Mapping Class Groups and CAT(0) Cube Complexes

Harry Petyt

9th December 2021

Overview

Goal

Use the median perspective to show that mapping class groups are quasiisometric to CAT(0) cube complexes.

Overview

Goal

Use the median perspective to show that mapping class groups are quasiisometric to CAT(0) cube complexes.

Outline

- Embedding hyperbolic spaces in finite products of trees.
- Medians.
- Hyperbolic spaces and mapping class groups.

Hyperbolic cones

Fact (Buyalo–Dranishnikov–Schroeder)

Let Z be a complete, bounded metric space. There is a hyperbolic graph $\mathrm{Co}\, Z$ with $\partial\, \mathrm{Co}\, Z = Z$. If Y is a *visual* hyperbolic space, then Y is quasiisometric to $\mathrm{Co}\, \partial Y$.

Hyperbolic cones

Fact (Buyalo–Dranishnikov–Schroeder)

Let Z be a complete, bounded metric space. There is a hyperbolic graph $\operatorname{Co} Z$ with $\partial \operatorname{Co} Z = Z$. If Y is a *visual* hyperbolic space, then Y is quasiisometric to $\operatorname{Co} \partial Y$.

Definition

A hyperbolic space Y is *visual* if there exist $y_0 \in Y$ and r such that every $y \in Y$ lies r–close to some geodesic ray emanating from y_0.

Hyperbolic cones

Fact (Buyalo–Dranishnikov–Schroeder)

Let Z be a complete, bounded metric space. There is a hyperbolic graph $\mathrm{Co}\, Z$ with $\partial\, \mathrm{Co}\, Z = Z$. If Y is a *visual* hyperbolic space, then Y is quasiisometric to $\mathrm{Co}\, \partial Y$.

Definition

A hyperbolic space Y is *visual* if there exist $y_0 \in Y$ and r such that every $y \in Y$ lies r–close to some geodesic ray emanating from y_0.

Observation

Given a hyperbolic space X, the hyperbolic space Y obtained by gluing a ray to each point of X is visual.

Fact (Buyalo–Dranishnikov–Schroeder)

Let Z be a complete, bounded metric space. There is a hyperbolic graph $\operatorname{Co} Z$ with $\partial \operatorname{Co} Z = Z$. If Y is a *visual* hyperbolic space, then Y is quasiisometric to $\operatorname{Co} \partial Y$.

Definition

A hyperbolic space Y is *visual* if there exist $y_0 \in Y$ and r such that every $y \in Y$ lies r–close to some geodesic ray emanating from y_0.

Observation

Given a hyperbolic space X, the hyperbolic space Y obtained by gluing a ray to each point of X is visual.

For any hyperbolic space X, there exists Y such that $X \hookrightarrow \operatorname{Co} \partial Y$.

Embeddings in products of trees

For any hyperbolic space X, there exists Y such that $X \hookrightarrow \operatorname{Co} \partial Y$.

Embeddings in products of trees

For any hyperbolic space X, there exists Y such that $X \hookrightarrow \operatorname{Co} \partial Y$.

Theorem (Buyalo–Dranishnikov–Schroeder)

If $\operatorname{capdim} Z < \infty$, then $\operatorname{Co} Z$ admits a quasiisometric embedding in a finite product of trees.

Embeddings in products of trees

For any hyperbolic space X, there exists Y such that $X \hookrightarrow \operatorname{Co} \partial Y$.

Theorem (Buyalo–Dranishnikov–Schroeder)

If $\operatorname{capdim} Z < \infty$, then $\operatorname{Co} Z$ admits a quasiisometric embedding in a finite product of trees.

Fact (Mackay–Sisto)

If Y is hyperbolic, then $\operatorname{capdim} \partial Y \leqslant \operatorname{asdim} Y$.

Embeddings in products of trees

For any hyperbolic space X, there exists Y such that $X \hookrightarrow \operatorname{Co} \partial Y$.

Theorem (Buyalo–Dranishnikov–Schroeder)

If $\operatorname{capdim} Z < \infty$, then $\operatorname{Co} Z$ admits a quasiisometric embedding in a finite product of trees.

Fact (Mackay–Sisto)

If Y is hyperbolic, then $\operatorname{capdim} \partial Y \leqslant \operatorname{asdim} Y$.

Summary

If X is hyperbolic and $\operatorname{asdim} X < \infty$, then

$$X \hookrightarrow \operatorname{Co} \partial Y \hookrightarrow \prod^{n} T_i.$$

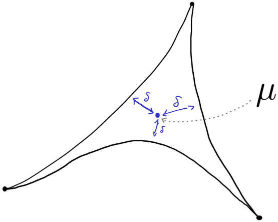
Medians

Recall that a graph is *median* if $\mu = [v_1, v_2] \cap [v_1, v_3] \cap [v_2, v_3]$ is a singleton for every three vertices.

Medians

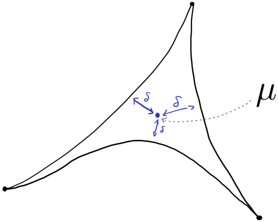
Recall that a graph is *median* if $\mu = [v_1, v_2] \cap [v_1, v_3] \cap [v_2, v_3]$ is a singleton for every three vertices.

Hyperbolic spaces also have a kind of median: μ is δ–close to all three geodesic edges.



Medians

Recall that a graph is *median* if $\mu = [v_1, v_2] \cap [v_1, v_3] \cap [v_2, v_3]$ is a singleton for every three vertices.



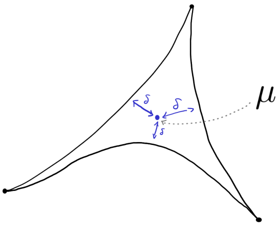
Hyperbolic spaces also have a kind of median: μ is δ–close to all three geodesic edges.

Coarse median spaces generalise this.

Medians

Recall that a graph is *median* if $\mu = [v_1, v_2] \cap [v_1, v_3] \cap [v_2, v_3]$ is a singleton for every three vertices.

Hyperbolic spaces also have a kind of median:
μ is δ–close to all three geodesic edges.



Coarse median spaces generalise this.

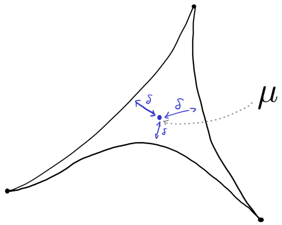
Definition

A map f is *quasimedian* if there exists k such that

$$\mathsf{d}\left(f\mu(x,y,z),\ \mu(f(x),f(y),f(z))\right) \leqslant k.$$

Medians

Recall that a graph is *median* if $\mu = [v_1, v_2] \cap [v_1, v_3] \cap [v_2, v_3]$ is a singleton for every three vertices.



Hyperbolic spaces also have a kind of median:
μ is δ–close to all three geodesic edges.

Coarse median spaces generalise this.

Definition

A map f is *quasimedian* if there exists k such that

$$\mathsf{d}\left(f\mu(x,y,z),\ \mu(f(x),f(y),f(z))\right) \leqslant k.$$

Examples

- Coordinate projection $\prod Q_i \to Q_i$ is 0–quasimedian.
- If X and Y are hyperbolic spaces, then any quasiisometric embedding $X \hookrightarrow Y$ is quasimedian. (Morse lemma.)

The key tool

Key tool (Bowditch, Hagen–P.)

If a coarsely connected coarse median space admits a quasimedian quasiisometric embedding in a CAT(0) cube complex, then it is quasimedian quasiisometric to a CAT(0) cube complex.

The key tool

Key tool (Bowditch, Hagen–P.)

If a coarsely connected coarse median space admits a quasimedian quasiisometric embedding in a CAT(0) cube complex, then it is quasimedian quasiisometric to a CAT(0) cube complex.

Proposition (P.)

The embedding $\operatorname{Co} Z \hookrightarrow \prod^n T_i$ is quasimedian.

The key tool

Key tool (Bowditch, Hagen–P.)

If a coarsely connected coarse median space admits a quasimedian quasiisometric embedding in a CAT(0) cube complex, then it is quasimedian quasiisometric to a CAT(0) cube complex.

Proposition (P.)

The embedding $\operatorname{Co} Z \hookrightarrow \prod^n T_i$ is quasimedian.

Thus, if X is a hyperbolic space with $\operatorname{asdim} X < \infty$, then

$$X \hookrightarrow \operatorname{Co} \partial Y \hookrightarrow \prod^n T_i$$

is a quasimedian quasiisometric embedding.

The key tool

Key tool (Bowditch, Hagen–P.)

If a coarsely connected coarse median space admits a quasimedian quasiisometric embedding in a CAT(0) cube complex, then it is quasimedian quasiisometric to a CAT(0) cube complex.

Proposition (P.)

The embedding $\operatorname{Co} Z \hookrightarrow \prod^n T_i$ is quasimedian.

Thus, if X is a hyperbolic space with $\operatorname{asdim} X < \infty$, then

$$X \hookrightarrow \operatorname{Co} \partial Y \hookrightarrow \prod^n T_i$$

is a quasimedian quasiisometric embedding.

Corollary

A hyperbolic space is quasiisometric to a CAT(0) cube complex if and only if it has finite asymptotic dimension.

Mapping class groups

Fact (Behrstock–Minsky, Bowditch)

Mapping class groups are coarse median spaces.

Mapping class groups

Fact (Behrstock–Minsky, Bowditch)

Mapping class groups are coarse median spaces.

Fact (Masur–Minsky, Bell–Fujiwara)

For any subsurface $U \sqsubset S$, the *curve graph* $\mathcal{C}U$ of U is a hyperbolic graph with finite asymptotic dimension.

Fact (Behrstock–Minsky, Bowditch)

Mapping class groups are coarse median spaces.

Fact (Masur–Minsky, Bell–Fujiwara)

For any subsurface $U \sqsubset S$, the *curve graph* $\mathcal{C}U$ of U is a hyperbolic graph with finite asymptotic dimension.

Theorem (Bestvina–Bromberg–Fujiwara)

There are hyperbolic graphs $X_1, \ldots, X_m$, built out of curve graphs of subsurfaces of S, such that there is a quasiisometric embedding $\mathrm{MCG}(S) \hookrightarrow \prod^m X_i$. Moreover, the $\operatorname{asdim} X_i$ are finite.

Conclusion

By the previous corollary, each X_i is quasimedian quasiisometric to some CAT(0) cube complex Q_i, so we have

$$\mathrm{MCG}(S) \hookrightarrow \prod^{m} X_i \to \prod^{m} Q_i.$$

Conclusion

By the previous corollary, each X_i is quasimedian quasiisometric to some CAT(0) cube complex Q_i, so we have

$$\mathrm{MCG}(S) \hookrightarrow \prod^{m} X_i \to \prod^{m} Q_i.$$

Theorem (Hagen–P.)

The embedding $\mathrm{MCG}(S) \hookrightarrow \prod^m X_i$ is quasimedian.

Conclusion

By the previous corollary, each X_i is quasimedian quasiisometric to some CAT(0) cube complex Q_i, so we have

$$\mathrm{MCG}(S) \hookrightarrow \prod^{m} X_i \to \prod^{m} Q_i.$$

Theorem (Hagen–P.)

The embedding $\mathrm{MCG}(S) \hookrightarrow \prod^m X_i$ is quasimedian.

Corollary

$\mathrm{MCG}(S)$ is quasimedian quasiisometric to a CAT(0) cube complex.