Metric Graph Theory and Related Topics
C. I. R. M.

Multiscale Substitution Tilings

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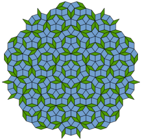
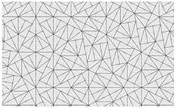
Based on a joint work with Yotam Smilansky (Rutgers)
6-10 December, 2021

A short introduction to my field

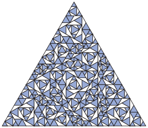
In *aperiodic order* we study properties **infinite** patterns, and of spaces of patterns, usually for their dynamical properties.



Penrose tiling - from Wikipedia



Pinwheel tiling - from Wikipedia



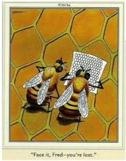
A multiscale tiling

The patterns are often non-periodic, but their "order" is manifested in other lattice-like properties, such as:

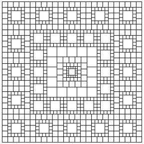
Finite local complexity (FLC):

the set of patches of a fixed size is finite.

Repetitivity:



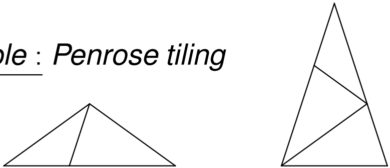
Self similarity



Constructing tilings - fixed scale substitution setup

Example : *Penrose tiling*

Substitution rule :



- By repeated subdivisions, and rescaling, one tile larger and larger regions, with the same set of tiles.
- As a limit of these, one gets a tiling of the whole space.

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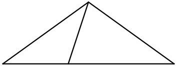

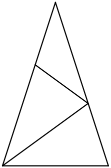

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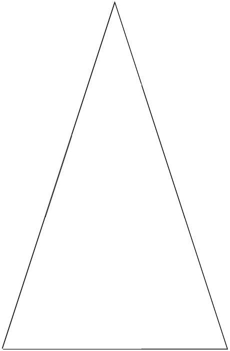

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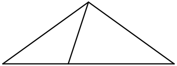
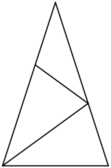
Substitution rule :  

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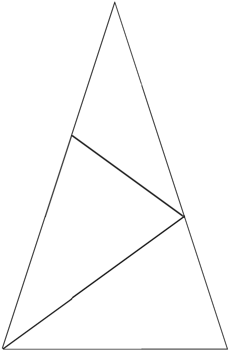

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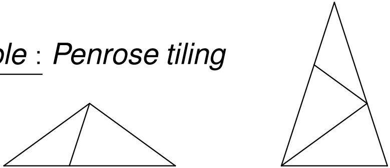

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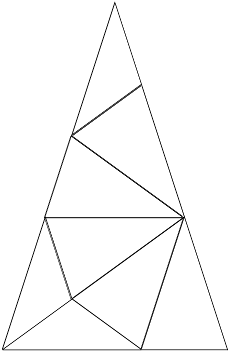

Constructing tilings - fixed scale substitution setup

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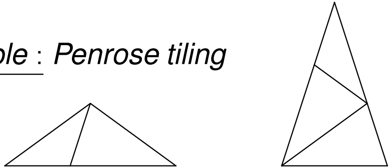

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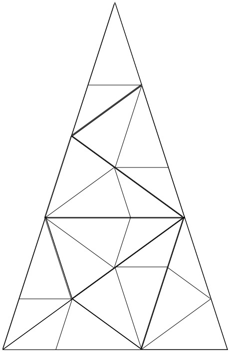

Constructing tilings - fixed scale substitution setup

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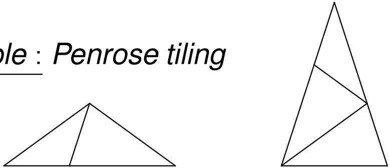

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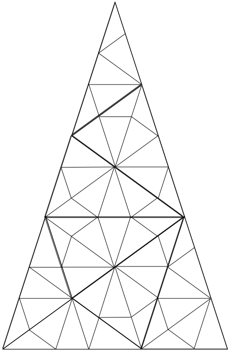

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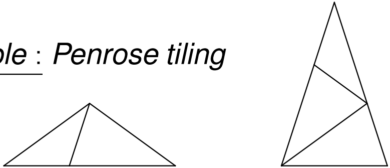
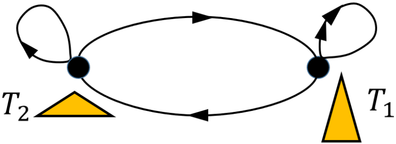
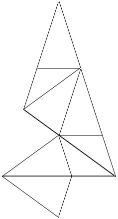


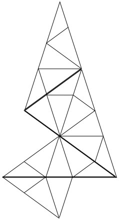


$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Adjacency matrix



$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} \Longrightarrow$$



$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

Definitions - Multiscale substitution scheme

A *multiscale substitution scheme* $\sigma = (\tau_\sigma, \varrho_\sigma)$ in $\mathbb{R}^d$ consists of a list of *prototiles* $\tau_\sigma = (T_1, \ldots, T_n)$, and a *substitution rule* defining a partition $\varrho_\sigma(T_i)$ of each prototile T_i, so that every tile in $\varrho_\sigma(T_i)$ is a translation of a rescaled copy of a prototile in τ_σ, where *multiple scales are allowed!*.

Example 1: $\tau_\sigma = \{S\}$:

$\varrho_\sigma(S)$

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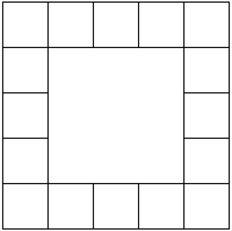
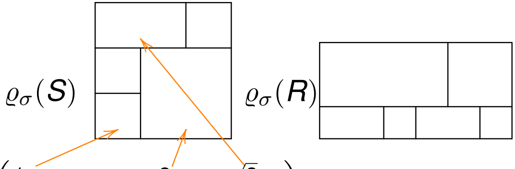


Definitions - Multiscale substitution scheme

Example 2: $\tau_\sigma = \{S, R\}$:
We assume that prototiles are of volume 1.

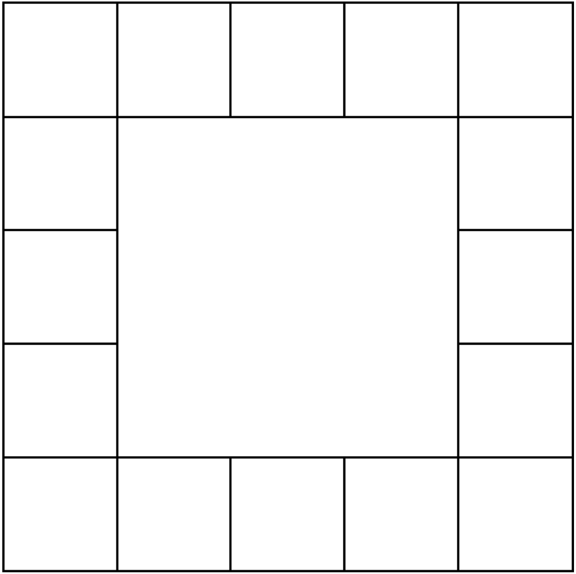


In $\varrho_\sigma(S)$: $\left(\frac{1}{3}S \ (\times 3),\ \ \frac{2}{3}S,\ \ \frac{\sqrt{2}}{3}R\right)$,

In $\varrho_\sigma(R)$: $\left(\frac{1}{3\sqrt{2}}S \ (\times 2),\ \ \frac{\sqrt{2}}{3}S,\ \ \frac{1}{3}R \ (\times 2),\ \ \frac{2}{3}R\right)$

Those numbers are referred to as the *scales*.

Repeated subdivisions fails for multiple scales



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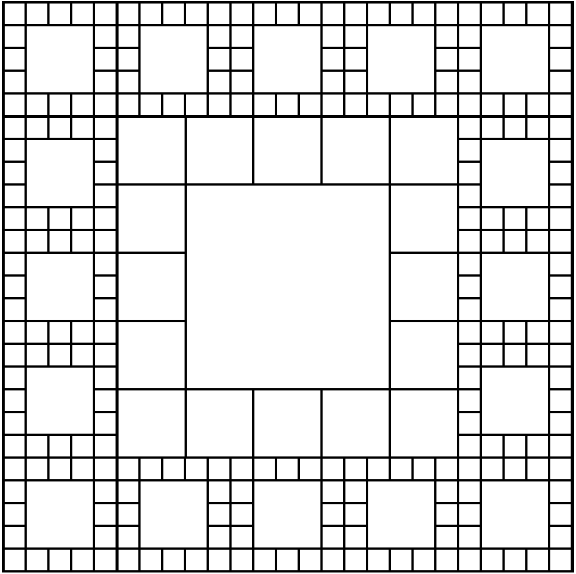

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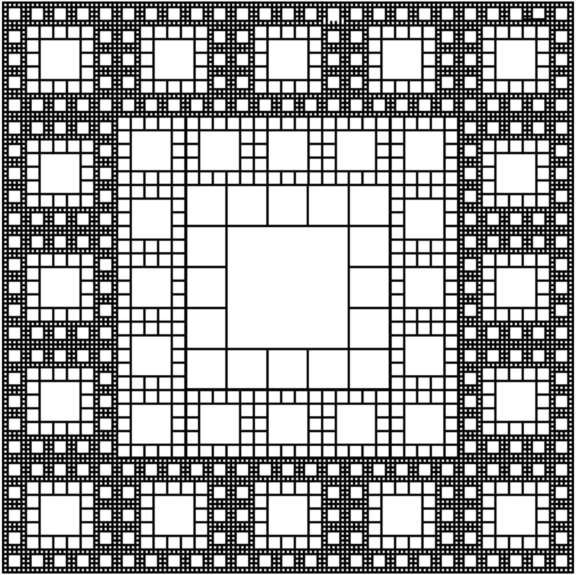
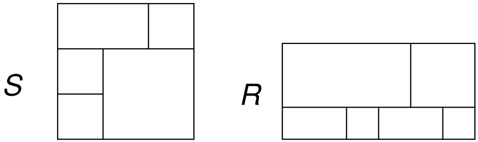


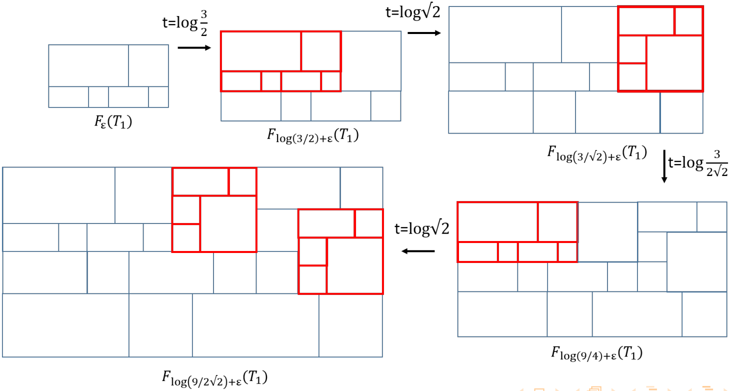
Multiscale substitution flow

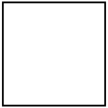
The *substitution flow* $F_t(T_i)$ is defined on the prototiles as follows:

- At time $t = 0$, $F_0(T_i) = T_i$.
- For $t > 0$, $F_t(T_i)$ is the patch obtained by inflating T_i by e^t and then repeatedly subdivide tiles, via the substitution rule, until all tiles are of volume $\leqslant 1$.

Multiscale substitution flow







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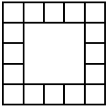

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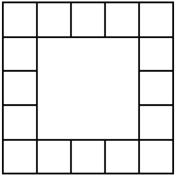

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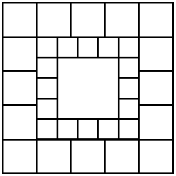

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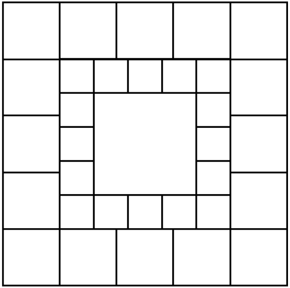

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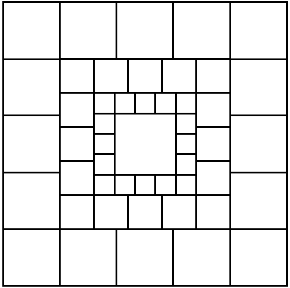

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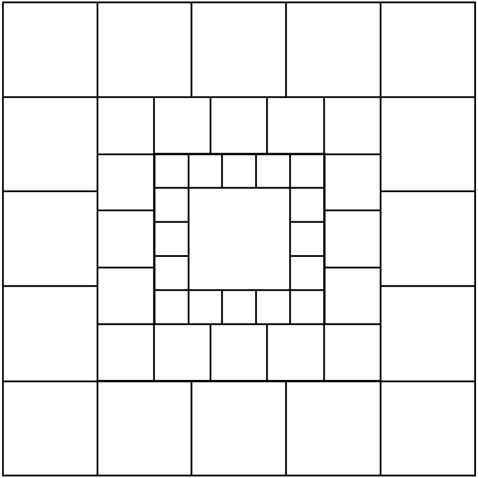

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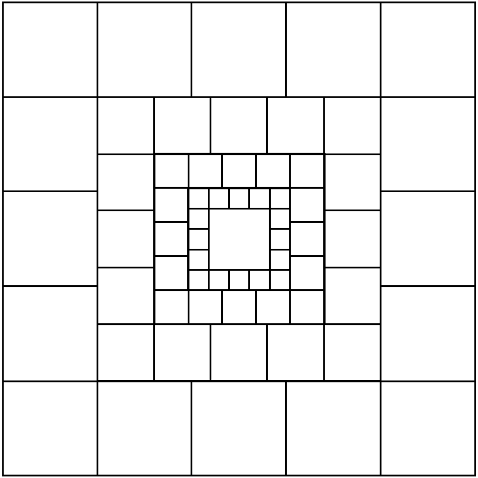

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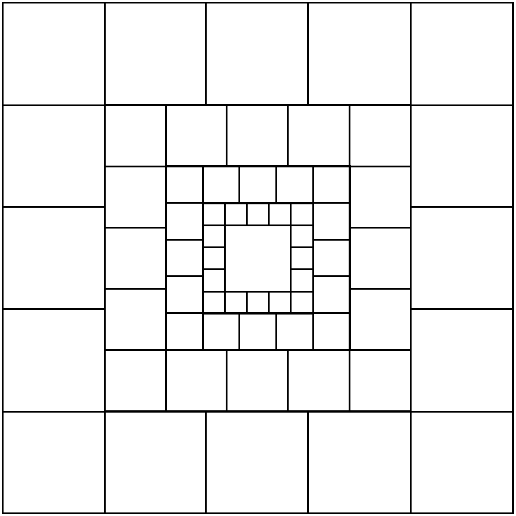

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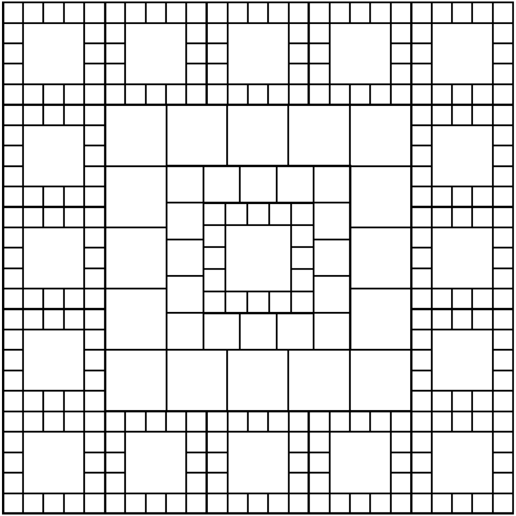

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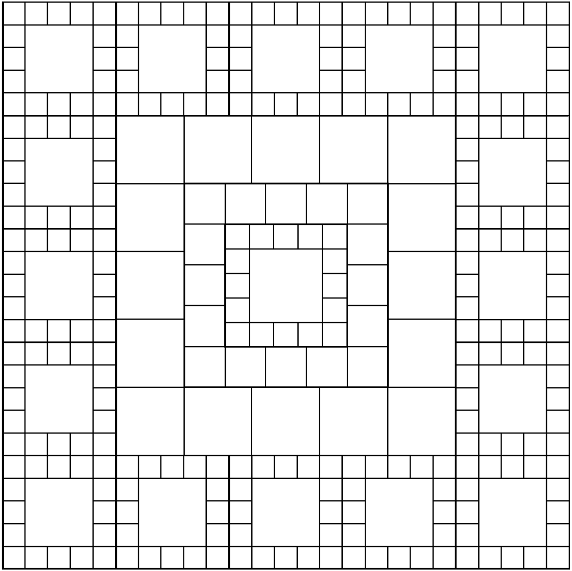

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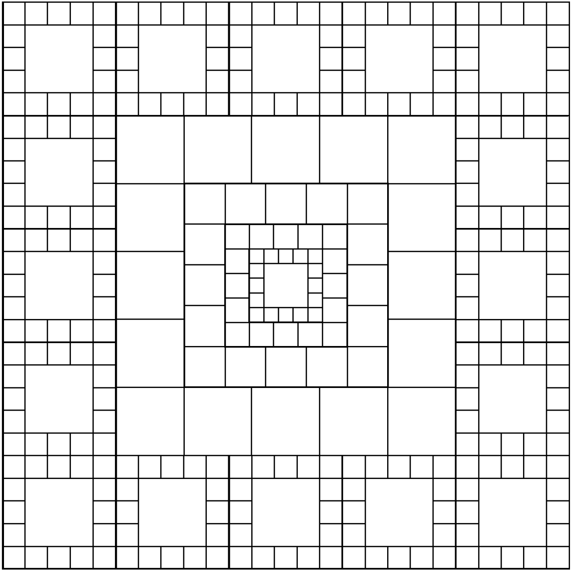

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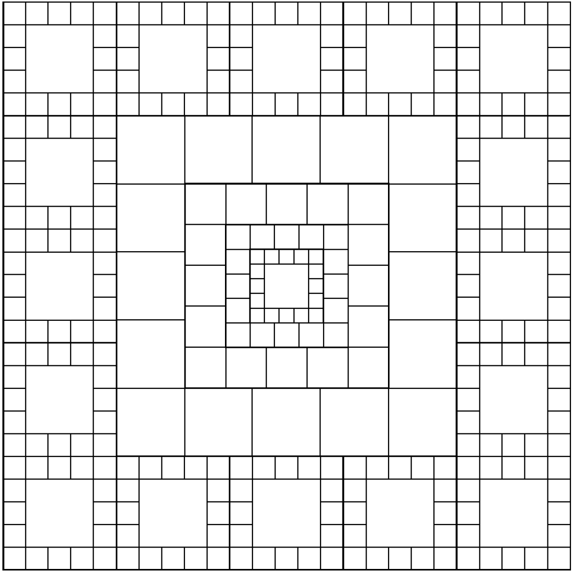
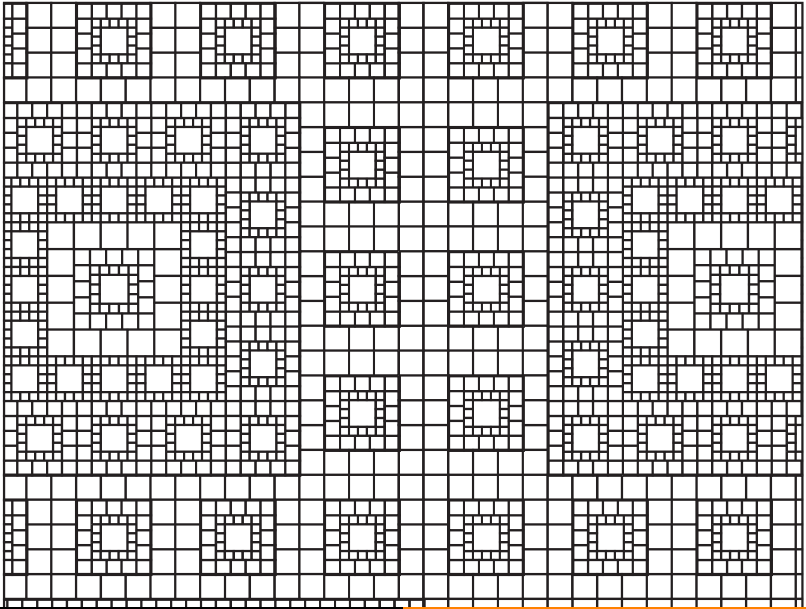


Limit objects - tilings of $\mathbb{R}^d$

- *Multiscale substitution tilings* are tilings of $\mathbb{R}^d$ that are limits of (translations of) those patches, $F_t(T_i)$.
- Limits are taken with respect to a natural compact topology in which two tilings / patches are close if when looking at a large centered ball they are close in the Hausdorff metric.
- The *tiling space* $\mathbb{X}_\sigma$ is the space of all these limit objects. With the action of $\mathbb{R}^d$ by translation, $(\mathbb{X}_\sigma, \mathbb{R}^d)$ becomes a nice, compact, dynamical system.

Limit objects - tilings of $\mathbb{R}^d$



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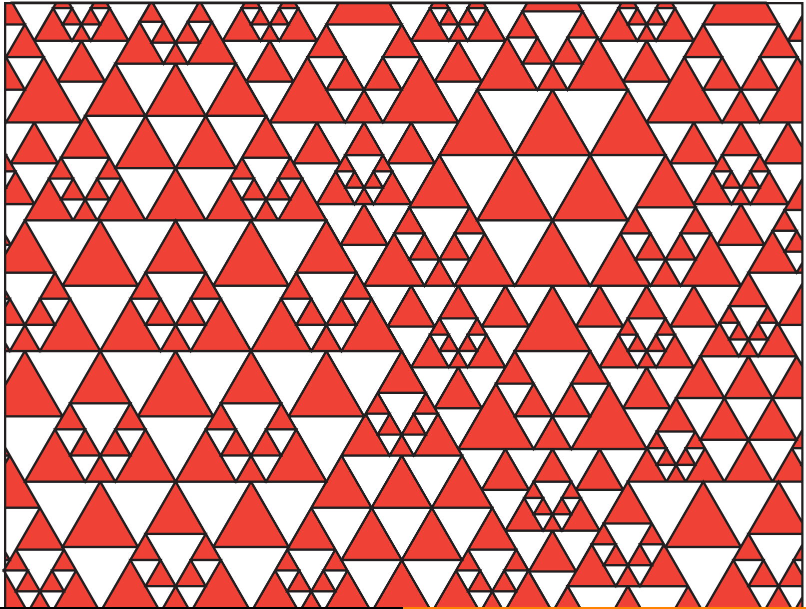
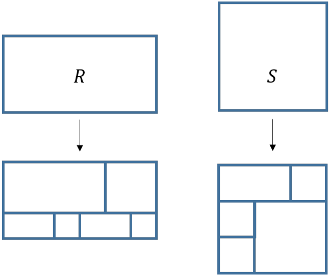
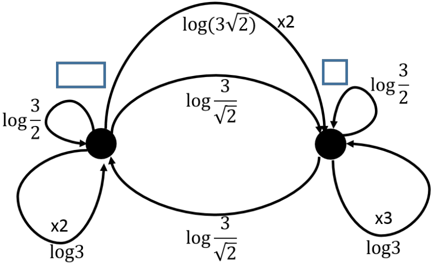


Graph model for multiscale substitution schemes





Vertices: are the prototiles

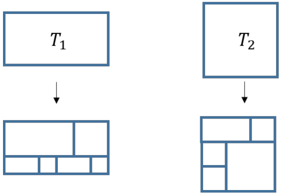
Edges: an outgoing edge for each tile in the subdivision

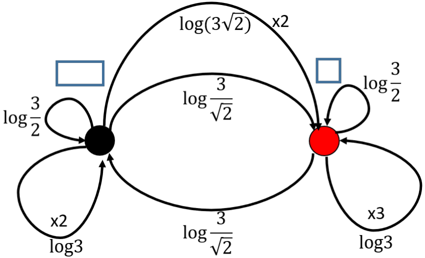
Lengths: a tile of scale α $\leftsquigarrow\rightsquigarrow$ an edge of length $\log(1/\alpha)$

Note: If $t = \log(1/\alpha)$, inflating a tile of scale α by e^t gives a tile of scale 1!

Graph Related Results

Stationary tilings





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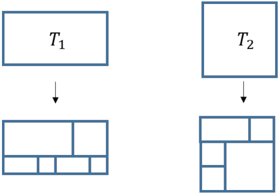

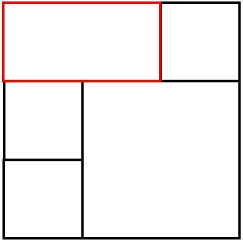

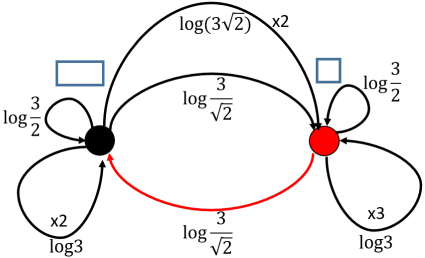

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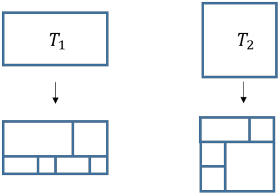

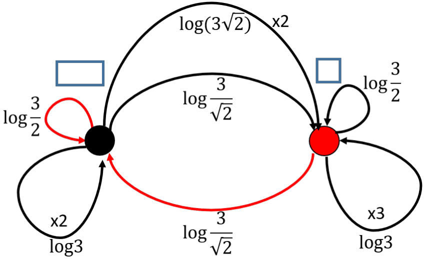

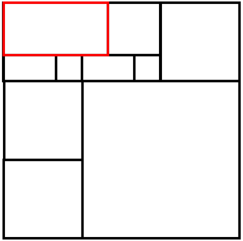

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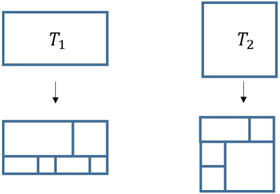

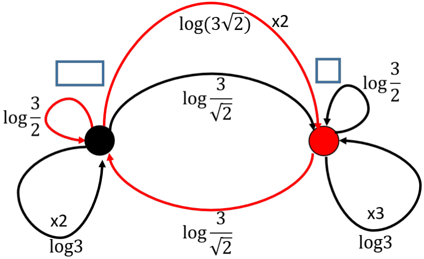

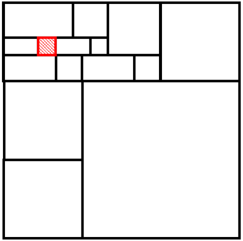

$$\boxed{s = \log\frac{3}{\sqrt{2}} + \log\frac{3}{2} + \log(3\sqrt{2})}$$

Stationary tilings

- Suppose that $s \in \mathbb{R}$ is such that $F_s(T_i)$ contains a copy of T_i in its interior, i.e. a length of a closed path in G_σ
- Then we can place the origin so that for every $k \in \mathbb{N}$: $F_{ks}(T_i)$ contains a copy of $F_{(k-1)s}(T_i)$.
- Then $\mathcal{S} := \bigcup_{k=0}^{\infty} F_{ks}(T_i)$ is a tiling of $\mathbb{R}^d$, and $F_s(\mathcal{S}) = \mathcal{S}$.

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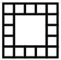
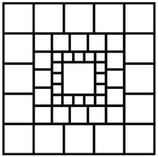
- Suppose that $s \in \mathbb{R}$ is such that $F_s(T_i)$ contains a copy of T_i in its interior, i.e. a length of a closed path in G_σ
- Then we can place the origin so that for every $k \in \mathbb{N}$: $F_{ks}(T_i)$ contains a copy of $F_{(k-1)s}(T_i)$.
- Then $\mathcal{S} := \bigcup_{k=0}^{\infty} F_{ks}(T_i)$ is a tiling of $\mathbb{R}^d$, and $F_s(\mathcal{S}) = \mathcal{S}$.

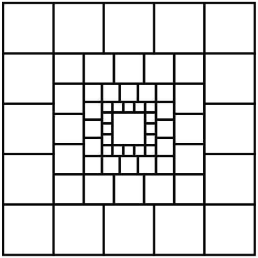
Stationary tilings



Stationary tilings



Stationary tilings



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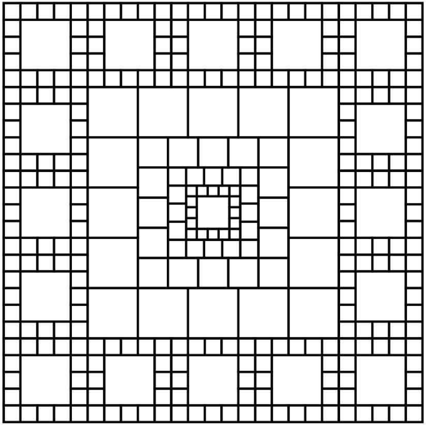
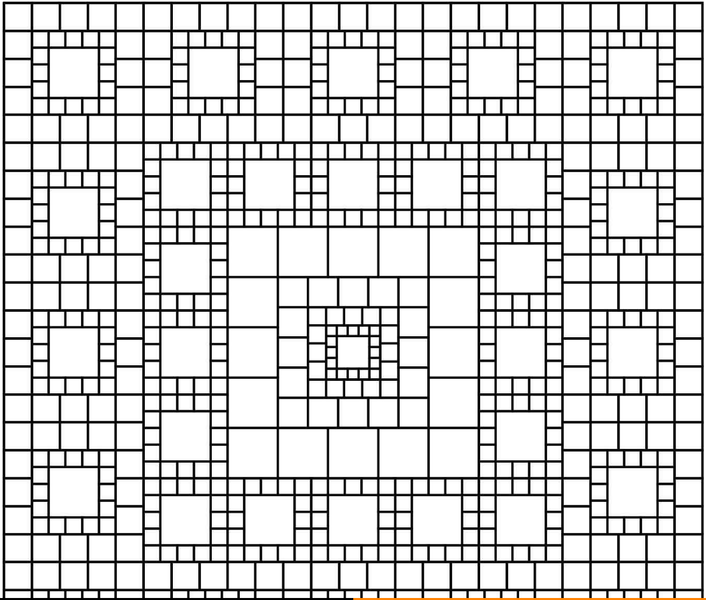


Stationary tilings



Graph model for multiscale substitution schemes

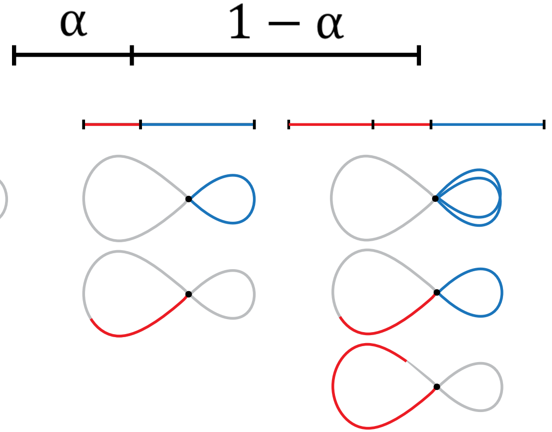
Let $\sigma = (\tau_\sigma, \varrho_\sigma)$ be a multiscale substitution scheme.

- σ is called *irreducible* if the associated graph G_σ is strongly connected.
- σ is called *incommensurable* if the associated graph G_σ contains two closed paths of lengths a, b, where $a \notin \mathbb{Q}b$.
- Our project focuses on schemes that are irreducible and incommensurable.

Important Observation

$\{\textit{tiles in } F_t(T_i)\} \leftrightsquigarrow \{\textit{paths of lengths } t \textit{ from } T_i \textit{ in } G_\sigma\}$

Example:



Theorem (Kiro-Smilansky$\times$2 '20, Smilansky $\geqslant'$ 21)

Let σ be an irreducible incommensurable scheme in $\mathbb{R}^d$. Then

$$\#\{\text{tiles in } F_t(T_i)\} = C_\sigma e^{dt} + ERROR,$$

where C_σ is explicit and $ERROR = o(e^{dt}), \quad t \to \infty$.

The constant C_σ is given explicitly in a recent paper of Smilansky in term of the following three matrices:

$$(S)_{ij} = \#\left\{\begin{matrix}\text{tiles of type} \\ j \text{ in } \varrho_\sigma(T_i)\end{matrix}\right\}, \quad (V)_{ij} = \sum_{\substack{\text{tiles of type} \\ j \text{ in } \varrho_\sigma(T_i)}} \mathrm{vol}(T),$$

$$(H)_{ij} = \sum_{\substack{\text{tiles of type} \\ j \text{ in } \varrho_\sigma(T_i)}} -\mathrm{vol}(T)\log(\mathrm{vol}(T))$$

Counting result

Theorem (Smilansky, S. ′21)

For every irreducible incommensurable scheme σ, $\exists k > 0$ s.t.

$$\underbrace{\left|\#\{\text{tiles in } F_t(T_i)\} - C_\sigma e^{dt}\right|}_{ERROR} \geqslant \Omega\left(\frac{e^{dt}}{t^k}\right).$$

Corollary (Smilansky, S. ′21)

Every irreducible, incommensurable multiscale substitution tiling $\mathcal{T}$ is **not uniformly spread**.

That is, if Λ is a point set obtained from the tiling $\mathcal{T}$ by placing a point in each tile, every bijection $\varphi : \Lambda \to \mathbb{Z}^d$ will have points that are mapped "arbitrarily far away" from where they are.

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Dynamics

- The irreducibility and incommensurability of G_σ can also be applied to study the dynamical system $(\mathbb{X}_\sigma, \mathbb{R}^d)$.

Theorem (Smilansky, S. ′21)

The dynamical system $(\mathbb{X}_\sigma, \mathbb{R}^d)$ is minimal.

- Geometric interpretation:

Corollary (Smilansky, S. ′21)

1. *Every multiscale tiling is (almost) repetitive.*
2. *Every two multiscale tilings in $\mathbb{X}_\sigma$ are (almost) locally indistinguishable.*

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