Hypergeometric Fourier transform associated with a root system of type BC

Nobukazu Shimeno

joint work with Tatsuo Honda and Hiroshi Oda

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Inversion of hypergeometric Fourier transform

- spherical Fourier transform on G/K (group case) **(Harish-Chandra, Gindikin-Karpelevich, Helgason, Rosenberg)**
- 1-dim K-type in group case **(Flensted-Jensen, Heckman, S)**
- hypergeometric Fourier transform in rank 1 case = Jacobi transform **(Koornwinder, Flensted-Jensen)**
- hypergeometric Fourier transform (non-negative multiplicity or negative multiplicity for reduced root systems) **(Heckman, Opdam)**

The case of BC_r $(r > 1)$ with general multiplicity has been open and it is the theme of this talk.

Hypergeom func for BC_1 = Jacobi func

$\mathscr{R} = \{\pm\beta_1, \pm 2\beta_1\} \subset \mathbb{R}$: root system of BC_1

$\boldsymbol{k} = (\boldsymbol{k}_s, \boldsymbol{k}_\ell) \in \mathbb{C}^2$ ($\boldsymbol{k}_s, \boldsymbol{k}_\ell$: multiplicities of $\pm\beta_1, \pm 2\beta_1$ resp.)

$$\boldsymbol{\alpha} := \boldsymbol{k}_s + \boldsymbol{k}_\ell - \tfrac{1}{2},\ \boldsymbol{\beta} := \boldsymbol{k}_\ell - \tfrac{1}{2},\ \rho(\boldsymbol{k}) = \boldsymbol{\alpha} + \boldsymbol{\beta} + 1$$

$F(\lambda, \boldsymbol{k}; x) = \phi^{(\boldsymbol{\alpha},\boldsymbol{\beta})}_{\sqrt{-1}\lambda}(x) \quad \longleftarrow$ **Jacobi function (Flensted-Jensen, Koornwinder)**

$$:= {}_2F_1\left(\tfrac{1}{2}(\lambda + \rho(\boldsymbol{k})), \tfrac{1}{2}(-\lambda + \rho(\boldsymbol{k})); \boldsymbol{\alpha} + 1; -\sinh^2 x\right)$$

It is a unique even eigenfunction of

$$L(\boldsymbol{k}) := \frac{d^2}{dx^2} + (\boldsymbol{k}_s \coth x + 2\boldsymbol{k}_\ell \coth 2x)\frac{d}{dx}$$

with e.v. $\lambda^2 - \rho(\boldsymbol{k})^2$ and value 1 at 0. $(\Leftarrow \tilde{\mathbf{c}}(\rho(\boldsymbol{k}), \boldsymbol{k}) \neq 0)$

$\mathbf{c}$-function

$\Phi(\lambda, k; x)$: eigenfunction of $L(\boldsymbol{k})$ on $(0, \infty)$

s.t. $\Phi(\lambda, \boldsymbol{k}; x) \sim e^{(\lambda - \rho(\boldsymbol{k}))x}$ $(\mathrm{Re}\,\lambda > 0, x \to \infty)$

Harish-Chandra series

$$F(\lambda, \boldsymbol{k}) = \mathbf{c}(\lambda, \boldsymbol{k})\Phi(\lambda, \boldsymbol{k}) + \mathbf{c}(-\lambda, \boldsymbol{k})\Phi(-\lambda, \boldsymbol{k})$$ (λ:generic)

hgf — Harish-Chandra expansion

$$\mathbf{c}(\lambda, \boldsymbol{k}) := \tilde{\mathbf{c}}(\lambda, \boldsymbol{k}) / \tilde{\mathbf{c}}(\rho(\boldsymbol{k}), \boldsymbol{k})$$ Harish-Chandra's $\mathbf{c}$-func

$$\tilde{\mathbf{c}}(\lambda, \boldsymbol{k}) := \frac{2^{\boldsymbol{\beta} - \boldsymbol{\alpha}} \Gamma\left(\frac{1}{2}\lambda\right) \Gamma\left(\frac{1}{2}(\lambda + 1)\right)}{\Gamma\left(\frac{1}{2}(\lambda + \boldsymbol{\alpha} + 1 - \boldsymbol{\beta})\right) \Gamma\left(\frac{1}{2}(\lambda + \boldsymbol{\alpha} + 1 + \boldsymbol{\beta})\right)}$$

Plancherel measure and L^2-HGFs are determined by $\mathbf{c}(\lambda, \boldsymbol{k})$.

Hypergeometric Fourier transform for BC_1

= Jacobi transform

$$\delta_{\boldsymbol{k}}(x) := |e^x - e^{-x}|^{2\boldsymbol{k}_s} |e^{2x} - e^{-2x}|^{2\boldsymbol{k}_\ell}$$

Assume $\boldsymbol{\alpha} > -1, \boldsymbol{\beta} \in \mathbb{R}$ ($\Leftrightarrow$ $\boldsymbol{k}$: real, $\delta_{\boldsymbol{k}} \in L^1_{\mathrm{loc}}(\mathbb{R})$)

$$\mathscr{F}_{\boldsymbol{k}} f(\lambda) := \frac{1}{2} \int_{\mathbb{R}} f(x) F(\lambda, \boldsymbol{k}; x) \delta_{\boldsymbol{k}}(x) dx \quad (f \text{:even})$$

Inversion formula (1st form)

$$f(x) = \frac{1}{2\pi\sqrt{-1}} \int_{\eta+\sqrt{-1}\mathbb{R}} \mathscr{F}_{\boldsymbol{k}} f(\lambda) \Phi(\lambda, \boldsymbol{k}; x) \frac{d\lambda}{\mathbf{c}(-\lambda, \boldsymbol{k})}$$

for $\eta \leq 0$ with $\eta < \boldsymbol{\alpha} - |\boldsymbol{\beta}| + 1$.

Set of simple poles of $\mathbf{c}(-\lambda, \boldsymbol{k})^{-1}$ for $\operatorname{Re}\lambda < 0$

$$D_{\boldsymbol{k}} := \{\lambda \in \mathbb{C} ; \operatorname{Re}\lambda \leq 0, \mathbf{c}(-\lambda, \boldsymbol{k}) = 0\}$$
$$= \{\boldsymbol{\alpha} - |\boldsymbol{\beta}| + 1 + 2j \; ; \; j \in \mathbb{N}, \boldsymbol{\alpha} - |\boldsymbol{\beta}| + 1 + 2j < 0\}$$

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

- $D_{\boldsymbol{k}} \neq \emptyset \Leftrightarrow |\boldsymbol{\beta}| > \boldsymbol{\alpha} + 1$
- $\lambda \in D_{\boldsymbol{k}} \Rightarrow F(\lambda, \boldsymbol{k})$ is square integrable.

$$F(\lambda, \boldsymbol{k}) = \mathbf{c}(\lambda, \boldsymbol{k}) \Phi(\lambda, \boldsymbol{k})$$

- $\{F(\lambda, \boldsymbol{k}); \lambda \in D_{\boldsymbol{k}}\}$ exhaust square integrable hypergeometric functions for BC_1.

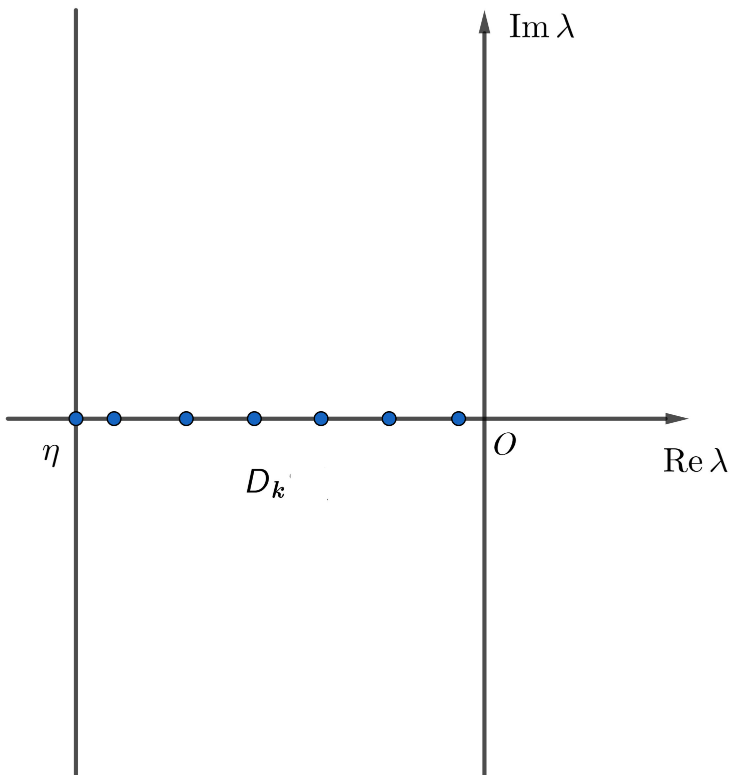
Above results are due to Flensted-Jensen77

Inversion formula (1st → final form)

$$f(x) = \frac{1}{2\pi\sqrt{-1}} \int_{\eta+\sqrt{-1}\mathbb{R}} \mathscr{F}_{\boldsymbol{k}} f(\lambda) \Phi(\lambda, \boldsymbol{k}; x) \frac{d\lambda}{\mathbf{c}(-\lambda, \boldsymbol{k})}$$

simple poles at $\lambda \in D_{\boldsymbol{k}}$

Let $\eta \to 0$ and apply Cauchy's theorem.



Inversion formula (case of BC_1) (Flensted-Jensen77)

$$f(x) = \frac{1}{4\pi\sqrt{-1}} \int_{\sqrt{-1}\mathbb{R}} \mathscr{F}_{\boldsymbol{k}} f(\lambda) F(\lambda, \boldsymbol{k}; x) \frac{d\lambda}{|\mathbf{c}(\lambda, \boldsymbol{k})|^2}$$

$$- \sum_{\lambda \in D_{\boldsymbol{k}}} d(\lambda, \boldsymbol{k}) \mathscr{F}_{\boldsymbol{k}} f(\lambda) F(\lambda, \boldsymbol{k}; x)$$

For $\lambda \in D_{\boldsymbol{k}}$

$$d(\lambda, \boldsymbol{k}) := -\mathbf{c}(\lambda, \boldsymbol{k})^{-1} \operatorname*{Res}_{\xi=\lambda} (\mathbf{c}(-\xi, \boldsymbol{k})^{-1}) = \tilde{\mathbf{c}}(\rho(\boldsymbol{k}), \boldsymbol{k})^2$$

$$\times \frac{-2^{2\boldsymbol{\alpha}-2\boldsymbol{\beta}-1}\lambda}{\pi} \frac{\Gamma\left(\frac{1}{2}(\lambda + \boldsymbol{\alpha} + |\boldsymbol{\beta}| + 1)\right)\Gamma\left(\frac{1}{2}(-\lambda + \boldsymbol{\alpha} + |\boldsymbol{\beta}| + 1)\right)}{\Gamma\left(\frac{1}{2}(\lambda - \boldsymbol{\alpha} + |\boldsymbol{\beta}| + 1)\right)\Gamma\left(\frac{1}{2}(-\lambda - \boldsymbol{\alpha} + |\boldsymbol{\beta}| + 1)\right)}.$$

Group case

G/K: Riemannian symmetric space of noncpt type

$G = K \exp \mathfrak{a} K$: Cartan decomposition, $\dim \mathfrak{a} = 1$

$\varphi_\lambda|_{\mathfrak{a}} = F(\lambda, \boldsymbol{k})$ $(\exists \boldsymbol{k})$

(φ_λ : zonal spherical function on G/K)

$L(\boldsymbol{k})$: radial part of the Laplacian on G/K

Example: $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$

$\boldsymbol{k}_s = 0$, $\boldsymbol{k}_\ell = \frac{1}{2}$ $(\Leftrightarrow \boldsymbol{\alpha} = \boldsymbol{\beta} = 0)$

π_ν $(\nu \in \mathbb{Z})$: one-dim K-type, $\boldsymbol{k}_s^{\pi_\nu} := \nu$, $\boldsymbol{k}_\ell^{\pi_\nu} := -\nu$

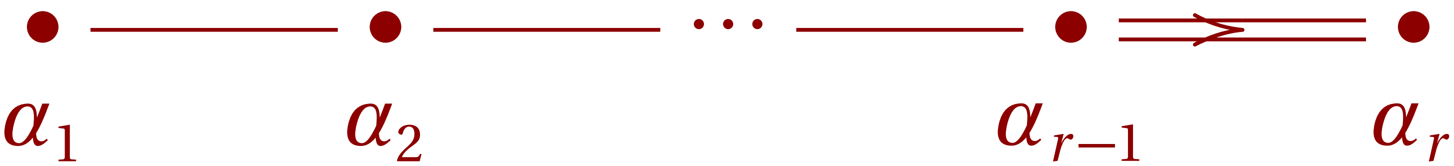
$\varphi_\lambda^{\pi_\nu}|_{\mathfrak{a}} = \cosh^{-\nu} F(\lambda, \boldsymbol{k}^{\pi_\nu})$ elementary π_ν-spherical func

Root system of type $BC_r\ (r \geq 2)$

$\mathscr{R} \subset \mathfrak{a}^* \simeq \mathbb{R}^r$: root system of type $BC_r\ (r \geq 2)$

W : Weyl group for $\mathscr{R}$

$\mathscr{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subset \mathscr{R}^+$: positive simple roots



$\alpha_i = \beta_{r+1-i} - \beta_{r-i}\ (1 \leq i \leq r-1),\ \alpha_r = \beta_1$

$(\{\frac{1}{2}\beta_1, \ldots, \frac{1}{2}\beta_r\}$: orthonormal basis of $\mathfrak{a}^*)$

$$\underbrace{\boldsymbol{k} = (\boldsymbol{k}_\alpha)_{\alpha \in \mathscr{R}}}_{W\text{-inv}},\ \boldsymbol{k}_s := \boldsymbol{k}_{\beta_1},\ \boldsymbol{k}_m := \boldsymbol{k}_{\beta_2 - \beta_1},\ \boldsymbol{k}_\ell := \boldsymbol{k}_{2\beta_1}$$

$$\boldsymbol{k}_s, \boldsymbol{k}_\ell \longleftrightarrow \boldsymbol{\alpha} := \boldsymbol{k}_s + \boldsymbol{k}_\ell - \frac{1}{2},\ \boldsymbol{\beta} := \boldsymbol{k}_\ell - \frac{1}{2}$$

Harish-Chandra's $\mathbf{c}$-functions

$\lambda \in \mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}^r$, $\lambda = \frac{1}{2}\sum_{i=1}^r \lambda_i \beta_i \quad (\lambda_i = \langle \lambda, \beta_i^\vee \rangle \in \mathbb{C})$

$$\tilde{\mathbf{c}}_{\beta_p \pm \beta_q}(\lambda, \boldsymbol{k}) := \frac{\Gamma\left(\frac{1}{2}(\lambda_p \pm \lambda_q)\right)}{\Gamma\left(\frac{1}{2}(\lambda_p \pm \lambda_q + 2\boldsymbol{k}_m)\right)},$$

$$\tilde{\mathbf{c}}_i(\lambda, \boldsymbol{k}) := \frac{2^{\boldsymbol{\beta}-\boldsymbol{\alpha}}\Gamma\left(\frac{1}{2}\lambda_i\right)\Gamma\left(\frac{1}{2}(\lambda_i + 1)\right)}{\Gamma\left(\frac{1}{2}(\lambda_i + \boldsymbol{\alpha} + 1 + \boldsymbol{\beta})\right)\Gamma\left(\frac{1}{2}(\lambda_i + \boldsymbol{\alpha} + 1 - \boldsymbol{\beta})\right)},$$

$$\tilde{\mathbf{c}}(\lambda, \boldsymbol{k}) := \prod_{1 \le q < p \le r} \tilde{\mathbf{c}}_{\beta_p - \beta_q}(\lambda, \boldsymbol{k})\tilde{\mathbf{c}}_{\beta_p + \beta_q}(\lambda, \boldsymbol{k}) \prod_{i=1}^r \tilde{\mathbf{c}}_i(\lambda, \boldsymbol{k})$$

$$\mathbf{c}(\lambda, \boldsymbol{k}) := \tilde{\mathbf{c}}(\lambda, \boldsymbol{k}) / \tilde{\mathbf{c}}(\rho(\boldsymbol{k}), \boldsymbol{k}) \quad \left(\rho(\boldsymbol{k}) = \frac{1}{2}\sum_{\alpha \in \mathscr{R}^+} \boldsymbol{k}_\alpha \alpha\right)$$

Harish-Chandra series

$$L(\boldsymbol{k}) := \Delta_{\mathrm{euc}} + \sum_{\alpha \in \mathscr{R}^+} \boldsymbol{k}_\alpha \coth\frac{\alpha}{2}\,\partial_\alpha$$

${}^\exists\, \mathbb{D}(\boldsymbol{k})$: commutative algebra of W-invariant diff ops on $\mathfrak{a}$ containing $L(\boldsymbol{k})$

$\Phi(\lambda, \boldsymbol{k}; x) \sim e^{(\lambda - \rho(\boldsymbol{k}))(x)}$ $(x \in \mathfrak{a}_+ = \{H \in \mathfrak{a}\,;\, \alpha(H) > 0\ (\alpha \in \mathscr{R}^+)\})$

Harish-Chandra series

eigenfunction of $L(\boldsymbol{k})$ with e.v. $\langle\lambda, \lambda\rangle - \langle\rho(\boldsymbol{k}), \rho(\boldsymbol{k})\rangle$

$\Phi(\lambda, \boldsymbol{k})$ is a joint eigenfunction of $\mathbb{D}(\boldsymbol{k})$.

Some of apparent singularities in λ are removed (Opdam).

Heckman-Opdam hypergeometric func

Assume $\tilde{\mathbf{c}}(\rho(\boldsymbol{k}),\boldsymbol{k})\neq 0$ and let

$$F(\lambda,\boldsymbol{k}) := \sum_{w\in W} \mathbf{c}(w\lambda,\boldsymbol{k})\Phi(w\lambda,\boldsymbol{k}).$$

It is analytically continued to a real analytic joint eigenfunction of $\mathbb{D}(\boldsymbol{k})$ on $\mathfrak{a}$ with $F(\lambda,\boldsymbol{k};0)=1$.

- $F(\lambda,\boldsymbol{k})$ is holomorphic in $\lambda\in\mathfrak{a}_{\mathbb{C}}^*\simeq\mathbb{C}^r$.
- $F(\lambda,\boldsymbol{k})$ is holomorphic in $\{\boldsymbol{k}\in\mathbb{C}^3;\tilde{\mathbf{c}}(\rho(\boldsymbol{k}),\boldsymbol{k})\neq 0\}$.
- $F(w\lambda,\boldsymbol{k})=F(\lambda,\boldsymbol{k})\quad(w\in W)$
- $F(\lambda,\boldsymbol{k};wx)=F(\lambda,\boldsymbol{k};x)\quad(w\in W)$

Group case (elementary spherical func)

(1) $G = K \exp \mathfrak{a} K$: Cartan decomposition

$\varphi_\lambda|_{\mathfrak{a}} = F(\lambda, \boldsymbol{k})$ ($\mathscr{R} = 2\Sigma(\mathfrak{g}, \mathfrak{a})$, $\boldsymbol{k}_{2\alpha} = \frac{1}{2}\boldsymbol{m}_\alpha = \frac{1}{2}\dim \mathfrak{g}_\alpha$)

$L(\boldsymbol{k})$: radial part of Laplacian on G/K

(2) G : non-cpt simple Lie gp, Hermitian type

π_ν : (suitably normalized) one-dim K-type

$$\varphi_\lambda^{\pi_\nu}\Big|_{\mathfrak{a}} = \prod_{i=1}^{r}\left(\cosh\frac{\beta_i}{2}\right)^{-\nu} F(\lambda, \boldsymbol{k}^{\pi_\nu})$$

$$\boldsymbol{\alpha}^{\pi_\nu} = \tfrac{1}{2}\boldsymbol{m}_s, \quad \boldsymbol{\beta}^{\pi_\nu} = -\nu, \quad \boldsymbol{k}_m^{\pi_\nu} = \tfrac{1}{2}\boldsymbol{m}_m$$

Hypergeometric Fourier transform $\mathscr{F}_{\boldsymbol{k}}$

$$\delta_{\boldsymbol{k}} := \prod_{\alpha \in \mathscr{R}^+} \left| e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha} \right|^{2\boldsymbol{k}_\alpha}$$

Assume $\boldsymbol{k}_s, \boldsymbol{k}_m \in \mathbb{R}$, $\boldsymbol{k}_s + \boldsymbol{k}_\ell > -\frac{1}{2}$, $\boldsymbol{k}_m \geq 0$.

$\Leftrightarrow \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{k}_m \in \mathbb{R}$, $\boldsymbol{\alpha} > -1$, $\boldsymbol{k}_m \geq 0$.

$\Rightarrow \tilde{\mathbf{c}}(\rho(\boldsymbol{k}), \boldsymbol{k}) \neq 0$, $\delta_{\boldsymbol{k}} \in L^1_{\mathrm{loc}}(\mathfrak{a})$

$f \in C_0^\infty(\mathfrak{a})^W$

$$\mathscr{F}_{\boldsymbol{k}} f(\lambda) := \frac{1}{|W|} \int_{\mathfrak{a}} f(x) F(\lambda, \boldsymbol{k}; x) \delta_{\boldsymbol{k}}(x) dx$$

It is the spherical transform in group case (1).

Inversion formula (1st form)

Choose $\eta \in -\mathrm{Cl}(\mathfrak{a}^*_+)$ s.t. $\mathbf{c}(-\lambda, \boldsymbol{k})^{-1}$ is regular on $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}}\,;\, \mathrm{Re}\,\lambda \in \eta - \mathrm{Cl}(\mathfrak{a}^*_+)\}$. $\quad (\mathfrak{a}^*_+ := \{\lambda \in \mathfrak{a}^* \mid \lambda(\alpha^\vee) > 0 \ (\alpha > 0)\})$

Theorem (Inversion formula, 1st form)**.**
$f \in C_0^\infty(\mathfrak{a})^W$, $x \in \mathfrak{a}_+$. $d\mu(\lambda) = (2\pi)^{-r}\, d(\mathrm{Im}\,\lambda)$

$$f(x) = \int_{\eta + \sqrt{-1}\mathfrak{a}^*} \mathscr{F}_{\boldsymbol{k}} f(\lambda) \Phi(\lambda, \boldsymbol{k}; x) \mathbf{c}(-\lambda, \boldsymbol{k})^{-1} d\mu(\lambda).$$

For $\boldsymbol{k} > 0$ this is due to Opdam. The result follows by using analytic continuation in $\boldsymbol{k}$ and the hypergeom shift operator.

Global inversion formula (a special case)

$\boldsymbol{k}_s \geq -1,\ \boldsymbol{k}_s + 2\boldsymbol{k}_\ell \geq 0,\ \boldsymbol{k}_m \geq 0 \implies$ may take $\eta = 0$ in

$$f(x) = \int_{\eta+\sqrt{-1}\mathfrak{a}^*} \mathscr{F}_{\boldsymbol{k}} f(\lambda) \Phi(\lambda, \boldsymbol{k}; x) \mathbf{c}(-\lambda, \boldsymbol{k})^{-1} d\mu(\lambda).$$

Thus we have the inversion formula

$$f(x) = \frac{1}{|W|} \int_{\sqrt{-1}\mathfrak{a}^*} \mathscr{F}_{\boldsymbol{k}} f(\lambda) F(\lambda, \boldsymbol{k}; x) |\mathbf{c}(\lambda, \boldsymbol{k})|^{-2} d\mu(\lambda) \quad (x \in \mathfrak{a})$$

The above condition on $\boldsymbol{k}$ is weaker than

$$\underbrace{\boldsymbol{k}_s > 0,\ \boldsymbol{k}_s + 2\boldsymbol{k}_\ell > 0,\ \boldsymbol{k}_m > 0}_{\text{std mutiplicity}} \implies \underbrace{\boldsymbol{k} > 0}_{\text{positive multiplicity}}$$

Toward inversion formula (general case)

$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{k}_m \in \mathbb{R}, |\boldsymbol{\beta}| > \boldsymbol{\alpha} + 1 > 0, \boldsymbol{k}_m \geq 0 \implies$ Move the domain of integration from $\eta + \sqrt{-1}\mathfrak{a}^*$ to $\sqrt{-1}\mathfrak{a}^*$ in

$$f(x) = \int_{\eta+\sqrt{-1}\mathfrak{a}^*} \mathscr{F}_{\boldsymbol{k}} f(\lambda) \Phi(\lambda, \boldsymbol{k}; x) \mathbf{c}(-\lambda, \boldsymbol{k})^{-1} d\mu(\lambda)$$

and compute residues (method in group case S94).

- take residues step by step: $\Theta_i \subset \mathscr{B}$ $(\langle \Theta_i \rangle = \mathscr{R}_{BC_i})$

$$\Theta_0 = \emptyset \to \Theta_1 \to \cdots \to \Theta_r = \mathscr{B}$$

most continuous $\longrightarrow \cdots \longrightarrow$ **discrete**

- $F_\Xi(\lambda, \boldsymbol{k}) = \sum_{s \in \mathfrak{S}_r} \mathbf{c}_\Xi(s\lambda, \boldsymbol{k}) \Phi(s\lambda, \boldsymbol{k})$ $(\langle \Xi \rangle = \mathscr{R}_{A_{r-1}} \subset \mathscr{R})$, its analytic property **(←Opdam, Pasquale, Olafsson)** and estimate

Parabolic subsets $\langle\Theta_i\rangle \subset \mathscr{R}$ $(0 \le i \le r)$

$\Theta_i := \{\alpha_j \,;\, r-i+1 \le j \le r\} \subset \mathscr{B}$

$\langle\Theta_i\rangle := \mathscr{R} \cap \mathfrak{a}(\Theta_i)^*$: root system of type BC_i

$\mathfrak{a}(\Theta_i)^* = \mathrm{Span}\{\beta_j ; 1 \le j \le i\}$, $\mathfrak{a}^*_{\Theta_i} = \mathrm{Span}\{\beta_j ; i+1 \le j \le r\}$

$W(\Theta_i)$: stabilizer of $\mathfrak{a}(\Theta_i)^*$ in W

residues step by step: $\Theta_0 = \emptyset \longrightarrow \Theta_1 \longrightarrow \cdots \longrightarrow \Theta_r = \mathscr{B}$

most continuous $\longrightarrow \cdots \longrightarrow$ **discrete**

Parameters of tempered spectra

$$D_{\boldsymbol{k}}(\Theta_i) := \{(\lambda \in \mathbb{R}^i ; \lambda_1 + |\boldsymbol{\beta}| - \boldsymbol{\alpha} - 1 \in 2\mathbb{N}, \lambda_i < 0, \\ \lambda_{j+1} - \lambda_j - 2\boldsymbol{k}_m \in 2\mathbb{N} \ (1 \le j \le \boldsymbol{i} - 1)\}$$

$$D_{\boldsymbol{k}}(\Theta_i) \neq \emptyset \iff |\boldsymbol{\beta}| > \boldsymbol{\alpha} + 2(\boldsymbol{i} - 1)\boldsymbol{k}_m + 1$$

- $\lambda \in D_{\boldsymbol{k}}(\Theta_i) \Longrightarrow F(\lambda, \boldsymbol{k})$ is tempered
- $\lambda \in D_{\boldsymbol{k}}(\mathscr{B}) \Longrightarrow F(\lambda, \boldsymbol{k}) \in L^2(\mathfrak{a}, \boldsymbol{\delta}_{\boldsymbol{k}}(x)dx)$
- If $\beta < 0$, $\lambda \in D_{\boldsymbol{k}}(\mathscr{B}) \Leftrightarrow \langle \lambda - \rho(\boldsymbol{k}), \alpha^\vee \rangle \in \mathbb{N} \ ({}^\forall \alpha > 0), \lambda_r < 0$
  (analytic continuation of the Jacobi polynomial)
- Group case: $D_{\boldsymbol{k}}(\mathscr{B}) \longleftrightarrow$ hol DS with 1-dim K-type

Residues

$\mathbf{c}_{\Theta_i}(\lambda, \boldsymbol{k})$: partial $\mathbf{c}$-function for $\langle \Theta_i \rangle \simeq \mathscr{R}_{BC_i}$

$$\mathbf{c}(\lambda, \boldsymbol{k}) = \mathbf{c}_{\Theta_i}(\lambda, \boldsymbol{k})\mathbf{c}^{\Theta_i}(\lambda, \boldsymbol{k})$$

$$\lambda \in D_{\boldsymbol{k}}(\Theta_i) \quad (1 \leq i \leq r)$$

$$d_{\Theta_i}(\lambda, \boldsymbol{k}) := (-1)^i \operatorname*{Res}_{\xi_i=\lambda_i} \cdots \operatorname*{Res}_{\xi_2=\lambda_2} \operatorname*{Res}_{\xi_1=\lambda_1} (\mathbf{c}_{\Theta_i}(\xi, \boldsymbol{k})^{-1} \mathbf{c}_{\Theta_i}(-\xi, \boldsymbol{k})^{-1})$$

$$= \tilde{\mathbf{c}}_{\Theta_i}(\rho(\boldsymbol{k}), \boldsymbol{k})^2$$

$$\times \prod_{j=1}^{i} \frac{-2^{2\boldsymbol{\alpha}-2\boldsymbol{\beta}-1}\lambda_j}{\pi} \frac{\Gamma\left(\frac{1}{2}(\lambda_j + \boldsymbol{\alpha} + |\boldsymbol{\beta}| + 1)\right)\Gamma\left(\frac{1}{2}(-\lambda_j + \boldsymbol{\alpha} + |\boldsymbol{\beta}| + 1)\right)}{\Gamma\left(\frac{1}{2}(\lambda_j - \boldsymbol{\alpha} + |\boldsymbol{\beta}| + 1)\right)\Gamma\left(\frac{1}{2}(-\lambda_j - \boldsymbol{\alpha} + |\boldsymbol{\beta}| + 1)\right)}$$

$$\times \prod_{1\leq q<p\leq i} \frac{(\lambda_q^2 - \lambda_p^2)\Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q + 2\boldsymbol{k}_m)\right)\Gamma\left(\frac{1}{2}(-\lambda_q - \lambda_p + 2\boldsymbol{k}_m)\right)}{4\Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q - 2\boldsymbol{k}_m + 2)\right)\Gamma\left(\frac{1}{2}(-\lambda_q - \lambda_p - 2\boldsymbol{k}_m + 2)\right)}$$

(If $r = 1$ or $\boldsymbol{k}_m = 0$, then the factors in the third line $= 1$.)

Wave packets

$\lambda = \lambda_{\mathfrak{a}(\Theta_i)} + \lambda_{\mathfrak{a}_{\Theta_i}} \quad (\lambda_{\mathfrak{a}(\Theta_i)} \in \mathfrak{a}(\Theta_i)^*,\ \lambda_{\mathfrak{a}_{\Theta_i}} \in \mathfrak{a}^*_{\Theta_i})$

$d\lambda_{\mathfrak{a}_{\Theta_i}} = d\lambda_{i+1} \cdots d\lambda_r$

$\nu_{\boldsymbol{k},\Theta_i}$: measure on $D_{\boldsymbol{k}}(\Theta_i) + \sqrt{-1}\mathfrak{a}^*_{\Theta_i}$ given by

$$d\nu_{\boldsymbol{k},\Theta_i}(\lambda) = (2\pi)^{-r+i} d_{\Theta_i}(\lambda_{\mathfrak{a}(\Theta_i)}, \boldsymbol{k}) \frac{d(\mathrm{Im}\,\lambda_{\mathfrak{a}_{\Theta_i}})}{|\mathbf{c}^{\Theta_i}(\lambda, \boldsymbol{k})|^2}$$

$\phi \in \mathscr{P}\mathscr{W}(\mathfrak{a}_{\mathbb{C}})^W$

$$\mathscr{J}_{\boldsymbol{k},\Theta_i}\phi(x) := \frac{1}{|W(\Theta_i)|} \int_{D_{\boldsymbol{k}}(\Theta_i)+\sqrt{-1}\mathfrak{a}^*_{\Theta_i}} \phi(\lambda) F(\lambda, \boldsymbol{k}; x)\, d\nu_{\boldsymbol{k},\Theta_i}(\lambda)$$

$D_{\boldsymbol{k}}(\Theta_i) = \emptyset \Rightarrow \nu_{\boldsymbol{k},\Theta_i} = 0$

$$\mathscr{J}_{\boldsymbol{k},\mathscr{B}}\phi(x) = \sum_{\lambda \in D_{\boldsymbol{k}}(\mathscr{B})} d_{\mathscr{B}}(\lambda, \boldsymbol{k}) \phi(\lambda) F(\lambda, \boldsymbol{k}; x)$$

Main result (Honda-Oda-S)

Assume $\boldsymbol{\alpha} > -1$, $\boldsymbol{\beta} \in \mathbb{R}$, $\boldsymbol{k}_m \geq 0$.

$$f(x) = \sum_{i=0}^{r} \mathscr{J}_{\boldsymbol{k},\Theta_i} \mathscr{F}_{\boldsymbol{k}} f(x) \quad (f \in C_c^\infty(\mathfrak{a})^W,\ x \in \mathfrak{a}),$$

$$\frac{1}{|W|}\int_{\mathfrak{a}} |f(x)|^2 \delta_{\boldsymbol{k}}(x)\,dx = \sum_{i=0}^{r} \frac{1}{|W(\Theta_i)|} \int_{D_{\boldsymbol{k}}(\Theta_i)+\sqrt{-1}\,\mathfrak{a}^*_{\Theta_i}} |\mathscr{F}_{\boldsymbol{k}} f(\lambda)|^2 \, d\nu_{\boldsymbol{k},\Theta_i}(\lambda)$$

$$F(\lambda, \boldsymbol{k}) \in L^2\left(\mathfrak{a}, \tfrac{1}{|W|}\delta_{\boldsymbol{k}}(x)dx\right) \Longleftrightarrow \lambda \in W D_{\boldsymbol{k}}(\mathscr{B})$$

$$\frac{1}{|W|}\int_{\mathfrak{a}} F(\lambda, \boldsymbol{k}; x)^2 \delta_{\boldsymbol{k}}(x)\,dx = \frac{1}{d_{\mathscr{B}}(\lambda, \boldsymbol{k})} \quad (\lambda \in D_{\boldsymbol{k}}(\mathscr{B}))$$

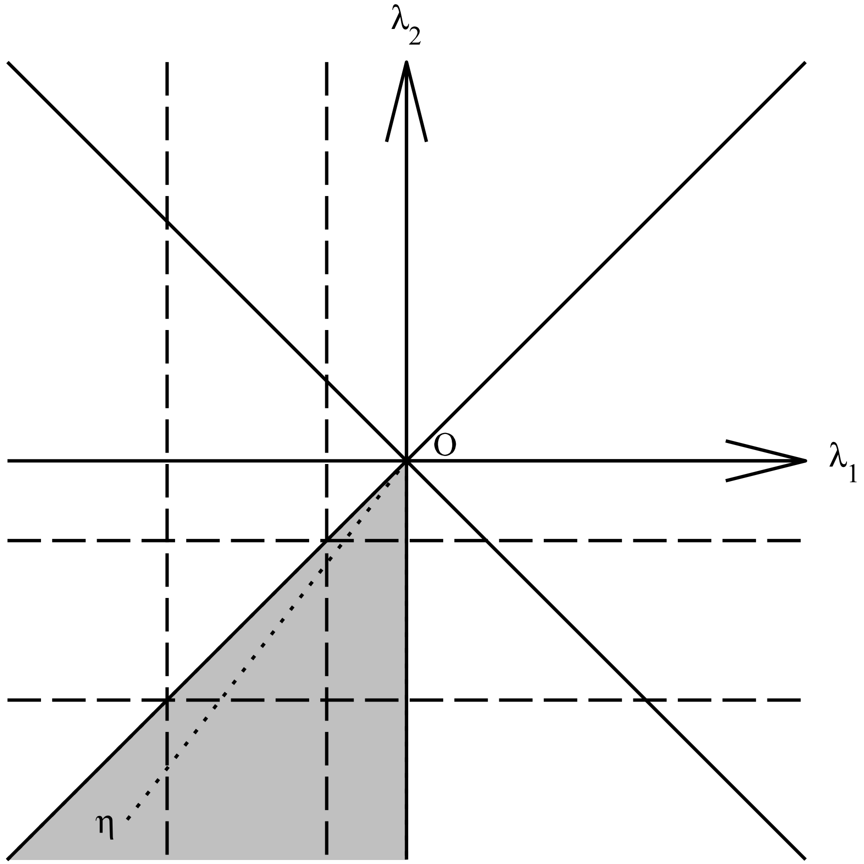
$$F(\lambda, \boldsymbol{k}) = \mathbf{c}(\lambda, \boldsymbol{k}) \Phi(\lambda, \boldsymbol{k}) \quad (\lambda \in D_{\boldsymbol{k}}(\mathscr{B}))$$

Rank two case

shaded region : $-\mathrm{Cl}(\mathfrak{a}_+^*)$,

dashed lines : $\lambda_i = \xi \ (\xi \in D_{\boldsymbol{k}}(\Theta_1),\ i = 1, 2)$ **($\#D_{\boldsymbol{k}}(\Theta_1) = 2$ for explanation)**

$\longleftrightarrow$ singularities of $\mathbf{c}(\lambda, \boldsymbol{k})^{-1}$ that concern as $\eta \to 0$



$$\int_{\eta+\sqrt{-1}\mathfrak{a}^*} F(\lambda)\Phi(\lambda,\boldsymbol{k};x)\,\frac{d\mu(\lambda)}{\mathbf{c}(-\lambda,\boldsymbol{k})} - \int_{\sqrt{-1}\mathfrak{a}^*} F(\lambda)\Phi(\lambda,\boldsymbol{k};x)\,\frac{d\mu(\lambda)}{\mathbf{c}(-\lambda,\boldsymbol{k})}$$

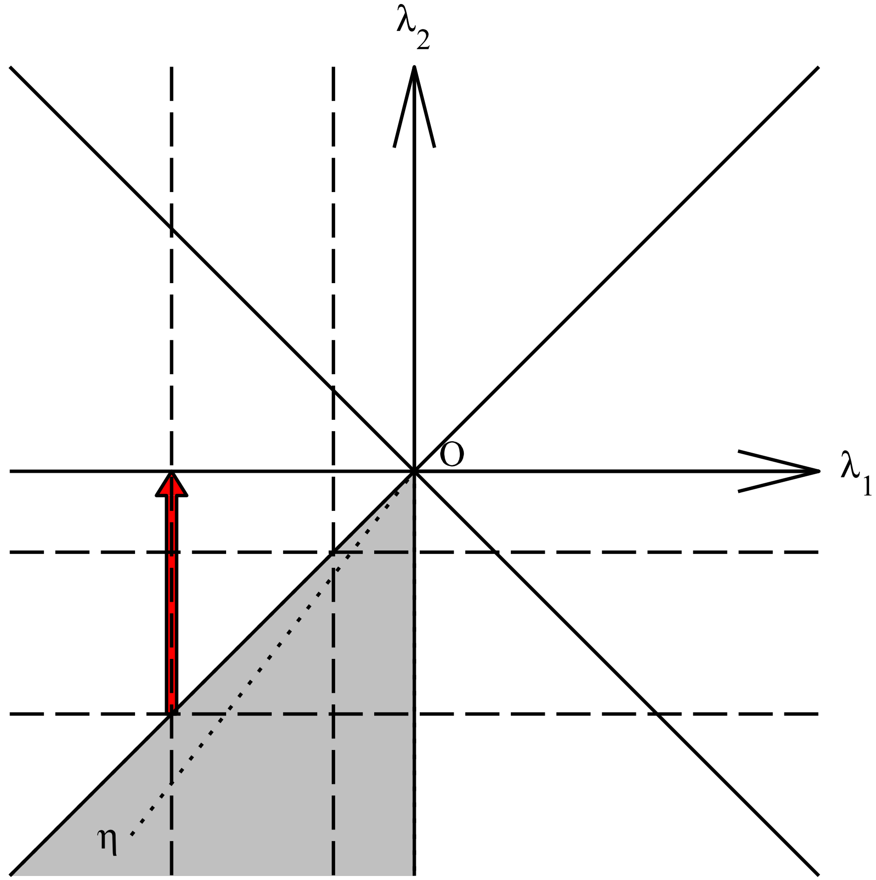
$$= -\frac{1}{8\pi\sqrt{-1}} \sum_{\xi\in D_{\boldsymbol{k}}(\Theta_1)} \sum_{i=1,2} \int_{\check{\xi}+\sqrt{-1}\mathbb{R}} (F(\lambda)\Phi(\lambda,\boldsymbol{k};x))|_{\lambda_i=\xi} \operatorname*{Res}_{\lambda_i=\xi}(\mathbf{c}(-\lambda,\boldsymbol{k})^{-1})\,d\lambda_{\bar{i}}$$

$$(F(\lambda)=\mathscr{F}(f)(\lambda),\ \check{\xi}=(\xi,\xi),\ \bar{1}=2,\ \bar{2}=1)$$

$$= -\frac{1}{8\pi\sqrt{-1}} \sum_{\xi\in D_{\boldsymbol{k}}(\Theta_1)} \int_{\check{\xi}+\sqrt{-1}\mathbb{R}} (F(\lambda)F_{\Xi}(\lambda,\boldsymbol{k};x))|_{\lambda_1=\xi} \operatorname*{Res}_{\lambda_1=\xi}(\mathbf{c}_{\Xi}(\lambda,\boldsymbol{k})^{-1}\mathbf{c}(-\lambda,\boldsymbol{k})^{-1})\,d\lambda_2$$

$F_{\Xi}(\lambda,\boldsymbol{k}) := \mathbf{c}_{\Xi}(\lambda,\boldsymbol{k})\Phi(\lambda,\boldsymbol{k}) + \mathbf{c}_{\Xi}(s_{12}\lambda,\boldsymbol{k})\Phi(s_{12}\lambda,\boldsymbol{k})$ **(regular on Re** $\lambda_i \leq 0\ (i=1,2)$**)**

$\tilde{\mathbf{c}}_{\Xi}(\lambda,\boldsymbol{k}) := \dfrac{\Gamma\left(\frac{1}{2}(\lambda_2-\lambda_1)\right)}{\Gamma\left(\frac{1}{2}(\lambda_2-\lambda_1+2\boldsymbol{k}_m)\right)}$ **(partial c-function of type** A_1**)**



Move the domain of integration from $\check{\xi}+\sqrt{-1}\mathbb{R}$ to $\xi\beta_1+\sqrt{-1}\mathbb{R}$. Singularities come from $\operatorname*{Res}_{\lambda_1=\xi}(\mathbf{c}(-\lambda,\boldsymbol{k})^{-1})$ and at λ_2 with $(\xi,\lambda_2)\in D_{\boldsymbol{k}}(\mathscr{B})$ of simple poles and we pick up residues.

Concluding remarks

- The preprint of this work is available in arXiv.

  `https://arxiv.org/abs/2007.08281`

- Future issues
  - non-symmetric case
  - applications to group cases

Thank you!