Strongly Shortcut Spaces

Nima Hoda

École Normale Supérieure

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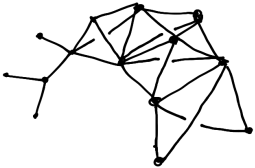
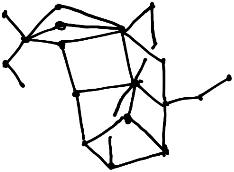
Motivation

Cycle distortion in combinatorial nonpositive curvature



A systolic complex



A quadric complex

Theorem (Chepoi)

A flag simplicial complex X is systolic iff its 1-skeleton X^1 is a bridged graph: i.e. every isometric cycle in X^1 has length 3.

Theorem (H.)

A flag square complex X is quadric iff every isometric cycle in X^1 has length 4.

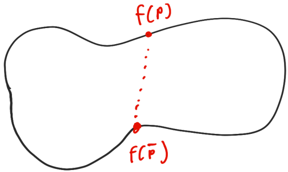
Motivation

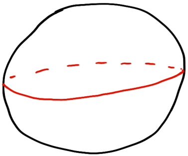
Circle distortion in Euclidean space

Observation

There is no distance-preserving map from a circle (with arc-length metric) into a Euclidean space. However, such a map exists to a sphere.

If $f\colon S^1 \to \mathbb{E}^n$ is a 1-Lipschitz map then for some pair of antipodal points $p, \bar{p} \in S^1$ we will have $d\big(f(p), f(\bar{p})\big) \leq \frac{2}{\pi} d(p, \bar{p})$.

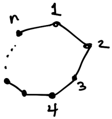




On the other hand, every great circle in a sphere is embedded in a distance-perserving way.

Graphs and Cycles

- We view graphs as geodesic metric spaces with edges isometric to $[0,1] \subset \mathbb{R}$. For example, the cycle graph S_n



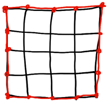
  is isometric to a circle with arc-length metric of length n.
- A *cycle* in a graph Γ is a graph homomorphism $f : S_n \to \Gamma$ for some n.

Graphs and Cycles

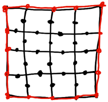
Note that a cycle $f\colon S \to \Gamma$ is isometrically embedded iff $d(f(p), f(\bar{p})) \geq \frac{|S|}{2}$ for every antipodal pair of points $p, \bar{p} \in S$.

Definition

A cycle $f\colon S \to \Gamma$ is $\frac{1}{K}$-*almost isometric* for some $K > 1$ if $d(f(p), f(\bar{p})) \geq \frac{1}{K} \cdot \frac{|S|}{2}$ for every antipodal pair of points $p, \bar{p} \in S$.



For any $\varepsilon > 0$, the inclusion of the red cycle is not $\frac{1}{2-\varepsilon}$-almost isometric, though it is $\frac{1}{2}$-almost isometric.



After subdivision of non-cycle edges, the inclusion of the red cycle is isometrically embedded, i.e., is $\frac{1}{K}$-almost isometric for any $K > 1$.

Strongly Shortcut Graphs

Definition

A graph Γ is *strongly shortcut* if, for some $K > 1$, there is a bound on the lengths of the $\frac{1}{K}$-almost isometric cycles of Γ.

Definition

A group G is *strongly shortcut* if G acts on a strongly shortcut graph properly (i.e. with finite vertex stabilizers) and cocompactly (i.e. with finitely many vertex orbits and edge orbits).

Strongly Shortcut Graphs

Examples

- Hyperbolic graphs
- Median graphs (i.e. 1-skel. of CAT(0) cube complexes) of fin. dim.
- Bridged graphs (i.e. 1-skel. of systolic complexes)
- Hereditary modular graphs (i.e. 1-skel. of quadric complexes)
- Standard Cayley graphs of Coxeter groups
- Every Cayley graph of $\mathbb{Z}^n$ (with Przytycki for $n \geq 3$)
- Helly graphs of finite max degree (with Haettel and Petyt)

Rough Geodesic Spaces

Let X and Y be metric spaces and let $f\colon Y \to X$ be a function.

Definition

R-roughly K-Lipschitz: $d\big(f(y_1), f(y_2)\big) \leq Kd(y_1, y_2) + R$

(K, R)-quasi-isometric embedding: R-roughly K-Lipschitz and $\frac{1}{K}d(y_1, y_2) - R \leq d\big(f(y_1), f(y_2)\big)$

roughly onto: some metric neighborhood of $f(Y)$ is equal to X

(K, R)-quasi-isometry: (K, R)-quasi-isom. embedding and roughly onto

R-rough isometric embedding: $(1, R)$-quasi-isom. embedding

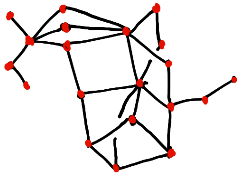
R-rough isometry: $(1, R)$-quasi-isometry

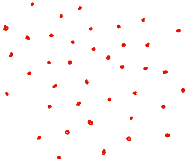
R-rough geodesic: R-rough isom. embedding and Y is a compact interval

Definition

Y is *R-roughly geodesic* if every pair of points of Y is joined by an R-rough geodesic.

Rough Geodesic Spaces





Example

A graph G is a geodesic space: i.e. a 0-roughly geodesic space. The vertex set of G with the subspace metric is 1-roughly geodesic.

Example

The Euclidean plane $\mathbb{E}^2$ is a geodesic metric space. If $A \subset \mathbb{E}^2$ is a subspace whose $\frac{R}{2}$-neighborhood is equal to $\mathbb{E}^2$ then A is R-roughly geodesic. Moreover, A is roughly isometric to $\mathbb{E}^2$.

Strongly Shortcut Spaces

Definition

An *R-circle* in a metric space X is an R-rough 1-Lipschitz function $f : S \to X$ where S is a circle (with arc-length metric). We denote the length of S by $|S|$.
An R-circle f is $\frac{1}{K}$-*almost isometric* for some $K > 1$ if $d\big(f(p), f(\bar{p})\big) \geq \frac{1}{K} \cdot \frac{|S|}{2}$ for every antipodal pair of points $p, \bar{p} \in S$.

Definition

An R-rough geodesic space X is *strongly shortcut* if, for some $K > 1$, there is a bound on the lengths of the $\frac{1}{K}$-almost isometric R-circles of X.

Fact

While not obvious as stated, the definition is independent of the choice of rough geodesicity constant R for X.

Strongly Shortcut Spaces

Examples

- Strongly shortcut graphs
- Asymptotically CAT(0) spaces
  - CAT(0) spaces
  - Gromov-hyperbolic spaces
  - $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with the Sasaki metric (Kar)
- Coarsely injective spaces of unif. bounded geom. (w/ Haettel, Petyt)
- $\mathbb{R}^n$ with a polyhedral norm (w/ Przytycki)
- Heisenberg groups of all dimensions (w/ Przytycki)
- In particular: all Thurston geometries except Sol.

Strongly Shortcut Spaces

Theorem (H.)

Let X be an R-rough geodesic space. The following conditions are equivalent:

1. *X is strongly shortcut.*
2. *There exists an $L > 1$ such that there is a bound on the lengths of the $(L, 4R)$-quasi-isometric embeddings of circles in X.*
3. *There exists an $L > 1$ such that for every $C \geq 0$ there is a bound on the lengths of the (L, C)-quasi-isometric embeddings of circles in X.*
4. *For some $L > 1$ and some $n \in \mathbb{N}$, there is a bound on the $\lambda > 0$ for which there exists an L-bilipschitz embedding of λS_n^0 in X, where S_n^0 is the vertex set of the cycle graph S_n of length n and λS_n^0 is S_n^0 with the metric scaled by λ.*
5. *No asymptotic cone of X contains an isometric copy of the circle of unit length.*

Strongly Shortcut Groups

Theorem (H.)

The following conditions are equivalent for a group G:

1. *G is strongly shortcut.*
2. *G acts metrically properly and coboundedly on a strongly shortcut rough geodesic space.*
3. *G has a finite generating set whose Cayley graph is strongly shortcut.*

Metrically proper: for any metric ball B, only finitely many $g \in G$ satisfy $gB \cap B \neq \emptyset$. *Cobounded*: the translates of some metric ball cover X.

Then, by a theorem of Riley, strongly shortcut groups have simply connected asymptotic cones. This has the following implications:

- Polynomial Dehn function (Gromov)
- Linear isodiametric function (Papasoglu)
- Linear filling length function (Riley)

Strongly Shortcut Groups

Examples

- CAT(0) groups
- Hyperbolic groups
- Helly groups (with Haettel and Petyt)
- Systolic groups and quadric groups
- Finitely presented small cancellation groups (C(6) via Wise)
- Discrete Heisenberg groups (with Przytycki)
- Hierarchically hyperbolic groups (with Haettel and Petyt)
    - In particular, mapping class groups of surfaces

Strongly Shortcut Groups

Closure properties

Theorem (H.)

The class of strongly shortcut groups is closed under the following operations.

- *Direct products.*
- *Graphs of groups with finite edge groups.*
  - *In particular: amalgams and HNN extensions over finite subgroups*

Theorem (Genevois)

Graph products of strongly shortcut groups are strongly shortcut.

Theorem (H., Krishna MS)

If a group G is hyperbolic relative to strongly shortcut subgroups then G is strongly shortcut.

Proof Ideas

The most difficult part of the proof of the theorem on equivalent conditions of the strong shortcut property for rough geodesic spaces is the implication (2) $\implies$ (1), which can be restated in contrapositive as follows.

Theorem

Suppose that X is not strongly shortcut: i.e. for every $K > 1$ there exist arbitrarily long $\frac{1}{K}$-almost isometric R-circles. Then for any $L > 1$, there are $(L, 4R)$-quasi-isometric embeddings of arbitrarily long circles in X.

This is an immediate consequence of a Circle Tightening Lemma.

Circle Tightening Lemma

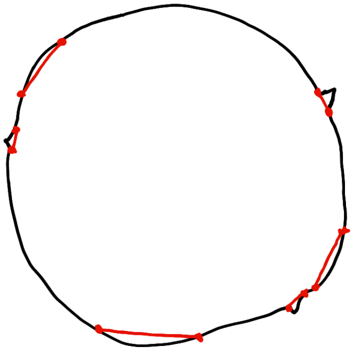
Theorem (H.)

Let $N > 1$, let $L > 1$, let $K > 1$ be small enough, let $R \geq 0$, let $M > 0$ be large enough and let $C \geq 4R$.
Let $\alpha\colon S \to X$ be a $\frac{1}{K}$-almost isometric R-circle of length $|S| > M$ in an R-rough geodesic space X. Then there exists a countable collection $\{Q_i\}_i$ of pairwise disjoint close segments in S such that.

1. $\sum_i |Q_i| < \frac{|S|}{N}$
2. *For each i there is a function $\gamma_i\colon \bar{Q}_i \to X$ from an interval $\bar{Q}_i \subset \mathbb{R}$ such that γ_i has the same endpoints as $\alpha|_{Q_i}$ and $|\bar{Q}_i| < |Q_i|$.*
3. *Replacing each $Q_i \to X$ in α with $\bar{Q}_i \to X$ results in an (L, C)-quasi-isometric embedding of an R-circle.*

Moreover, if $C > 0$ then $\{Q_i\}_i$ is a finite collection.

Circle Tightening Lemma



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