The axiomatic characterization of the interval function of Ptolemaic and bridged graphs

Lekshmi Kamal

University of Kerala, India

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Joint work with Manoj Changat & P.G.Narasimha Shenoi

Outline

Transit function

- Transit functions is due to Mulder (2008) to generalize the notions of interval, betweenness and convexity in an axiomatic approach.
- Interval operator- Calder, Nebeský, Mulder, Victor Chepoi, H J Bandelt, van De Vel, M.Changat
- Organizing function- Nebeský, Mulder
- Recombination operator- Peter F. Stadler - evolutionary biology.

Definition of transit function

Definition

A transit function on a nonempty finite set V is a function $R : V \times V \to 2^V \backslash \phi$ satisfying the three transit axioms.

(t1) $u \in R(u, v)$, for all $u, v \in V$ (law of extension)

(t2) $R(u, v) = R(v, u)$, for all $u, v \in V$ (law of symmetry)

(t3) $R(u, u) = u$, for all $u \in V$. (law of idempotent)

Transit function on graph

- The axiomatic approach using transit functions in graphs is a tool for studying and characterizing graph classes.
- If V is the vertex set of a graph G and R a transit function on V, then R is called a transit function on G.
- The underlying graph G_R of a transit function R is the graph with vertex set V , where two distinct vertices u and v are joined by an edge if and only if $R(u,v) = \{u,v\}$.

Path transit function on graph

The transit function may be defined in terms of paths in the graph G, such transit functions are called path transit functions on G. Prime examples of transit functions on graphs are

- Interval function
- Induced path function
- All-paths function.

Interval function and induced path function

Definition

The interval function I_G of a connected graph G is defined as $I : V \times V \longrightarrow 2^V$

$$I_G(u,v) = \{w \in V : \text{ w lies on some shortest u,v - path in G } \}$$
$$= \{w \in V : d(u,w) + d(w,v) = d(u,v)\}$$

Interval function and induced path function

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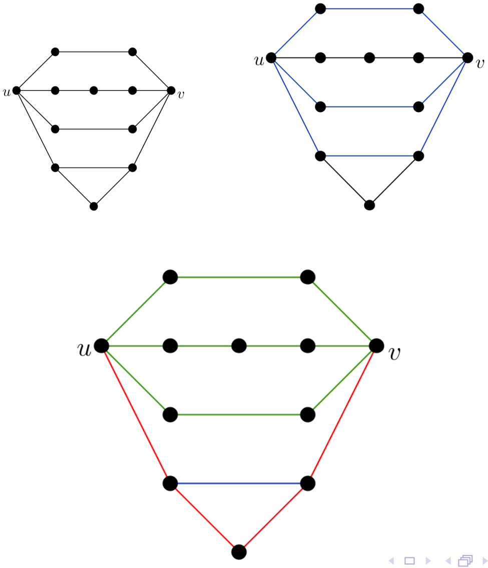
Definition

The induced path function J_G of a connected graph G is defined as $J : V \times V \longrightarrow 2^V$
$J_G(u, v) = \{w \in V : \text{ w lies on some induced u,v - path in G } \}$

Example



Betweenness axiom on I

Some of the betweenness axiom satisfied by interval function are

(b1) $x \in R(u,v), x \neq v \implies v \notin R(u,x), \forall u, v \in V.$

(b2) $x \in R(u,v)$ and $y \in R(u,x) \implies y \in R(u,v),$

(b3) $x \in R(u,v)$ and $y \in R(u,x) \implies x \in R(y,v),$

(b4) $x \in R(u,v) \implies R(u,x) \cap R(x,v) = \{x\}$

- $(b3) \implies (b4) \implies (b1)$

Problem

Is it possible to give a characterization of I for a connected graph using a set of first order axioms defined on R?

- Sholander (1952)
- Nebeský (1994).
- Nebeský (1995,1998,2001).
- H.J Bandelt, Victor Chepoi (1996).

axiomatic characterization of the interval function of connected graph

Theorem (H.M.Mulder, L.Nebeský, 2009)

Let $R : V \times V \longrightarrow 2^V$ be a function on V, satisfying the axioms $(t1), (t2), (b2), (b3), (b4)$ with the underlying graph G_R and let I be the interval function of G_R. The following statements are equivalent.

(a) $R = I$.

(b) R *satisfies axioms* $(s1)$ *and* $(s2)$.

- $(s1)$ $R(u, \bar{u}) = \{u, \bar{u}\}, R(v, \bar{v}) = \{v, \bar{v}\}, u \in R(\bar{u}, \bar{v})$ *and* $\bar{u}, \bar{v} \in R(u, v)$, *then* $v \in R(\bar{u}, \bar{v})$.
- $(s2)$ $R(u, \bar{u}) = \{u, \bar{u}\}, R(v, \bar{v}) = \{v, \bar{v}\}, \bar{u} \in R(u, v)$, $v \notin R(\bar{u}, \bar{v}), \bar{v} \notin R(u, v)$, *then* $\bar{u} \in R(u, \bar{v})$.

axiomatic characterization of the interval function of tree

- Sholander (1952) A function $R : V \times V \to 2^V$ is the interval function of a tree if and only if it satisfies $(t3), (C)$and (Mod).

(C) $x \in R(u,v), y \in R(x,z) \implies x \in R(v,y)$ or $x \in R(z,u)$,for all $u,v,x,y,z \in V$.

(mod) $\mid R(u,v) \cap R(v,w) \cap R(w,u) \mid \neq \phi$, for all $u,v,w \in V$.

- Sholander proved that axioms $(t3)$ and (C) imply axioms $(t1), (t2)$, $(b1), (b2)$ and (γ).

$(J0)$ $y \in R(u,x), x \in R(y,v), y \neq x \implies x \in R(u,v)$, for all $u,v,x,y \in V$

- $(J0)$ For any pair of distinct vertices $u, v, x, y \in V$ we have $x \in R(u, y), y \in R(x, v) \implies x \in R(u, v)$.
- $(J0')$ For any pair of distinct vertices $x \in R(u, y), y \in R(x, v), R(u, y) \cap R(x, v) \subseteq \{u, x, y, v\} \implies x \in R(u, v)$.

(J0) For any pair of distinct vertices $u, v, x, y \in V$ we have $x \in R(u, y), y \in R(x, v) \implies x \in R(u, v)$.

(J0′) For any pair of distinct vertices $x \in R(u, y), y \in R(x, v), R(u, y) \cap R(x, v) \subseteq \{u, x, y, v\} \implies x \in R(u, v)$.

- From the definitions of $(J0)$ and $(J0')$ it follows that $(J0) \implies (J0')$.

Example ($(J0') \nRightarrow (J0)$)

Let $V = \{a, b, c, d, e\}$ Let $R : V \times V \to 2^V$ defined as follows. $R(a, e) = \{a, e\}, R(a, b) = \{a, b\}, R(b, e) = \{b, e\}, R(b, c) = \{b, c\}, R(c, e) = \{c, e\}, R(c, d) = \{c, d\}, R(d, e) = \{d, e\}, R(a, c) = \{a, b, c, e\}, R(a, d) = \{a, e, d\}, R(b, d) = \{b, c, d, e\}$. We can see that $b \in R(a, c)$ and $c \in R(b, d)$ but $b \notin R(a, d)$, so that R doesnot satisfy $(J0)$ axiom.

Lemma

If R is a transit function on V satisfying the axioms $(J0')$ and $(b3)$, then R satisfies axiom $(b2)$.

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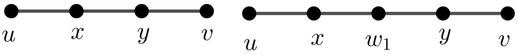

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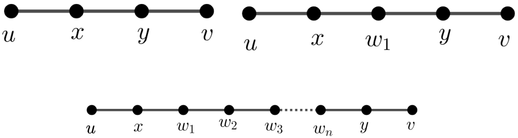


Corollary

If R is a transit function on V satisfying the axioms $(J0)$ and $(b3)$, then R satisfies axiom $(b2)$ and G_R is connected.

Ptolemaic graphs

A graph is Ptolemaic if and only if -

- the distances obey Ptolemy's inequality. For every four vertices u, v, w and x the inequality

$$d(u,v)d(w,x) + d(u,x)d(v,w) \geq d(u,w)d(v,x)$$

  holds. (David, Chartrand- 1965)
- it is both chordal and distance-hereditary (Edward-1981).
- it is chordal and 3 - fan - free.

- (Chepoi- 1986), characterization of ptolemaic graph using α_0-metric. A graph G has an α_0-*metric* if for any edge vw of G and any two vertices u, x such that $v \in I(u, w)$ and $w \in I(v, x)$, the inequality $d(u, x) \geq d(u, v) + d(w, x) + d(w, v)$ holds.

Theorem (K.Balakrishnan et al, 2015)

Let G be a graph. The interval function I_G satisfies the axiom $(J0)$ if and only if G is a Ptolemaic graph.

(*J*0) For any pair of distinct vertices $u, v, x, y \in V$ we have $x \in R(u, y), y \in R(x, v) \implies x \in R(u, v)$.

(*J*2) $R(u, x) = \{u, x\}, R(x, v) = \{x, v\}, R(u, v) \neq \{u, v\} \implies x \in R(u, v)$.

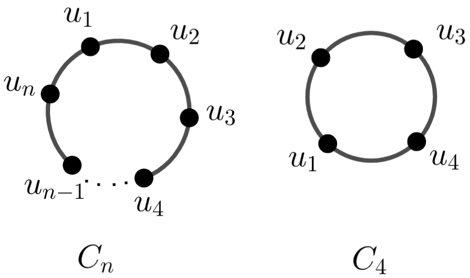
(*b*3) $x \in R(u, v)$ and $y \in R(u, x) \implies x \in R(y, v)$.

Theorem

Let R be a transit function on the vertex set V of a connected graph G. Then R satisfies the axioms $(b3), (J0)$ and $(J2)$ if and only if G_R is a Ptolemaic graph and R coincides the interval function I_{G_R} .

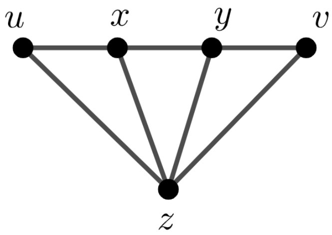
Theorem

Let R be any transit function defined on a non-empty set V. If R satisfies $(J0)$ and $(J2)$ then the underlying graph G_R of R is C_n-free for $n \geq 4$.



Theorem

Let R be any transit function satisfying the axioms $(b3), (J0)$ *and* $(J2)$ *then* G_R *is Ptolemaic and* $R(u, v) = I(u, v)$.



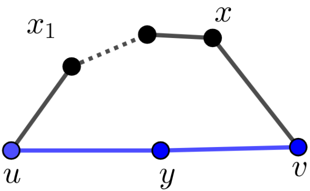
Proof

$R(u, v) = I(u, v)$

$d(u, v) = 2$

$I(u, v) \subseteq R(u, v)$

$R(u, v) \subseteq I(u, v)$



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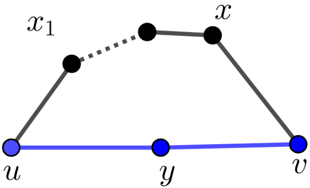

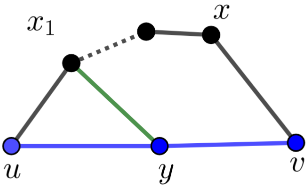

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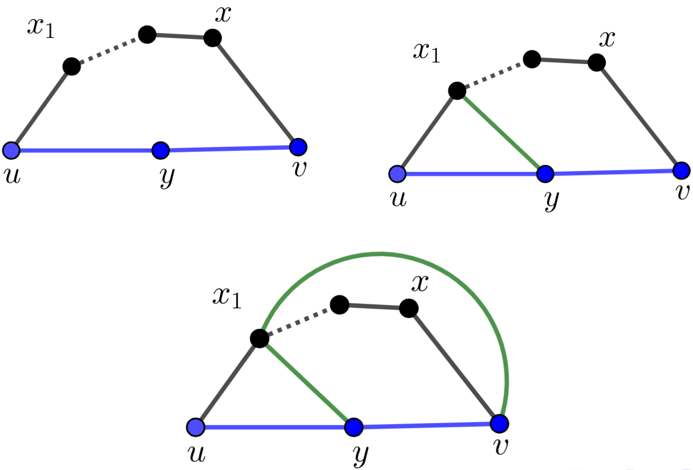

Result

Theorem

Let R be a transit function on the vertex set V of a connected graph G. Then R satisfies the axioms $(b3), (J0)$ and $(J2)$ if and only if G_R is a Ptolemaic graph and R coincides the interval function I_{G_R} .

The following examples show that the axioms $(J0), (J2)$ and $(b3)$ are independent.

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Example ($(J0), (J2)$ but not $(b3)$)

Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}, R(a, c) = \{a, c\}, R(a, d) = \{a, b, c, d\}, R(a, e) = V, R(b, c) = \{b, c\}, R(b, d) = \{b, d\}, R(b, e) = \{b, e\}, R(c, d) = \{c, d\}, R(c, e) = \{b, c, d, e\}, R(d, e) = \{d, e\}$. We can see that R satisfies $(J0)$ and $(J2)$. But $d \in R(a, e), b \in R(a, d)$, but $d \notin R(b, e)$. Therefore R doesnot satisfy the $(b3)$ axiom.

Example $((J2),(b3)$ but not $(J0))$

Let $V = \{a,b,c,d,e\}$ and define a transit function R on V as follows: $R(a,b) = \{a,b\}, R(a,c) = \{a,c\}, R(a,d) = \{a,b,c,d\}, R(a,e) = \{a,b,e\}, R(b,c) = \{b,c\}$, $R(b,d) = \{b,d\}, R(b,e) = \{b,e\}, R(c,d) = \{c,d\}, R(c,e) = \{b,c,d,e\}, R(d,e) = \{d,e\}$. Here R Satisfies $(J2)$ and $(b3)$. We can see that $c \in R(a,d), d \in R(c,e)$ but $c \notin R(a,e)$. So R doesnot satisfy $(J0)$.

Example ($(J2), (b3)$ but not $(J0)$)

Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}, R(a, c) = \{a, c\}, R(a, d) = \{a, b, c, d\}, R(a, e) = \{a, b, e\}, R(b, c) = \{b, c\}$, $R(b, d) = \{b, d\}, R(b, e) = \{b, e\}, R(c, d) = \{c, d\}, R(c, e) = \{b, c, d, e\}, R(d, e) = \{d, e\}$. Here R Satisfies $(J2)$ and $(b3)$. We can see that $c \in R(a, d), d \in R(c, e)$ but $c \notin R(a, e)$. So R doesnot satisfy $(J0)$.

Example ($(J0), (b3)$ but not $(J2)$)

Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, e) = \{a, e\}, R(b, e) = \{b, e\}, R(a, b) = \{a, b, c\}$ and for all other pair $R(x, y) = \{x, y\}$ we can see that R satisfies $(J0), (b3)$. But since $e \notin R(a, b)$ we can see that R fails to satisfy $(J2)$.

Bridged graph

Definition

A graph G is bridged if every cycle of length at least 4 has a bridge (i.e. the only isometric cycles in G can be of length 3).

- A graph G is called bridged if it is weakly modular graphs without C_4 and C_5 as induced subgraphs (Chepoi, 1989).
- A graph G is called bridged if all neighborhoods of convex sets are convex (Martin Farber-1986).

Interval function of bridged graphs

(J0′) $x \in R(u,y), y \in R(x,v), R(u,y) \cap R(x,v) \subseteq \{u,x,y,v\} \implies x \in R(u,v)$.

(b3) $x \in R(u,v)$ and $y \in R(u,x) \implies x \in R(y,v)$

(s1) $R(u,\bar{u}) = \{u,\bar{u}\}, R(v,\bar{v}) = \{v,\bar{v}\}, u \in R(\bar{u},\bar{v})$ and $\bar{u},\bar{v} \in R(u,v) \implies v \in R(\bar{u},\bar{v})$.

(s2) $R(u,\bar{u}) = \{u,\bar{u}\}, R(v,\bar{v}) = \{v,\bar{v}\}, \bar{u} \in R(u,v),\ v \notin R(\bar{u},\bar{v}), \bar{v} \notin R(u,v) \implies \bar{u} \in R(u,\bar{v})$.

Theorem

Let G be a connected graph. The interval function I_G satisfies the axiom $(J0')$ if and only if G is a bridged graph.

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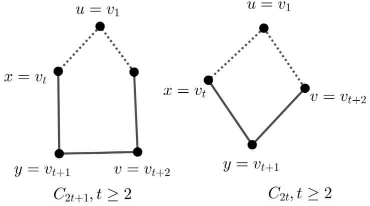
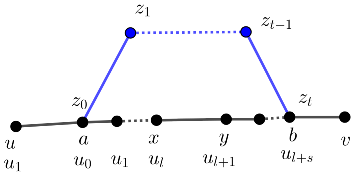
Proof $\rightarrow$



proof $\leftarrow$



- An a, b-subpath of R, $R_{a,b} : a = z_0 z_1 \dots z_{t-1} z_t = b, (t \geq 1)$ a, b-induced path $P : a = u_0 u_1 \dots u_\ell = x u_{\ell+1} = y \dots u_{\ell+s} = b$, $(\ell + s \geq 3)$ containing x and y
- claim: The index $t = \ell + s - 1$
- z_ℓ, which is different from u, x, y, v also belongs to $I_G(u, y) \cap I_G(x, v)$

Result

Theorem

Let R be a transit function on the vertex set V of a connected graph G. Then R satisfies the axioms $(b3), (J0'), (s1)$, $(s2)$ if and only if G_R is a bridged graph and R coincides the interval function I_{G_R}

Example (($(J0')$, $(s1)$, $(s2)$ but not $(b3)$)

Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}, R(a, c) = \{a, b, e, c\}$, $R(a, d) = V, R(a, e) = \{a, e\}, R(b, c) = \{b, c\}$, $R(b, d) = \{b, c, e, d\}$, $R(b, e) = \{b, e\}, R(c, d) = \{c, d\}, R(c, e) = \{c, e\}, R(d, e) = \{d, e\}$. We can see that R satisfies $(J0')$, $(s1)$ and $(s2)$. But $b \in R(a, d), e \in R(b, d)$ and $b \notin R(a, e)$. Therefore R does not satisfy the $(b3)$ axiom.

Example $((J0'), (b3), (s2)$ but not $(s1)$)

Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}, R(a, c) = \{a, e, c, d\}$, $R(a, d) = \{a, d\}, R(a, e) = \{a, e\}, R(b, c) = \{b, c\}$, $R(b, d) = V$, $R(b, e) = \{b, e\}, R(c, d) = \{c, d\}, R(c, e) = \{c, e\}, R(d, e) = \{d, e\}$. Then R satisfies $(J0')$, $(b3)$ and $(s2)$. But $R(a, d) = \{a, d\}, R(b, c) = \{b, c\}, a, c \in R(b, d), d \in R(a, c)$ and $b \notin R(a, c)$. Therefore R does not satisfy the $(s1)$ axiom.

Example ($(J0')$, $(b3)$, $(s1)$ but not $(s2)$)

Let $V = \{a, b, c, d\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}, R(b, c) = \{b, c\}, R(c, d) = \{c, d\}, R(a, c) = \{a, b, c\}$, $R(a, d) = \{a, d\}$, $R(b, d) = \{b, d\}$. Then R satisfies $(J0')$, $(b3)$ and $(s1)$. But $R(a, b) = \{a, b\}$, $R(c, d) = \{c, d\}$, $b \in R(a, c), c \notin R(b, d), d \notin R(a, c)$ and $b \notin R(a, d)$. Therefore R does not satisfy the $(s2)$ axiom.

Example $((b3), (s1), (s2)$ but not $(J0')$)

Let $V = \{a, b, c, d\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}, R(b, c) = \{b, c\}, R(c, d) = \{c, d\}, R(a, d) = \{a, d\}, R(a, c) = V, R(b, d) = V$. Then R satisfies $(b3)$, $(s1)$ and $(s2)$. But $b \in R(a, c), c \in R(b, d), R(a, c) \cup R(b, d) \subseteq \{a, b, c, d\}$ and $b \notin R(a, d)$. Therefore R does not satisfy the $(J0')$ axiom.

Induced path function of chordal graph

Theorem

Let R be a transit function on the vertex set V of a connected graph G. Then R satisfies the axioms $(b1), (b2)(J0), (J1)$ and $(J2)$ if and only if G_R is a chordal graph and R coincides the induced path function J_{G_R}.

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THANK YOU...