(q^2u^2 - q^{-2}u^{-2}) & qu\kappa^* + q^{-1}u^{-1}\kappa \end{pmatrix}, \quad K(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix},$$

the starting point is the identification of a **reference state** $|\Omega\rangle$ such that

$$\bar{\pi}(\mathcal{C}(u))|\Omega\rangle = 0 \quad \text{where} \quad \mathcal{C}(u) = \rho\chi^{-1}(u^2 - u^{-2})\left(\rho^{-1}\left(\left[\mathsf{A}, \mathsf{A}^*\right]_q + \frac{\omega}{q - q^{-1}}\right) + \frac{qu^2 + q^{-1}u^{-2}}{q^2 - q^{-2}}\right).$$

.

Problem : for a Leonard pair $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*)$, no solution !

Idea :

→ **Introduce a (gauge) transformation** such that $t(u) = \text{tr}\left(\tilde{K}^+(u|m)K(u|m,\epsilon)\right)$

$$K(u|m,\epsilon) = \begin{pmatrix} \mathscr{A}^\epsilon(u|m) & \mathscr{B}^\epsilon(u|m) \\ \mathscr{C}^\epsilon(u|m) & \mathscr{D}^\epsilon(u|m) \end{pmatrix}, \quad \epsilon = \pm, \to \{\mathscr{A}^\epsilon, \ldots\} \in \mathbb{K}[u, u^{-1}] \otimes AW$$ **'dynamical' operators**

→ **Identify references states** (= pseudo-vacuum states) $|\Omega^\pm\rangle$ such that

$$\bar{\pi}(\mathscr{C}^\pm(u|m))|\Omega^\pm\rangle = 0.$$

→ **Gauge transformations :**

Let $\epsilon = \pm 1$, α, β be generic complex parameters and m be an integer. Introduce the covariant (resp. contravariant) vectors [Cao et al., '03]

$$|X^\epsilon(u,m)\rangle = \begin{pmatrix} \alpha q^{\epsilon m} u^\epsilon \\ 1 \end{pmatrix}, \quad |Y^\epsilon(u,m)\rangle = \begin{pmatrix} \beta q^{-\epsilon m} u^\epsilon \\ 1 \end{pmatrix}$$

$$\langle \tilde{X}^\epsilon(u,m)| = -\epsilon \frac{q^{-\epsilon} u^{-\epsilon}}{\gamma_{m-1}} \begin{pmatrix} -1 & \alpha q^{\epsilon m} u^\epsilon \end{pmatrix}, \quad \langle \tilde{Y}^\epsilon(u,m)| = -\epsilon \frac{q^{-\epsilon} u^{-\epsilon}}{\gamma_{m+1}} \begin{pmatrix} 1 & -\beta q^{-\epsilon m} u^\epsilon \end{pmatrix}$$

where $\gamma^\epsilon(u,m) = \alpha^{\frac{1-\epsilon}{2}} \beta^{\frac{\epsilon+1}{2}} q^{-m} u - \alpha^{\frac{\epsilon+1}{2}} \beta^{\frac{1-\epsilon}{2}} q^m u^{-1}$.

Applying the gauge transformation to $K(u)$, the entries of $K(u|m)$ are given by :

$$\mathscr{A}^\epsilon(u,m) = \langle \tilde{Y}^\epsilon(u,m-2)|K(u)|X^\epsilon(u^{-1},m)\rangle, \quad \mathscr{B}^\epsilon(u,m) = \langle \tilde{Y}^\epsilon(u,m)|K(u)|Y^\epsilon(u^{-1},m)\rangle,$$

$$\mathscr{C}^\epsilon(u,m) = \langle \tilde{X}^\epsilon(u,m)|K(u)|X^\epsilon(u^{-1},m)\rangle,$$

$$\mathscr{D}^\epsilon(u,m) = \frac{\gamma^\epsilon(1,m+1)}{\gamma^\epsilon(1,m)} \langle \tilde{X}^\epsilon(u,m+2)|K(u)|Y^\epsilon(u^{-1},m)\rangle - \frac{(q-q^{-1})\gamma^\epsilon(u^{-2},m+1)}{(qu^2 - q^{-1}u^{-2})\gamma^\epsilon(1,m)} \mathscr{A}^\epsilon(u,m).$$

Example :

$$\mathscr{C}^+(u,m) = \frac{\alpha b(u^2)}{\alpha - \beta q^{2-2m}} \Big(\frac{\alpha q^m}{u\chi} [\mathsf{A}, \mathsf{A}^*]_q - \frac{\chi q^{-m}}{\alpha \rho u} [\mathsf{A}^*, \mathsf{A}]_q - \frac{(q^2+1)}{qu} \mathsf{A} + (\frac{1}{qu^3} + qu)\mathsf{A}^* \Big) + f_0(u)\ .$$

→ **Reference state.**

The gauge transformation depends on some parameters α, β and ϵ. ⇒ We fix α, β such that $|\Omega^{\pm}\rangle$ is a reference state. For a **Leonard pair of q-Racah type** [Terwilliger,'99], use :

$$\bar{\pi}(\mathsf{A})|\theta_M\rangle = \theta_M|\theta_M\rangle\,, \quad \bar{\pi}(\mathsf{A}^*)|\theta_M\rangle = a_{M,M+1}|\theta_{M+1}\rangle + a_{M,M}|\theta_M\rangle + a_{M,M-1}|\theta_{M-1}\rangle\,,$$
$$\bar{\pi}(\mathsf{A}^*)|\theta^*_M\rangle = \theta^*_M|\theta^*_M\rangle\,, \quad \bar{\pi}(\mathsf{A})|\theta^*_M\rangle = a^*_{M,M+1}|\theta^*_{M+1}\rangle + a^*_{M,M}|\theta^*_M\rangle + a^*_{M,M-1}|\theta^*_{M-1}\rangle\,,$$

where $a_{0,-1} = a_{2s,2s+1} = a^*_{0,-1} = a^*_{2s,2s+1} = 0$ and

$$\theta_M = bq^{2M} + cq^{-2M}\,, \quad \theta^*_M = b^*q^{2M} + c^*q^{-2M}\,,$$

Lemma 1. Let m_0 be an integer. If the parameters α, β are such that :

$$(q^2 - q^{-2})\alpha c^* q^{m_0} = 1 \qquad (\text{resp. } (q^2 - q^{-2})\beta c^* q^{-m_0} = 1)$$

then $|\Omega^+\rangle \equiv |\theta^*_0\rangle$ satisfies

$$\bar{\pi}(\mathscr{C}^+(u, m_0))|\Omega^+\rangle = 0 \qquad (\text{resp. } \bar{\pi}(\mathscr{B}^+(u, m_0))|\Omega^+\rangle = 0).$$

Lemma 2. Other constraints on α, β for $\bar{\pi}(\mathscr{C}^-(u, m_0))|\Omega^-\rangle = 0$ (resp. $\bar{\pi}(\mathscr{B}^-(u, m_0))|\Omega^-\rangle = 0$).

⇒ **Two families of eigenstates will be considered, starting from $|\Omega^{\pm}\rangle$**

Step 2-b : Diagonalization of $\bar{\pi}(t(u))$ using ABA (dynamical operators/Bethe states)

Given a gauge transformation (α, β, ϵ fixed), the **generating function** now reads as a combination of 'dynamical' operators. $t(u) = \mathbb{K}[u, u^{-1}] \otimes AW$ is :

$$t(u) = a(u,m)\mathscr{A}^{\epsilon}(u,m) + d(u,m)\mathscr{D}^{\epsilon}(u,m) + b(u,m)\mathscr{B}^{\epsilon}(u,m) + c(u,m)\mathscr{C}^{\epsilon}(u,m)$$

In the **algebraic Bethe ansatz approach**, the idea is to start from candidates for eigenstates of $\bar{\pi}(t(u))$ of the form :

$$|\Psi^M_{\pm}(\bar{u}, m_0)\rangle = \bar{\pi}(\mathscr{B}^{\pm}(u_1, m_0 + 2(M-1)) \cdots \mathscr{B}^{\pm}(u_M, m_0))|\Omega^{\pm}\rangle$$

Then, provided we know :

→ Action of the dynamical operators $\{\mathscr{A}^{\epsilon}(u,m), \ldots\}$ on the reference states $|\Omega^{\pm}\rangle$
→ Commutation relations between dynamical operators $\{\mathscr{A}^{\epsilon}(u,m), \ldots\}$

one can compute the action of $\bar{\pi}(t(u))$ on the candidates for eigenstates. It reads :

$$\bar{\pi}(t(u))|\Psi^M_{\pm}(\bar{u}, m_0)\rangle = \Lambda^M_{\pm}(u; \bar{u})|\Psi^M_{\pm}(\bar{u}, m_0)\rangle \; + \; \underbrace{\sum_i E_i |\Psi^M_{\pm}(\{u, \bar{u}_i\}, m_0)\rangle}_{E_i \equiv 0 \text{ for some } \{u_i\} \text{ solving Bethe equations}}$$

$(\bar{u}_i = \bar{u} \backslash u_i)$

→ Action of the dynamical operators

For some α, β, recall that one has :

$$\bar{\pi}(\mathscr{C}^{\pm}(u, m_0))|\Omega^{\pm}\rangle = 0 \ .$$

Lemma 3. Let α, β be fixed such that $|\Omega^{\pm}\rangle$ are references states. Then :

$$\begin{aligned} \bar{\pi}(\mathscr{A}^{\pm}(u, m_0))|\Omega^{\pm}\rangle &= \Lambda_1^{\pm}(u)|\Omega^{\pm}\rangle \ , \\ \bar{\pi}(\mathscr{D}^{\pm}(u, m_0))|\Omega^{\pm}\rangle &= \Lambda_2^{\pm}(u)|\Omega^{\pm}\rangle \ , \end{aligned}$$

where $\Lambda_1^{\pm}(u), \Lambda_2^{\pm}(u)$ are ratios of Laurent polynomials in the variable u.

Proof : Use Leonard pair's properties.

→ Commutation relations between the dynamical operators

$$\begin{aligned} \mathscr{B}^{\epsilon}(u, m+2)\mathscr{B}^{\epsilon}(v, m) &= \mathscr{B}^{\epsilon}(v, m+2)\mathscr{B}^{\epsilon}(u, m), \\ \mathscr{A}^{\epsilon}(u, m+2)\mathscr{B}^{\epsilon}(v, m) &= f(u, v)\mathscr{B}^{\epsilon}(v, m)\mathscr{A}^{\epsilon}(u, m) \\ &\quad + g(u, v, m)\mathscr{B}^{\epsilon}(u, m)\mathscr{A}^{\epsilon}(v, m) + w(u, v, m)\mathscr{B}^{\epsilon}(u, m)\mathscr{D}^{\epsilon}(v, m), \\ \mathscr{D}^{\epsilon}(u, m+2)\mathscr{B}^{\epsilon}(v, m) &= h(u, v)\mathscr{B}^{\epsilon}(v, m)\mathscr{D}^{\epsilon}(u, m), \\ &\quad + k(u, v, m)\mathscr{B}^{\epsilon}(u, m)\mathscr{D}^{\epsilon}(v, m) + n(u, v, m)\mathscr{B}^{\epsilon}(u, m)\mathscr{A}^{\epsilon}(v, m), \\ \mathscr{C}(u, m+2)\mathscr{B}(v, m) &= \mathscr{B}(v, m-2)\mathscr{C}(u, m) \\ &\quad + q(u, v, m)\mathscr{A}(v, m)\mathscr{D}(u, m) + r(u, v, m)\mathscr{A}(u, m)\mathscr{D}(v, m) \\ &\quad + s(u, v, m)\mathscr{A}(u, m)\mathscr{A}(v, m) + x(u, v, m)\mathscr{A}(v, m)\mathscr{A}(u, m) \\ &\quad + y(u, v, m)\mathscr{D}(u, m)\mathscr{A}(v, m) + z(u, v, m)\mathscr{D}(u, m)\mathscr{D}(v, m). \end{aligned}$$

Goal : Diagonalize $\bar{\pi}(\mathsf{I}^{AW})$.

$$t(u) = (q^2u^2 - q^{-2}u^{-2})(u^2 - u^{-2}) \underbrace{\left(\kappa\,\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\left[\mathsf{A},\mathsf{A}^*\right]_q + \kappa_-\left[\mathsf{A}^*,\mathsf{A}\right]_q\right)}_{\equiv \mathsf{I}^{AW}}$$

$$\bar{\pi}(t(u))|\Psi^M_\pm(\bar{u},m_0)\rangle = \Lambda^M_\pm(u;\bar{u})|\Psi^M_\pm(\bar{u},m_0)\rangle \; + \; \underbrace{\sum_i E_i|\Psi^M_\pm(\{u,\bar{u}_i\},m_0)\rangle}$$

$(\bar{u}_i = \bar{u}\backslash_{u_i})$

$E_i \equiv 0$ for some $\{u_i\}$ solving Bethe equations

for
- → Special cases ('a'=sp) $\kappa^* = \kappa_\pm = 0$ or $\kappa = \kappa_\pm = 0$;
- → Diagonal case ('a'=d) $\kappa_\pm = 0,\ \kappa,\kappa^* \neq 0$;
- → Generic case ('a'=g) $\kappa,\kappa^*,\kappa_\pm \neq 0$.

Goal : Diagonalize $\bar{\pi}(\mathsf{I}^{AW})$.

$$t(u) = (q^2u^2 - q^{-2}u^{-2})(u^2 - u^{-2}) \underbrace{\left(\kappa \mathsf{A} + \kappa^* \mathsf{A}^* + \kappa_+ [\mathsf{A}, \mathsf{A}^*]_q + \kappa_- [\mathsf{A}^*, \mathsf{A}]_q\right)}_{\equiv \mathsf{I}^{AW}}$$

$$\bar{\pi}(t(u))|\Psi^M_\pm(\bar{u}, m_0)\rangle = \Lambda^M_\pm(u; \bar{u})|\Psi^M_\pm(\bar{u}, m_0)\rangle \ + \underbrace{\sum_i E_i |\Psi^M_\pm(\{u, \bar{u}_i\}, m_0)\rangle}_{E_i \equiv 0 \text{ for some } \{u_i\} \text{ solving Bethe equations}}$$

$(\bar{u}_i = \bar{u} \backslash u_i)$

for
- → Special cases ('a'=sp) $\kappa^* = \kappa_\pm = 0$ or $\kappa = \kappa_\pm = 0$;
- → Diagonal case ('a'=d) $\kappa_\pm = 0,\ \kappa, \kappa^* \neq 0$;
- → Generic case ('a'=g) $\kappa, \kappa^*, \kappa_\pm \neq 0$.

Solution :

- **Spectrum :** $\Lambda^M_{a,\pm} = f(\{u_1, ..., u_M\})$
- **Bethe eigenstates :**

$$|\Psi^M_\pm(\bar{u}, m_0)\rangle = \bar{\pi}(\mathscr{B}^\pm(u_1, m_0 + 2(M-1)) \cdots \mathscr{B}^\pm(u_M, m_0))|\Omega^\pm\rangle$$

- **Bethe equations :**

$$P_a(U_i) = 0 \ \ i = 1, ..., dim(\bar{V}) \quad \text{where} \quad U_i = \frac{qu_i^2 + q^{-1}u_i^{-2}}{q + q^{-1}} \quad \text{with} \quad a \in \{sp, d, g\}.$$

Special case : $\kappa^* = \kappa_\pm = 0$ or $\kappa = \kappa_\pm = 0$

Recall that $\mathsf{I}^{AW}|_{(\kappa,\kappa^*,\kappa_+,\kappa_-)} = \kappa\,\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\,[\mathsf{A},\mathsf{A}^*]_q + \kappa_-\,[\mathsf{A}^*,\mathsf{A}]_q$

Proposition 1 :

$$\bar{\pi}\left(\mathsf{I}^{AW}|_{(\kappa,0,0,0)}\right)|\Psi^M_{sp,-}(\bar{u},m_0)\rangle = \frac{\kappa}{2}q^{\frac{1}{2}(\nu+\nu')}\left(e^{-\mu}q^{-2s+2M} + e^{\mu}q^{2s-2M}\right)|\Psi^M_{sp,-}(\bar{u},m_0)\rangle$$

where the set $\bar{u} = \{u_1, u_2, \ldots, u_M\}$ satisfies the Bethe equations for $i = 1, \ldots, M$:

$$\prod_{j=1, j\neq i}^{M}\left(\frac{b(u_i/(qu_j))b(u_iu_j)}{b(qu_i/u_j)b(q^2u_iu_j)}\right) = \frac{\left(qe^{\mu'}u_i + q^{-1}e^{\mu}u_i^{-1}\right)\left(qe^{-\mu}u_i + q^{-1}e^{\mu'}u_i^{-1}\right)b\left(q^{\frac{1}{2}-s}vu_i\right)b\left(q^{\frac{1}{2}-s}v^{-1}u_i\right)}{\left(e^{\mu'}u_i + e^{-\mu}u_i^{-1}\right)\left(e^{\mu}u_i + e^{\mu'}u_i^{-1}\right)b\left(q^{s+\frac{1}{2}}vu_i\right)b\left(q^{s+\frac{1}{2}}v^{-1}u_i\right)}.$$

Remark :

$\longrightarrow$ Spectrum has the typical form for Leonard pairs ($\theta_M = bq^{2M} + cq^{-2M}$)

$\longrightarrow$ Similar result for the special case $\kappa = 0$ (**Proposition 1* :**) :

$$\bar{\pi}\left(\mathsf{I}^{AW}|_{(0,\kappa^*,0,0)}\right)|\Psi^M_{sp,+}(\bar{u},m_0)\rangle = \frac{\kappa^*}{2}q^{\frac{1}{2}(\nu+\nu')}\left(e^{-\mu'}q^{2s-2M} + e^{\mu'}q^{-2s+2M}\right)|\Psi^M_{sp,+}(\bar{u},m_0)\rangle$$

Diagonal case : $\kappa_\pm = 0,\ \kappa, \kappa^* \neq 0$

Recall that $\mathsf{I}^{AW}|_{(\kappa,\kappa^*,\kappa_+,\kappa_-)} = \kappa\,\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\,[\mathsf{A},\mathsf{A}^*]_q + \kappa_-\,[\mathsf{A}^*,\mathsf{A}]_q$

Proposition 2 : For $\epsilon = \pm 1$, one has :

$$\bar{\pi}\left(\mathsf{I}^{AW}|_{(\kappa,\kappa^*,0,0)}\right)|\Psi^{2s}_{d,\epsilon}(\bar{u},m_0)\rangle = \Lambda^{2s}_{d,\epsilon}|\Psi^{2s}_{d,\epsilon}(\bar{u},m_0)\rangle$$

with

$$\Lambda^{2s}_{d,+} = \kappa^*\theta^*_{2s} + \kappa e^{\mu-\mu'} b\Big((v^2+v^{-2})[2s]_q + 2e^{\mu'}\cosh(\mu) - q\sum_{j=1}^{2s}(qu_j^2+q^{-1}u_j^{-2})\Big)\,,$$

$$\Lambda^{2s}_{d,-} = \kappa\theta_{2s} + \kappa^* e^{\mu'-\mu} c^*\Big((v^2+v^{-2})[2s]_q + 2e^{\mu}\cosh(\mu') - q^{-1}\sum_{j=1}^{2s}(qu_j^2+q^{-1}u_j^{-2})\Big)\,,$$

where the set $\bar{u}$ satisfies the (inhomogeneous) Bethe equations for $i = 1,\ldots,2s$:

$$\frac{b(u_i^2)}{b(qu_i^2)}(\kappa\,u_i+\kappa^*u_i^{-1})\prod_{j=1,j\neq i}^{2s} f(u_i,u_j)\Lambda_1^\epsilon(u_i) - q^{-\epsilon}u_i^{-2\epsilon}(q\kappa^*u_i+q^{-1}\kappa u_i^{-1})\prod_{j=1,j\neq i}^{2s} h(u_i,u_j)\Lambda_2^\epsilon(u_i)$$
$$\underbrace{+(-1)^{2s}\,\epsilon(q-q^{-1})^{-1}q^\epsilon\kappa^{(1+\epsilon)/2}\kappa^{*(1-\epsilon)/2}\delta_d\frac{u_i^{-2\epsilon}b(u_i^2)\prod_{k=0}^{2s}b(q^{1/2+k-s}vu_i)b(q^{1/2+k-s}v^{-1}u_i)}{\prod_{j=1,j\neq i}^{2s}b(u_iu_j^{-1})b(qu_iu_j)}}_{\text{inhomogeneous term}} = 0.$$

Generic case : $\kappa_\pm \neq 0,\ \kappa, \kappa^* \neq 0$

Recall that $\mathsf{I}^{AW}|_{(\kappa,\kappa^*,\kappa_+,\kappa_-)} = \kappa\,\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\,[\mathsf{A},\mathsf{A}^*]_q + \kappa_-\,[\mathsf{A}^*,\mathsf{A}]_q$

Proposition 3 : For $\epsilon = \pm 1$, one has :

$$\bar\pi\left(\mathsf{I}^{AW}|_{(\kappa,\kappa^*,\kappa_+,\kappa_-)}\right)|\Psi^{2s}_{g,\epsilon}(\bar u, m_0)\rangle = \Lambda^{2s}_{g,\epsilon}|\Psi^{2s}_{g,\epsilon}(\bar u, m_0)\rangle$$

with

$$\Lambda^{2s}_{g,-} = \kappa\theta_{2s} + \kappa\theta_{2s}|_{\mu\to-\xi'}\frac{\cosh(\mu')}{\cosh(\xi)} + \left(-\frac{\kappa}{2\cosh(\xi)}\theta_{3s+1/2}|_{\mu\to\mu-\xi'} + (-1)^{2s+1}\delta_g[2s+1]_q\right)(v^2+v^{-2})$$
$$-\omega\frac{(\chi^{-1}\kappa_+ + \chi\kappa_-)}{(q-q^{-1})} + \left(\frac{\kappa}{2\cosh(\xi)}(q-q^{-1})\theta_{2s}|_{\mu\to\mu-\xi} - (-1)^{2s+1}\delta_g\right)\sum_{j=1}^{2s}(qu_j^2 + q^{-1}u_j^{-2})\,,$$

where the set $\bar u$ satisfies the (inhomogeneous) Bethe equations for $i = 1,\dots,2s$:

$$-\frac{b(u_i^2)}{b(qu_i^2)}\Delta_g(u_i)\prod_{j=1,j\neq i}^{2s} f(u_i,u_j)\Lambda_1^\epsilon(u_i) + \Delta_g(q^{-1}u_i^{-1})\prod_{j=1,j\neq i}^{2s} h(u_i,u_j)\Lambda_2^\epsilon(u_i)$$
$$-(-1)^{2s}\delta_g(q-q^{-1})^{-1}\frac{u_i^{-\epsilon}b(u_i^2)\prod_{k=0}^{2s} b(q^{1/2+k-s}vu_i)b(q^{1/2+k-s}v^{-1}u_i)}{\prod_{j=1,j\neq i}^{2s} b(u_iu_j^{-1})b(qu_iu_j)} = 0$$

Examples : 3-sites Heisenberg spin-$\frac{1}{2}$ chain

$$\mathsf{I} = \kappa \mathsf{A} + \kappa^* \mathsf{A}^* + \kappa_+ \left[\mathsf{A}, \mathsf{A}^*\right]_q + \kappa_- \left[\mathsf{A}^*, \mathsf{A}\right]_q \quad \text{with}$$

$$\mathsf{A} \rightarrow \Delta(C) \otimes I\!I \,, \quad \mathsf{A}^* \rightarrow I\!I \otimes \Delta(C) \,, \quad \text{Casimir } U_q(sl_2): \; C = (q-q^{-1})^2 S_- S_+ + q^{2S_z+1} + q^{-2S_z-1}$$

Fix $\bar{\mathcal{V}} = \mathbb{C}^2 \Rightarrow a, b, c = x, y, z, \; i, j = 1, 2, 3, \; \bar{\pi}(S_a) \rightarrow \sigma^a$.

$$\frac{\bar{\pi}(\mathsf{I}|_{(\kappa,\kappa^*,\kappa_+,\kappa_-)})}{2(q-q^{-1})^2} = \sum_a \sum_{i,j} J^a_{ij} \sigma^a_i \sigma^a_j + \sum_{a\neq b} \sum_{i,j} K^{ab}_{ij} \sigma^a_i \sigma^b_j + \sum_{\{a,b,c\}} \sum_{i,j,k} L^{abc}_{ijk} \sigma^a_i \sigma^b_j \sigma^c_k + \sum_i b^z_i \sigma^z_i + J_0 \,,$$

Spin-$\frac{1}{2}$ chain	Direct diagonalization (degeneracy)	Diagonalization via ABA $\Lambda^{2s}_{a,+}$ (s)	Bethe roots $\{U_1, \ldots, U_{2s}\}$
$\kappa = 1$, $\kappa^* = 1/3$, $\kappa_\pm = 0$	32.5 (4) 14.4069 (2) 28.0931 (2)	32.5 (0) 14.4069 (1/2) 28.0931 (1/2)	- {-1.0344} {-1.4906}
$\kappa = -\frac{5}{4\sqrt{2}}$, $\kappa^* = -\frac{9}{4\sqrt{2}}$, $\kappa_+ = -\frac{2}{15}$, $\kappa_- = \frac{1}{60}$	-0.200512 (4) -6.25895 + 3.32745 i (2) -6.25895 - 3.32745 i (2)	-0.200512 (0) -6.25895 + 3.32745 i (1/2) -6.25895 - 3.32745 i (1/2)	- {-0.793147 - 1.40509 i} {-0.793147 + 1.40509 i}

$\longrightarrow$ Here for $s = 1/2$ only. Results for any s OK.
$\longrightarrow$ Results for arbitrary alternating spin chain j_1, j_2, j_3 OK.
$\longrightarrow$ Alternative ABA solution for the special case $\kappa = \kappa^* = 1$ [Sklyanin,88].

PROBLEM 2

Let $(\bar{\pi}, \bar{V})$ be an irreducible finite dimensional representation of the **Askey-Wilson algebra** with generators A, A^*. Denote $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*)$ the corresponding **Leonard pair of Askey-Wilson type**.

$\Rightarrow$ Express the q-Racah polynomials as scalar products of Bethe states

Remark : Towards determinental representations for q-Racah polynomials.

Strategy : Eigenvectors $|\theta_M\rangle, |\theta^*_N\rangle$ of Leonard pairs $\bar{\pi}(\mathsf{A}), \bar{\pi}(\mathsf{A}^*) \sim$ **Bethe states**

$\Rightarrow$ Compute $\langle \Psi_+^N(\bar{v}, m_0) | \Psi_-^M(\bar{u}, m_0) \rangle \sim R_M(\theta^*_N)$

$\Rightarrow$ Determinantal expression for q-Racah polynomials ? (in progress)

q-Racah OP, Leonard pairs and scalar products in ABA

From previous results, explicit **Bethe eigenbases for Leonard pairs of AW type** are obtained. From [Zhedanov,'92,Terwilliger,'04], q-Racah polynomials $\sim$ entries of transition matrix between Leonard pairs' eigenbases.

Idea : Compute scalar products of Bethe eigenstates

$\Rightarrow$ Determinantal representations for q-Racah polynomials ?

Reminder : For a **Leonard pair of q-Racah type** [Terwilliger,'04], use :

$$\bar{\pi}(\mathsf{A})|\theta_M\rangle = \theta_M|\theta_M\rangle\ ,\quad \bar{\pi}(\mathsf{A}^*)|\theta_M\rangle = A^*_{M+1,M}|\theta_{M+1}\rangle + A^*_{M,M}|\theta_M\rangle + A^*_{M-1,M}|\theta_{M-1}\rangle\ ,$$
$$\bar{\pi}(\mathsf{A}^*)|\theta^*_N\rangle = \theta^*_N|\theta^*_N\rangle\ ,\quad \bar{\pi}(\mathsf{A})|\theta^*_N\rangle = A_{N+1,N}|\theta^*_{N+1}\rangle + A_{N,N}|\theta^*_N\rangle + A_{N-1,N}|\theta^*_{N-1}\rangle\ ,$$

where

$$\theta_M = bq^{2M} + cq^{-2M}\ ,\quad \theta^*_M = b^*q^{2M} + c^*q^{-2M}\ ,$$

$$A^*_{M,M-1} = q^{2-4s}\frac{(1-q^{2M})(c-bq^{2M+4s})(b^*q^{2s-1}\zeta^{-2}+bq^{2M-2})(cq^{2s-1}\zeta^2+c^*q^{2M-2})}{(c-bq^{4M-2})(c-bq^{4M})}\ ,$$
$$A^*_{M-1,M} = \frac{(1-q^{2M-4s-2})(c-bq^{2M-2})(c+b^*\zeta^{-2}q^{2M+2s-1})(c^*+b\zeta^2q^{2M+2s-1})}{(c-bq^{4M-4})(c-bq^{4M-2})}\ ,$$
$$A^*_{M,M} = \theta^*_0 - A^*_{M,M+1} - A^*_{M,M-1}\ .$$

The **q-Racah polynomials** determine the entries of the transition matrices relating the eigenbases $\{|\theta_M\rangle\}_{M=0}^{2s}$, $\{|\theta^*_N\rangle\}_{N=0}^{2s}$ [Zhedanov,'92, Terwilliger,'04] $(R_0(\theta^*_N)=1)$:

$$|\theta_M\rangle = \sum_{N=0}^{2s} \langle\theta^*_N|\theta_M\rangle|\theta^*_N\rangle \quad \text{with} \quad R_M(\theta^*_N) = \frac{\langle\theta^*_N|\theta_M\rangle}{\langle\theta^*_N|\theta_0\rangle}$$

$$R_M(\theta^*_N) = {}_4\phi_3\left[\begin{matrix} q^{-2M},\ \frac{b}{c}q^{2M},\ q^{-2N},\ \frac{b^*}{c^*}q^{2N} \\ -\frac{b}{c^*}q^{2s-1}\zeta^2,\ -\frac{b^*}{c}q^{2s-1}\zeta^{-2},\ q^{-4s} \end{matrix};q^2,q^2\right].$$

Scalar products and q-Racah OP : From previous results, recall the Bethe eigenstates for some $\{u_i, v_j\}$ satisfying certain Bethe equations :

$$\begin{aligned} |\Psi^M_-(\bar u, m_0)\rangle &= \bar\pi(\mathscr{B}^-(u_1, m_0+2(M-1))\cdots\mathscr{B}^-(u_M, m_0))|\Omega^-\rangle\ , \\ \langle\Psi^N_+(\bar v, m_0)| &= \langle\Omega^+|\mathscr{C}^+(v_1, m_0+2))\cdots\mathscr{C}^+(v_N, m_0+2N))\ . \end{aligned}$$

$$\begin{aligned} \bar\pi(\mathsf{A})|\Psi^M_-(\bar u, m_0)\rangle &= \theta^M_-|\Psi^M_-(\bar u, m_0)\rangle \quad \text{with} \quad \theta^M_- = \frac{1}{2}q^{\frac{1}{2}(\nu+\nu')}\left(e^{-\mu}q^{-2s+2M} + e^{\mu}q^{2s-2M}\right)\ , \\ \langle\Psi^N_+(\bar v, m_0)|\bar\pi(\mathsf{A}^*) &= \langle\Psi^N_+(\bar v, m_0)|\theta^{*N}_+ \quad \text{with} \quad \theta^{*N}_+ = \frac{1}{2}q^{\frac{1}{2}(\nu+\nu')}\left(e^{-\mu'}q^{2s-2N} + e^{\mu'}q^{-2s+2N}\right)\ , \end{aligned}$$

$\Rightarrow$ Compute $\langle\Psi^N_+(\bar v, m_0)|\Psi^M_-(\bar u, m_0)\rangle \sim R_M(\theta^*_N)$

$\Rightarrow$ Determinant expression for q-Racah polynomials ? (hint : Recent works [Belliard-Slavnov,'19]) (in progress)

Comments : ABA generates q-difference operators

For each case $a \in \{sp, d, g\}$, define the **Baxter Q-polynomial** :

$$Q^a_M(U) = \prod_{j=1}^{M}(U - U_j) \quad \text{with} \quad U_i = \frac{qu_i^2 + q^{-1}u_i^{-2}}{q + q^{-1}} \ .$$

From ABA results, it is found that $Q^a_M(U)$ solves a **second-order q-difference equation**. The corresponding T-Q relations can be written :

$$\begin{array}{lrcl} (Special) & \pi(\mathsf{I}(0,\kappa^*,0,0))Q^{sp}_M(U) & = & \Lambda^M_{sp,+}Q^{sp}_M(U)\ ,\ Q^{sp}_M(U) \sim \text{AW polynom.} \\ (Diagonal) & \pi(\mathsf{I}(\kappa,\kappa^*,0,0))Q^{d}_{2s}(U) & = & \Lambda^{2s}_{d,+}Q^{d}_{2s}(U) - \kappa\delta^{2s}_d H(U)\ , \\ (Generic) & \pi(\mathsf{I}(\kappa,\kappa^*,\kappa_+,\kappa_-))Q^{g}_{2s}(U) & = & \Lambda^{2s}_{g,+}Q^{g}_{2s}(U) - \kappa\delta^{2s}_g \underbrace{H(U)}_{\text{inhomogeneous term}} \end{array}$$

with $H(U) = \prod_{k=0}^{2s} b(q^{1/2+k-s}vu)b(q^{1/2+k-s}v^{-1}u)$, $\qquad b(x) = x - x^{-1}$.

where the **Heun-Askey-Wilson operator** $\pi(\mathsf{I}^{AW})$ is introduced.

$\Rightarrow$ **Extract $\pi(\mathsf{A}), \pi(\mathsf{A}^*), \ldots$** using $\mathsf{I}^{AW} = \kappa\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\,[\mathsf{A},\mathsf{A}^*]_q + \kappa_-\,[\mathsf{A}^*,\mathsf{A}]_q$

From the T-Q relations, one extracts :

Example : Denote $T_\pm(f(z)) = f(q^{\pm 1}z)$:

$$\begin{aligned}\pi(\mathsf{A}) &= q^{-1}z^{-1}\phi(z)(T_+ - 1) + q^{-1}z\phi(z^{-1})(T_- - 1)\\ &\quad + \frac{1}{2}q^{\frac{\nu+\nu'}{2}}e^{-\mu'}q^{2s}\left(2e^{\mu'}\cosh(\mu) - (v^2+v^{-2})q^{-2s-1} + q^{-1}(z+z^{-1})\right),\\ \pi(\mathsf{A}^*) &= \phi(z)(T_+ - 1) + \phi(z^{-1})(T_- - 1) + \frac{1}{2}q^{(\nu+\nu')/2}(e^{\mu'}q^{-2s} + e^{-\mu'}q^{2s})\end{aligned}$$

and

$$\pi([\mathsf{A}^*,\mathsf{A}]_q) = -\frac{q^{\nu+\nu'}(q-q^{-1})}{4}\left((q+q^{-1})(z+z^{-1}) - (q^{2s+1}+q^{-2s-1})(v^2+v^{-2}) + 4\cosh(\mu)\cosh(\mu')\right)$$

where

$$\phi(z) = \frac{1}{2}q^{\frac{\nu+\nu'}{2}}e^{-\mu'}q^{2s}\frac{(1+qe^{-\mu+\mu'}z)(1+qe^{\mu+\mu'}z)(1-q^{-2s}v^2z)(1-q^{-2s}v^{-2}z)}{(1-z^2)(1-q^2z^2)}.$$

$\Rightarrow$ **second order q-difference operators** that satisfy the Askey-Wilson relations.

Perspective 1 : Integrable systems related with the q-Onsager algebra O_q

Definition :[Terwilliger,'99],[B,'04] The q-Onsager algebra is generated by A, A* subject to the relations :

$$\big[\mathsf{A},\big[\mathsf{A},\big[\mathsf{A},\mathsf{A}^*\big]_q\big]_{q^{-1}} = \rho\big[\mathsf{A},\mathsf{A}^*\big]\ ,$$
$$\big[\mathsf{A},\big[\mathsf{A}^*,\big[\mathsf{A}^*,\mathsf{A}\big]_q\big]_{q^{-1}} = \rho\big[\mathsf{A}^*,\mathsf{A}\big]$$

Example : L-sites open XXZ chain with generic integrable boundary conditions

$$H_{XXZ} = \sum_{k=0}^{L} F_k \underbrace{\bar{\pi}(\mathsf{I}_{2k+1})}_{\text{Generalizations of HAW op.}} + F_0$$

Case $k=0$: $\quad \mathsf{I}_1 \mapsto \kappa\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\,[\mathsf{A},\mathsf{A}^*]_q + \kappa_-\,[\mathsf{A}^*,\mathsf{A}]_q$.

$\Rightarrow$ **Diagonalization of $\mathsf{I}_{2k+1}, k=0,1,...,N-1$ via modified ABA ?**

Reflection algebra for O_q (known) + Tridiagonal pairs (known). ABA (OK)

Perspective 2 : Integrable systems related with higher rank AW algebras

For the **Askey-Wilson algebra**, one has the following realization : [Granovskii-Zhedanov,'93] One has the map $AW \to U_q(sl_2) \otimes U_q(sl_2) \otimes U_q(sl_2)$.

$$\begin{aligned} \mathsf{A} \quad &\to \quad \Delta(C) \otimes I\!I \ , \\ \mathsf{A}^* \quad &\to \quad I\!I \otimes \Delta(C) \ . \end{aligned}$$

Recently, **higher-rank Askey-Wilson algebras** $AW(N)$ have been introduced [DeBie et al.,19], with generators $\mathsf{A}_{12}, \mathsf{A}_{23}, ..., \mathsf{A}_{N-1\ N}$. One has the map $AW(N) \to U_q(sl_2) \otimes U_q(sl_2) \otimes \cdots \otimes U_q(sl_2)$

$$\begin{aligned} \mathsf{A}_{12} \quad &\to \quad \Delta(C) \otimes I\!I \otimes I\!I \otimes \cdots \otimes I\!I \ , \\ \mathsf{A}_{23} \quad &\to \quad I\!I \otimes \Delta(C) \otimes I\!I \otimes \cdots \otimes I\!I \ , \\ \mathsf{A}_{N-1\ N} \quad &\to \quad I\!I \otimes \cdots \otimes I\!I \otimes \Delta(C) \end{aligned}$$

Consider the element : $\mathsf{I}_N = \kappa_1 \mathsf{A}_{12} + \kappa_2 \mathsf{A}_{23} + \cdots \kappa_{N-1N} \mathsf{A}_{N-1N}$

$\Rightarrow$ **Diagonalization of I_N via modified ABA ?**

$\exists$ **Reflection algebra using connection with generalized q-Onsager algebras ?**

THANK YOU FOR YOUR ATTENTION !!!