Median geometry, non-positive curvature and actions on Banach spaces

Cornelia Druțu

University of Oxford

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A conjecture. Main reasons to study median geometry

Conjecture: Median geometry is so popular among people in Geometric Group Theory because Victor is such a nice person.

Main reasons to study median geometry:

- because to my first email message (a perfect stranger, maître de conference in Lille working in a completely different area), Victor answered with a long message, full of details and references, crafted in elegant Romanian;
- because like that I get many messages that begin with 'Buna ziua' and end up with 'Cu prietenie';
- because we are both from Moldavia, about which Cantemir wrote: "Nu se poate afla nicăieri în vreo altă țară cît Moldova de mică atâtea ape și natură împodobită de asemenea locuri minunate ca aici. Apele clocotesc de pești, pe care oștenii îi prind cu sulița, aducându-i vii la masa domnească. În Ineu oile pasc o iarbă înrourată, grasă ca untul. Solul mustește de fierberea mineralelor...Aurul curge pe rîuri, fierul se coagulează în bulgări mari la suprafața pământului."

Further reasons to study median geometry

- Cantemir wrote also:

  Aici este urbea Iași [...]. Aici e scaunul țării pe care l-a mutat Ștefan Voevod din Suceava, pentru ca să poată apăra țara mai bine din mijlocul ei [...], observând că-i venea mai greu a face aceasta din Suceava.

- Victor is one of the few people capable of saying/writing things like this:

  "I do not know much about this subject, but as far as I know ....avalanche of precise and detailed information very much to the point and supported by many references...I hope this may be of any use."

Less significant reasons to be interested in median geometry

Because of its close connection to:

- non-positive curvature;
- actions by isometries on Banach spaces;
- the asymptotic geometry of certain important groups and spaces;
- graph theory, computer science, optimization theory.

Connection to non-positive curvature

Chepoi: a graph $\Gamma = (V, E)$ is the 1-skeleton of a CAT(0) cube complex $\Leftrightarrow$ V with the simplicial distance is median.

A median space = a metric space (X, d) such that every triple of points $x_1, x_2, x_3 \in X$ admits a unique median point $m \in X$ satisfying

$$d(x_i, m) + d(m, x_j) = d(x_i, x_j)$$

for all $i, j \in \{1, 2, 3\}, i \neq j$.

Other examples

1. *Real trees;*
2. $\mathbb{R}^n$ *with the norm* ℓ^1*;*
3. $L^1(X, \mu)$.

- (Assouad) Every median space embeds isometrically into an L^1–space.
- (Verheul) A complete median normed space is linearly isometric to an L^1–space.

Median spaces: non-discrete versions of CAT(0)-c.c.

Median spaces = non-discrete versions of 0-skeleta of CAT(0) cube complexes.
Analogous to real trees = non-discrete versions of simplicial trees.
Bowditch: The metric of a complete connected finite rank median metric space has a bi-Lipschitz equivariant deformation that is CAT(0) and has the same collection of convex subsets.

The rank of a median metric space X = the supremum over the set of integers k such that X contains an isometric copy of the set of vertices $\{-a, a\}^k$ of the cube of edge length $2a$, for some $a > 0$.

Convention

From now on we assume all median spaces to be complete and connected (hence geodesic).

Another metric notion of “negative curvature”

Fact

In every geodesic triangle in $\mathbb{H}^n$, each edge is contained in the tubular neighbourhood of radius $\ln 3$ *of the union of the other two edges.*

Example

Let G be a finitely generated group with properly discontinuous actions by isometries on the real hyperbolic space $\mathbb{H}^n$ such that $\mathbb{H}^n/G$ is compact. For every Cayley graph of G, there exists a constant $\delta > 0$ such that for every geodesic triangle, each edge is contained in the tubular neighbourhood of radius δ of the union of the other two edges.

Such a group is called a hyperbolic group. Similar terminology for metric spaces.

Hyperbolic spaces are everywhere

Examples

1. Any tree is 0-hyperbolic.
2. Any metric space X with finite diameter is δ-hyperbolic (for example take δ to be the diameter of X).
3. $\mathbb{R}^2$ is not hyperbolic.

Several good reasons to be interested in hyperbolic groups and spaces:

- Random (finitely presented) groups are hyperbolic (M. Gromov).
- Given an oriented surface Σ with genus at least 2, its curve complex $\mathcal{C}(\Sigma)$ is hyperbolic. (H. Masur- Y. Minsky).

Connection with actions on Banach spaces

Let G be locally compact second countable.
G has Kazhdan's property (T) iff every action by affine isometries on a Hilbert space has a global fixed point.

G is a-(T)-menable iff there exists an action by affine isometries on a Hilbert space that is proper.
In the equivalence above one can replace 'Hilbert space' by $L^1(X, \mu)$. Or by $L^p(X, \mu)$, for a fixed $p \in [1, 2]$.

1. G locally compact second countable has property (T) $\Leftrightarrow$ any continuous action by isometries on a median space has bounded orbits (Chatterji -D. - Haglund).
2. G a-(T)-menable $\Leftrightarrow$ it admits a proper continuous action by isometries on a median space (Chatterji -D. - Haglund).

Back to the matter of hyperbolicity *versus* median geometry

A group is said to be cubulable if it acts properly discontinuously cocompactly on a CAT(0) cube complex.

Theorem (Bergeron-Wise, Kahn-Markovic)

Every fundamental group of a hyperbolic 3*-manifold is cubulable.*

Using this and the theory of special cube complexes (Haglund-Wise), Agol proved the virtual Haken conjecture.

Degrees of compatibility with median geometry

A group is said to be

- cubulable if it acts properly discontinuously cocompactly on a CAT(0) cubical complex;
- strongly medianizable if it acts properly discontinuously cocompactly on a median space of finite rank;
- (weakly) medianizable if it acts properly discontinuously cocompactly on a median space of infinite rank.

If a finitely generated group is strongly (or weakly) medianizable then the median space on which it acts is also locally compact.

Cubulable $\Rightarrow$ strongly medianizable $\Rightarrow$ (weakly) medianizable.

Strongly medianizable versus cubulable

Possibly strongly medianizable $\Rightarrow$ cubulable.

They share the same properties:

- Tits alternative, super-rigidity (Caprace-Sageev, Fioravanti);
- irreducible uniform lattices in $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ with $k \geq 2$ act on every finite dimensional CAT(0)-c.c. with a global fixed point (Chatterji-Fernos-Iozzi)
- also true for actions on finite rank median spaces (Fioravanti).

Theorem (Chatterji-D.)

The above lattices (with $k \geq 1$) are medianizable.

In view of Fioravanti's result, this is the best that one can get for these lattices, in terms of compatibility with a median geometry.

It is unknown if the same lattices can act properly on an infinite dimensional CAT(0) cube complex.

Medianizable lattices

The result is interesting in the case of one factor ($k = 1$) too:

- not known if all arithmetic uniform lattices in $SO(n, 1)$, with n odd and larger than 3, are cubulable;
- in particular, there is an example of arithmetic uniform lattices in $SO(7, 1)$, constructed using octaves, that is thought not to be cubulable.

Rips-type Theorems

When can one extract from an action on a real tree $T \neq \mathbb{R}$ (minimal non-trivial) an action on a simplicial tree ?

Stable action = the family of stabilizers of non-trivial arcs satisfies the ACC (Ascending Chain Condition).

- (Bestvina-Feighn) If G is finitely presented then there exists an action on a simplicial tree with stabilizers of edges = stabilizers of arcs-by-cyclic.
- (Sela) Same with f.p. replaced by "trivial stabilizers of tripods".

Interest of a median version: it would relate the negation of property (T) ($\Leftrightarrow$ existence of an action with infinite orbits on a median space) to actions on CAT(0) cube complexes.

Rips-type Theorems for median spaces

Our theorem emphasizes that one cannot expect, for actions on median spaces, a theorem similar to Bestvina-Feighn:

- uniform lattices in $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ are finitely presented;
- they act properly, minimally and cocompactly on median spaces;
- they cannot act non-trivially cocompactly with amenable stabilizers on a CAT(0) cube complex (which would therefore have to be of finite dimension), by Chatterji-Fernos-Iozzi.

Rips-type Theorems for median spaces

- Still possible to obtain Rips-type theorems for actions on median spaces of finite rank.
- consistent with the case of real trees, since these are median spaces of rank one;
- a good candidate for the condition "tree not a line" might be "median space with no global fixed point at infinity under the full isometry group, not within bounded Hausdorff distance from a space $\mathbb{R}^n$ with ℓ^1 norm"

Where does the median geometry come from ?

Theorem (Chatterji-D.)

The real hyperbolic space $\mathbb{H}^n$ embeds isometrically and $\mathrm{Isom}(\mathbb{H}^n)$–equivariantly into a proper median space at finite Hausdorff distance from the embedded $\mathbb{H}^n$.

The embedding is constructed using another structure closely connected with the median geometry: measured walls (introduced by Cherix-Martin-Valette).

A key property: the metric on $\mathbb{H}^n$ coincides with the metric induced by a structure of measured walls.

A metric space can be isometrically embedded into a median space iff its metric is induced by a structure of measured walls.

Complex hyperbolic space

- $(\mathbb{H}^n_{\mathbb{C}}, \mathrm{dist})$ cannot be isometrically embedded into a median space. In particular, dist cannot be a wall metric.
- Faraut and Harzallah: $\mathbb{H}^n_{\mathbb{C}}$ equipped with $\mathrm{dist}^{\frac{1}{2}}$ has a structure of measured walls.
- For any $\alpha \in [1/2, 1)$, whenever dist^α is a metric induced by a structure of measured walls, $(\mathbb{H}^n_{\mathbb{C}}, \mathrm{dist}^\alpha)$ cannot be at bounded Hausdorff distance from a median space.

Acylindrically hyperbolic groups

The action of a group G on a metric space Y is $((D, B)$-)acylindrical if there are functions $D = D(\epsilon), B = B(\epsilon) > 0$ so that for any $\epsilon > 0$ if $x, y, x', y' \in Y$ have $d_Y(x, y) \geq D(\epsilon)$ then

$$|\{g \in G : d_Y(x', gx), d_Y(y', gy) \leq \epsilon\}| \leq B(\epsilon).$$

A group G is acylindrically hyperbolic if it admits an acylindrical, non-elementary action on a Gromov hyperbolic space.

Examples

1. *non-elementary hyperbolic groups;*
2. *mapping class groups that are not virtually abelian;*
3. *groups of outer automorphisms of free non-abelian groups,* $Out(F_n)$, $n \geq 2$.

Actions acylindrically hyp. groups cannot have I

Acylindrically hyp. groups cannot act by affine isometries on L^p–spaces, $p > 1$, such that their orbits are infinite.

1. Minasyan and Osin proved that there exists a finitely generated acylindrically hyperbolic group AH that is quotient of all hyperbolic groups.
2. With John Mackay we proved that random groups have the fixed point property for larger and larger classes of L^p–spaces.
3. De Laat and de la Salle proved that random groups satisfy fixed point properties for larger and larger classes of uniformly curved Banach spaces.
4. This implies that the Minasyan-Osin example cannot act by affine uniformly bi-Lipschitz transformations on L^p–spaces such that their orbits are infinite.

Actions acylindrically hyp. groups cannot have II

Acylindrically hyp. groups cannot act properly by affine uniformly bi-Lipschitz transformations on L^1–spaces.

1. This is due to examples of graphical small cancellation groups.
2. Graphical small cancellation groups have been introduced by Gromov with the view to construct groups with prescribed embedded subgraphs in their Cayley graphs. Gromov used this technique to construct the so called Gromov monsters, groups that contain families of expanders uniformly embedded into their Cayley graphs (Arzhantseva-Delzant, Osajda).
3. Gromov monsters cannot embed uniformly into any L^p–space, for $p \in [1, 2]$.
4. Gromov monsters are acylindrically hyperbolic (Gruber-Sisto).

Actions acylindrically hyp. groups can have

Theorem (D.-Mackay)

Any acylindrically hyperbolic group admits an affine uniformly Lipschitz action on ℓ_1 with unbounded orbits.

Corollary

Every infinitely presented graphical Gr(7) small cancellation group and every cubical small cancellation group admits a uniformly Lipschitz action on ℓ_1 with unbounded orbits.

This follows from the result of Gruber and Sisto, and that of Arzhantseva-Hagen proving that cubical small cancellation groups are acylindrically hyperbolic.

Happy Birthday!

La Mulți Ani!