Weakly modular graphs in group theory

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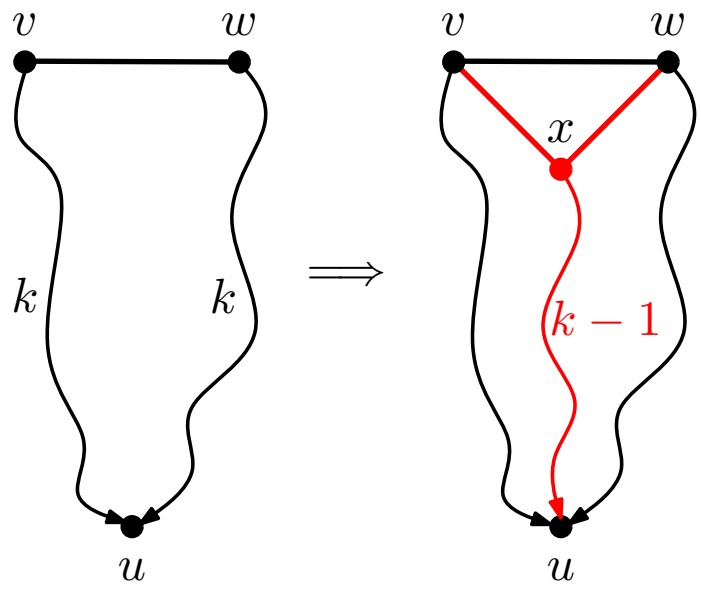
Main source:

Chalopin-Chepoi-Hirai-O., *Weakly modular graphs and nonpositive curvature*, Mem. Amer. Math. Soc., 268 (2020), no. 1309

Weakly modular graph

Definition (Triangle condition)

A (simplicial) graph satisfies the *triangle condition at distance* $k > 0$ *with respect to a vertex* u, denoted $\mathrm{TC}(u, k)$, if for any two vertices v, w with $1 = d(v, w) < d(u, v) = d(u, w) = k$ there exists a common neighbor x of v and w such that $d(u, x) = k - 1$.

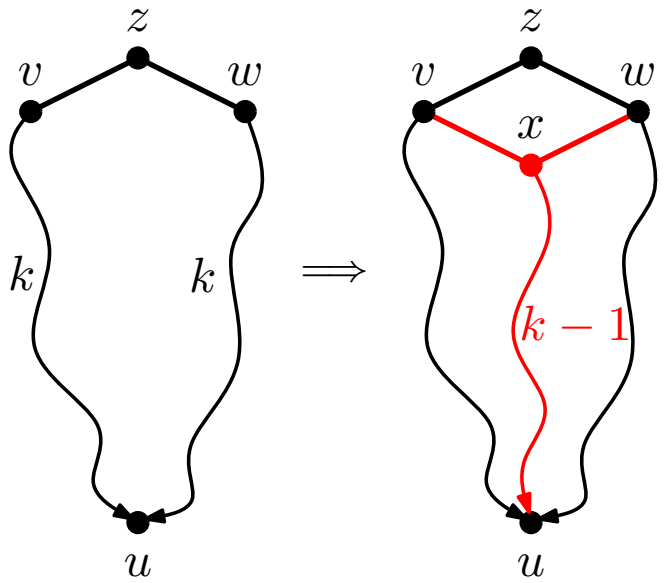


$\mathrm{TC}(u, k)$

Weakly modular graph

Definition (Quadrangle condition)

A (simplicial) graph satisfies the *quadrangle condition at distance* $k > 0$ *with respect to a vertex* u, denoted $\mathrm{QC}(u, k)$, if for any three vertices v, w, z with $d(v, z) = d(w, z) = 1$ and $2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, z) - 1 = k$, there exists a common neighbor x of v and w such that $d(u, x) = k - 1$.

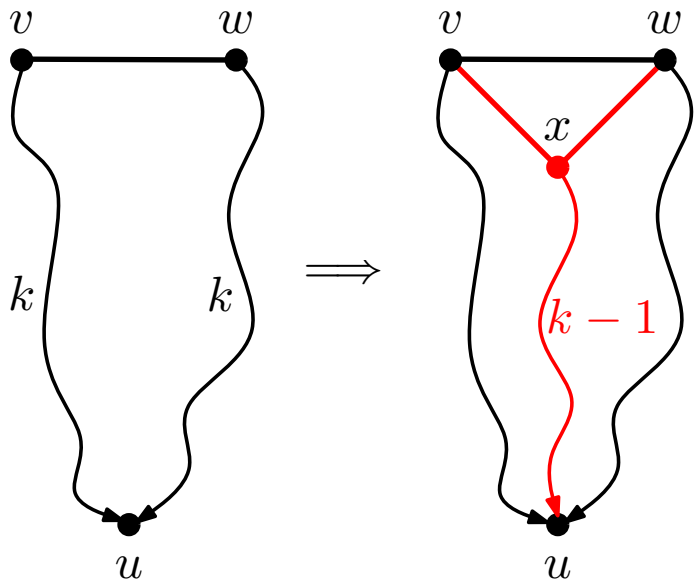


$\mathrm{QC}(u, k)$

Weakly modular graph

Definition (Weakly modular graph)

A (simplicial) graph is *weakly modular* if it satisfies conditions $\mathrm{TC}(u,k)$ and $\mathrm{QC}(u,k)$ for all u and k.



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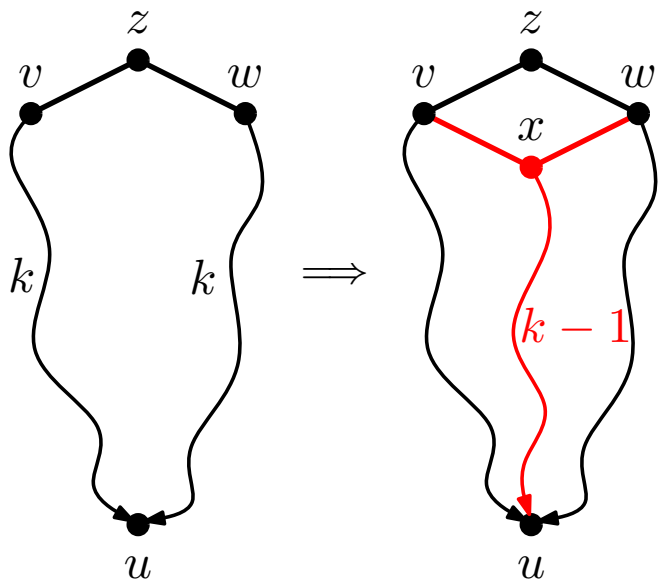

$\mathrm{QC}(u,k)$

Genesis

МЕТОДЫ ДИСКРЕТНОГО АНАЛИЗА
В ОПТИМИЗАЦИИ УПРАВЛЯЮЩИХ СИСТЕМ

Сборник трудов

1989 г. Института математики СО АН СССР Выпуск 49

УДК 519.17

КЛАССИФИКАЦИЯ ГРАФОВ С ПОМОЩЬЮ МЕТРИЧЕСКИХ ТРЕУГОЛЬНИКОВ

В.Д.Чепой

Вводится понятие метрического треугольника и рассматриваются некоторые типы треугольников в графах. Характеризуются графы, в которых все треугольники принадлежат одному и тому же типу, исследуются также некоторые метрические свойства этих графов. Данная классификация обусловлена тем, что во многих известных классах графов треугольники обладают этими специальными свойствами. К этим классам относятся медианные графы и их обобщения (см. [1-6] и цитированную там литературу), мостовые графы [7-9], подкласс слаботриангулированных графов [10], включающий триангулированные [11] и двудольные хордовые [12] графы, наследственные по расстоянию и птолемеевы графы [13-15], абсолютные ретракты n-хроматических [16-19] и рефлексивных [20,21] графов (последние называются также графами Хелли) и др.

1. Основные понятия и определения

Пусть $G=(X,U)$ – обыкновенный связный граф с произвольным, не обязательно конечным множеством вершин X. Снабдим G естественной метрикой $d(x,y)$, равной количеству ребер крат-

Examples

The following graphs are weakly modular:

- median graphs = 1-skeleta of CAT(0) cube complexes
- bridged graphs = 1-skeleta of systolic complexes
- (weakly) bridged graphs = 1-skeleta of (weakly) systolic complexes [Chepoi-O.]
- modular graphs
- bucolic graphs [Chalopin-Chepoi-Brešar-Gologranc-O.]
- quasi-median graphs
- Helly graphs

Geometric Group Theory

Applications in Geometric Group Theory:

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A) constructing examples of groups with interesting properties

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B) studying existing (classical) groups

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- groups containing expander graphs (small cancellation) [O.]
- interesting Burnside (infinite, torsion) groups (median graphs) [O.]

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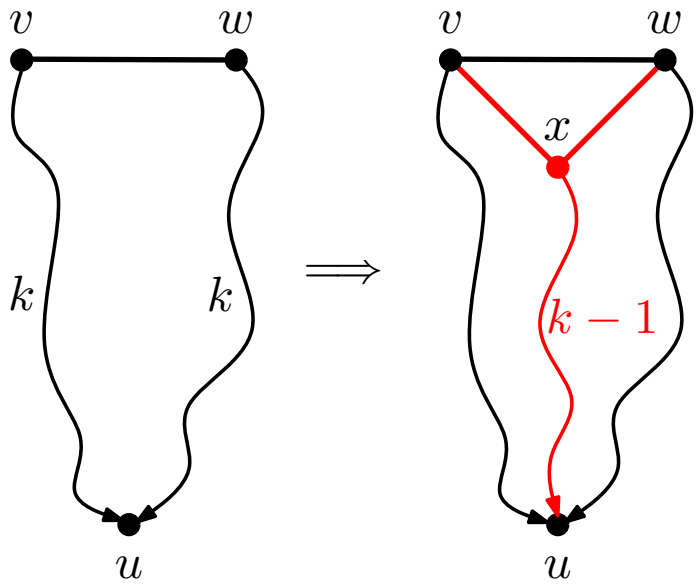
- some Coxeter groups (weakly modular graphs) [Munro]
- Helly groups (Helly graphs) [...Hirai, Huang, Hoda, Haettel...]

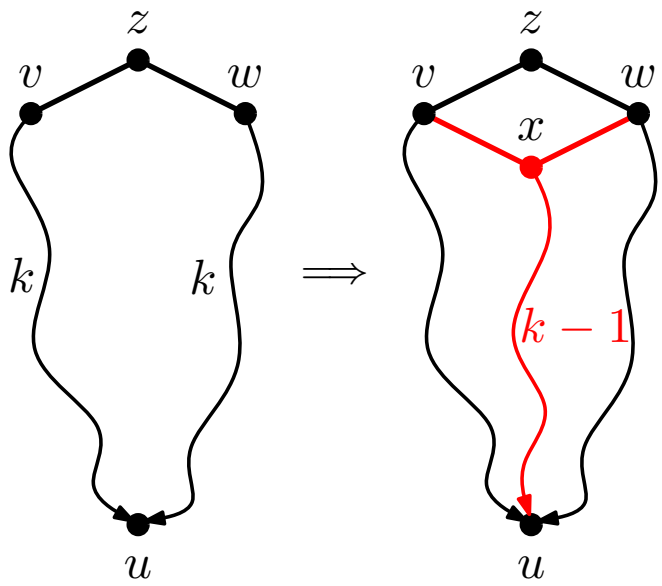
Local-to-global characterization

Definition (Locally weakly modular graph)

A graph is *locally weakly modular* if it satisfies conditions $\mathrm{TC}(u,k)$ and $\mathrm{QC}(u,k)$ for all u and $k = 2$.



$\mathrm{TC}(u,k)$



$\mathrm{QC}(u,k)$

Local-to-global characterization

Theorem (CCHO)

A graph is weakly modular iff it is locally weakly modular and its triangle-square complex is simply connected.

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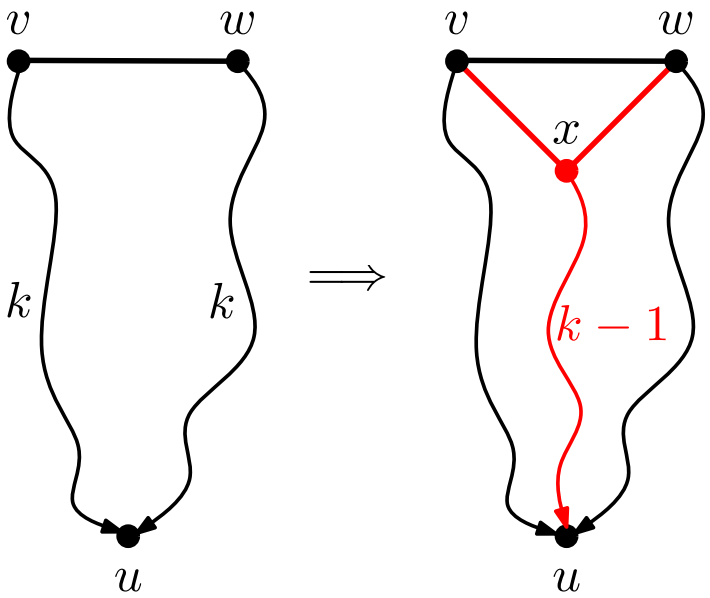
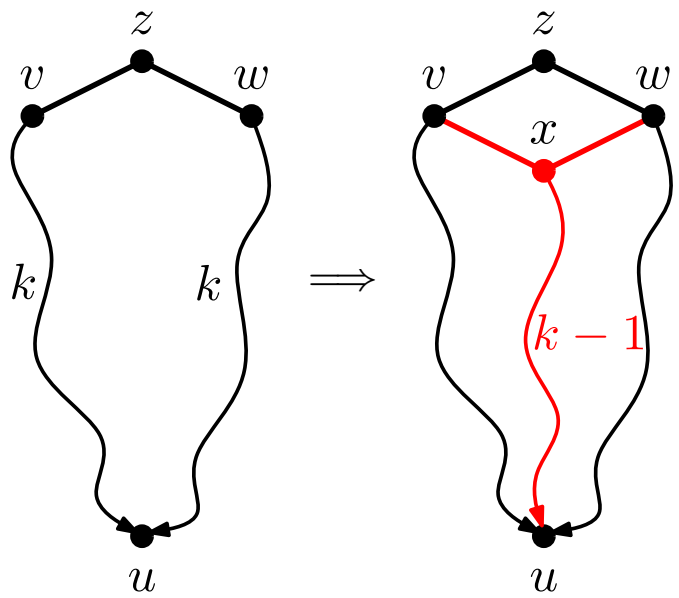
Theorem (CCHO)

A graph is Helly iff it is clique-Helly and its triangle complex is simply connected.

On a proof:



$\mathrm{TC}(u,k)$



$\mathrm{QC}(u,k)$

Weakly modular complex?

Question: Does there exist a reasonable "weakly modular complex"?

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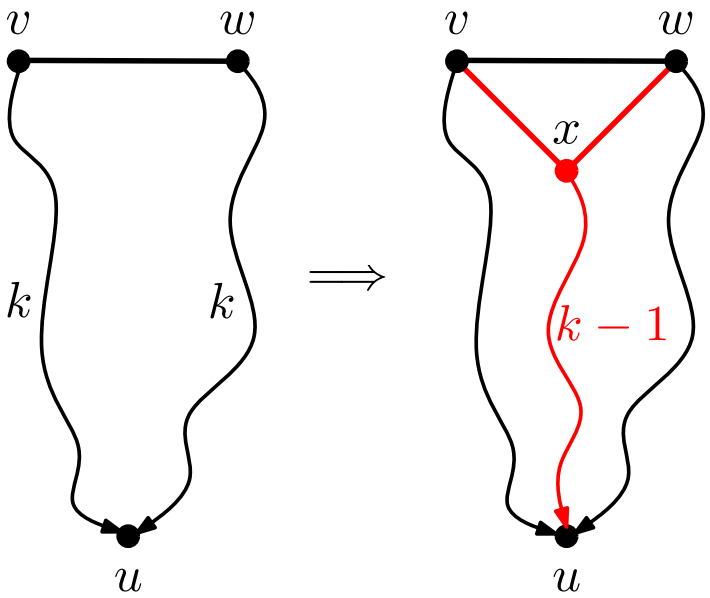
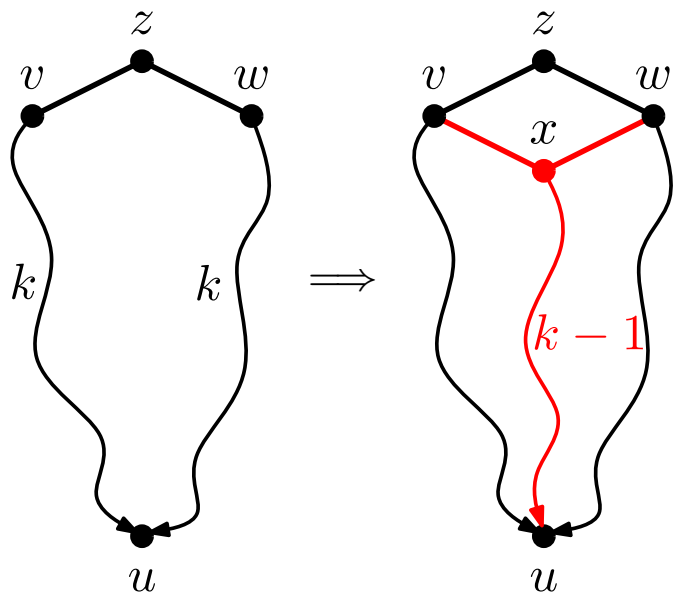
- median graphs — fill with cubes
- (weakly) bridged and Helly graphs — fill with simplices
- (weakly) bucolic graphs — fill with prisms

Weakly modular graph



$\mathrm{TC}(u,k)$



$\mathrm{QC}(u,k)$

Plate


TC: ∀u,v,w 1=d(v,w)<d(u,v)=d(v,w)=k ∃x~v,w d(u,x)=k-1
OC: ∀u,v,w,z z~v,w 2=d(v,w)≤d(u,v)=d(u,w)=d(u,z)-1=k ∃x~v,w d(u,x)=k-1
u
v
w
x
z

Plate