HARDYS INEQUALITY IN PERSPECTIVE OF ROOT SYSTEMS

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Abstract

Hardy and Littlewood proved the following inequality for Fourier coefficients

$$\sum_{k\in\mathbb{Z}} \frac{|\hat{f}(k)|}{|k|+1} \lesssim \|f\|_{\mathrm{Re}H^1}, \tag{1}$$

where $\mathrm{Re}H^1$ denotes the real Hardy space constituted by the boundary values of the real parts of functions in the Hardy space $H^1(\mathbb{D})$, where $\mathbb{D}$ is the unit disk on the plane. Two decades ago Kanjin initiated investigation of analogues of (1) for orthogonal expansions with respect to the Hermite and standard Laguerre functions.

More generally, Hardy's inequality associated with orthogonal expansions, for a measure space (X,μ), where $X \subset \mathbb{R}^d$, and $p \in (0,1]$, is of the form

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f,\varphi_n\rangle|^p}{(n_1+\ldots+n_d+1)^E} \lesssim \|f\|^p_{H^1(X,\mu)}, \qquad f\in H^p(X,\mu), \tag{2}$$

where $n=(n_1,\ldots,n_d)$, $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(X,\mu)$, $\{\varphi_n\}_{n\in\mathbb{N}^d}$ is an orthonormal basis in $L^2(X,\mu)$, and $H^p(X,\mu)$ denotes an appropriate Hardy space.

We introduce a method of proving Hardy's inequality for a large class of orthogonal expansions. It relies on study of the corresponding integral kernel. The strength of this method is evidenced by its generality and sharpness. The latter means that in the classical systems (e.q. Hermite, Laguerre, Jacobi) it leads to the smallest possible exponent E.

Moreover, in the spirit of the conference topic, we mention some aspects of the proof which involve the theory of root systems.

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