Artin groups, hyperplane complement arrangements and Helly groups

Jingyin Huang (joint work with D. Osajda)

Metric Graph Theory and Related Topics, Dec 2021

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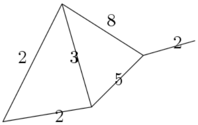

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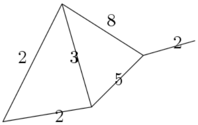


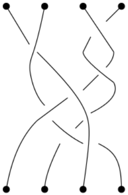
Vertex set of $\Gamma = \{v_1, v_2, \dots, v_n\}$

$$A_\Gamma = \{v_1, v_2, \dots, v_n \mid \underbrace{v_i v_j v_i \cdots}_{m} = \underbrace{v_j v_i v_j \cdots}_{m} \text{ if } v_i \overset{m}{-} v_j\}$$

Artin group with defining graph Γ

Braid groups

Braid diagrams (on n strands):
in $\mathbb{R}^3$, n-points on top, n points on bottom, and n disjoint height-monotone arcs connecting them.



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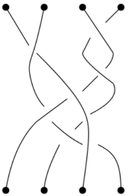

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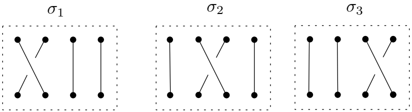

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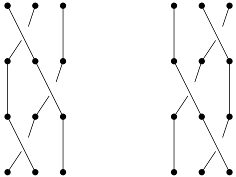

$$B_4 = \{\sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3\}$$

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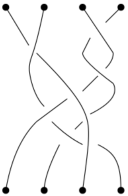

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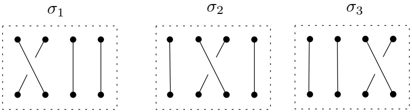

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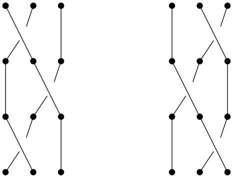

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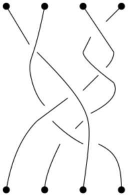
Homomorphism: $B_4 \to S_4$

Artin groups and hyperplane complements

Each braid induced a loop in the configuration spaces of n unordered pts in $\mathbb{R}^2 = \mathbb{C}$.



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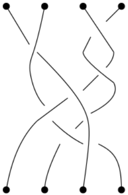

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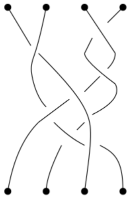

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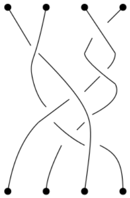

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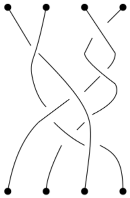

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Homomorphism $A_\Gamma \to C_\Gamma$ (eg. $B_4 \to S_4$).

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NPC Conjecture: Artin groups are non-positively curved in an appropriate sense.

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$G \curvearrowright X$ is geometric if the action is properly discontinuous and cocompact by isometries.

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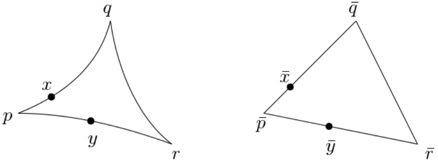

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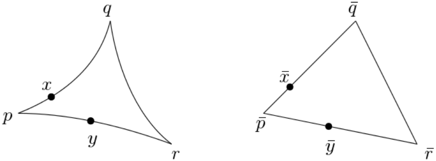

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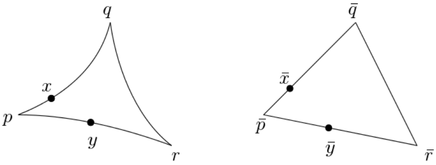

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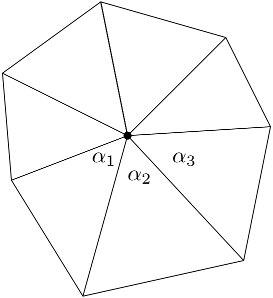

$CAT(0) \Leftrightarrow \Sigma_i \alpha_i \geq 2\pi$

CAT(0) Cube complexes

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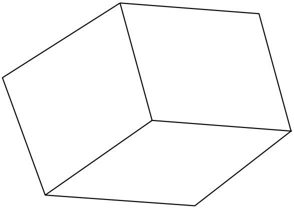

Figure: missing corner

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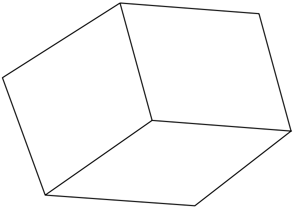

Figure: missing corner

Observation (Gromov) If a cube complex has no "missing corners" and is simply-connected, then it is CAT(0).

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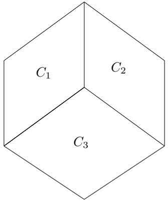

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Theorem (H.-Osajda, Chalopin - Chepoi - Hirai - Osajda)

Suppose X is a locally finite s.c. cell Helly complex. Then the thickening of X is a Helly graph. In particular, groups acting geometrically on a s.c. cell Helly complex also acts geometrically on Helly graphs.

Main theorem

Theorem (H.-Osajda 2019)

Let G be a Garside group of finite type or an Artin groups of type FC. Then G acts geometrically on a Helly graphs.

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Definition

An Artin A_Γ is of type FC if every complete subgraph of Γ is spherical.

Immediate consequences (by combining with known work)

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Geometric consequences:

6. G has Euclidean higher order homological Dehn function.
7. G has a "Tits boundary".

The Salvetti complex for hyperplane arrangement

Let $\mathcal{A}$ be a central arrangement of hyperplanes in $\mathbb{R}^n$, i.e. we are given a finite collection of linear hyperplanes $\{H_\lambda\}_{\lambda \in \Lambda}$ in $\mathbb{R}^n$. (e.g. $\{x_i = x_j\}_{j \neq i}$)

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Let $\mathcal{A}\otimes\mathbb{C}$ be the complexification of $\mathcal{A}$, i.e. a collection of complex linear hyperplanes $\{H_\lambda\otimes\mathbb{C}\}_{\lambda\in\Lambda}$ in $\mathbb{C}^n$ (e.g. $\{z_i = z_j\}_{j\neq i}$).

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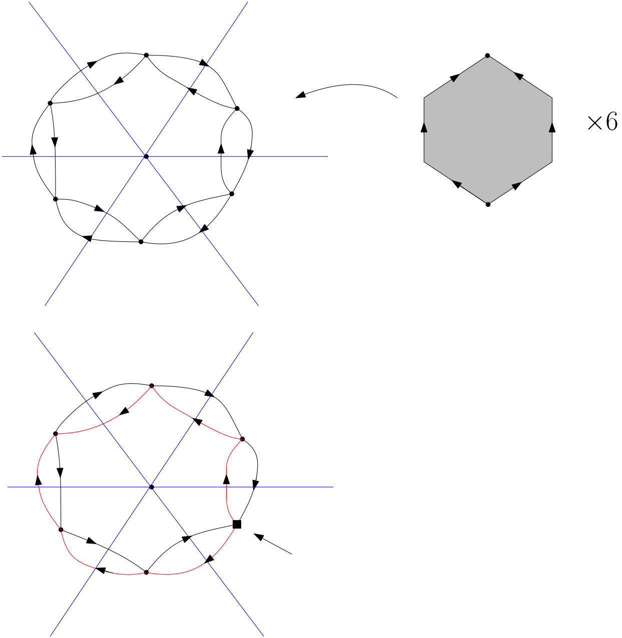
Salvetti constructed a cell complex which is a deformation retract $M(\mathcal{A} \otimes \mathbb{C})$.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Theorem (H.-Osajda 2021)

If the arrangement $\mathcal{A}$ is simplicial, i.e. the hyperplanes cut the unit sphere into simplices, then the associated Salvetti complex has a cell-Helly universal cover. Hence $\pi_1 M(\mathcal{A} \otimes \mathbb{C})$ is Helly.

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Each central arrangement of hyperplane gives rise an *oriented matroid*. By the Folkman–Lawrence topological representation theorem, each oriented matroid gives an "arrangements of pseudo-hyperplanes"

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(Oriented Matroid version of the above theorem) If the oriented matroid is "simplicial", then the Salvetti complex is cell-Helly, and its fundamental group is Helly.

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Theorem (H.-Osajda)

$\mathcal{A}$ is cell-Helly if

1. *for any distinct triples of vertices $\{v_1, v_2, v_3\}$ and $\{C_i\}_{i=1}^3$ with each C_i being a (v_i, v_{i+1})-component, if $\cap_{i=1}^3 C_i \neq \emptyset$, then there exists vertex $v \in \Gamma$ such that $\mathcal{O}_v$ coincident with the orientation on $C_1 \cup C_2 \cup C_3$;*
2. *for any distinct triples of vertices $\{w_1, w_2, w_3\}$ of Γ and any base point w_0, there is a vertex w of Γ such that w is contained in any special component of Γ of type $(w_0, *)$ that contains two vertices of $\{w_1, w_2, w_3\}$;*

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Question: Are there any other kinds arrangements satisfying the conditions of the above theorem besides simplicial arrangements?

Happy Birthday, Victor!